A New Extension of Bessel Matrix Functions

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt
Department of Mathematics, College of Science and Arts in Unaizah, Qassim University, Qassim, Kingdom of Saudi Arabia
Email: drshehata2006@yahoo.com

Received 16 February 2014
Accepted 9 March 2015

Communicated by H.M. Srivastava

AMS Mathematics Subject Classification(2000): 33C10, 33C45, 33C60, 15A60

Abstract. The main aim of this paper is to define a new family of special matrix functions, say, \( p \)-Hypergeometric, \( p \)-Bessel, \( p \)-modified Bessel and \( p \)-Tricomi matrix functions, where \( p \) is a positive integer. The convergence properties of the \( p \)-Hypergeometric matrix function are established and the matrix differential equation is constructed. The differential properties, integral representations, integrals and various particular cases on the \( p \)-Bessel matrix function have been obtained. The \( p \)-modified Bessel matrix functions, and some properties of this function are investigated. Finally, the \( p \)-Tricomi matrix function and several results of the function are defined and studied.

Keywords: \( p \)-Hypergeometric matrix function; \( p \)-Bessel; \( p \)-modified Bessel and \( p \)-Tricomi matrix functions; A Jordan block; Differential properties; Integral representations.

1. Introduction and Preliminaries

Special matrix functions appear in connection with statistics [4], mathematical physics, theoretical physics, group representation theory, Lie groups theory [11], orthogonal matrix polynomials are closely related [14]. The Bessel functions are among the most important special functions with very diverse applications to physics, engineering and mathematical analysis [10, 17, 40]. In [15, 16, 19, 24, 33], the hypergeometric matrix function has been introduced as a matrix power series, hypergeometric matrix differential equation and an integral representation. In [1, 2, 3, 12, 13, 21], the authors introduced the
Bessel matrix functions and gave some results with Bessel matrix functions. In [8, 9, 20, 26, 27, 28, 30, 31, 32, 34, 35, 36, 37, 38, 39, 41], extension to the matrix function framework of the classical families of $p$-Kummers, $p$ and $q$-Appell, Humbert, Tricomi and Hermite-Tricomi matrix functions, and Gegenbauer, Rice, Humbert, Rainville’s, Bessel and Konhauser matrix polynomials have been proposed. The author has earlier studied the $p$ and $q$-Horn’s $H_2$, $pl(m,n)$-Kummer matrix functions of two complex variables under differential operators [25, 29]. The reason of interest for this family of Bessel function is due to their intrinsic mathematical importance and the fact that these polynomials have applications in physics. The present investigation is motivated essentially by several recent works [2, 3, 12, 13, 20, 21, 25, 26, 29, 32]. From this motivation, we prove some new properties for the new class of $p$-hypergeometric, $p$-Bessel, $p$-modified Bessel and $p$-Tricomi matrix functions with complex analysis.

The outline of this paper is organized as follows: Section 2 some basic relations involving the $p$-hypergeometric matrix function, convergence properties and matrix differential equation have been obtained. Some new properties, integral representations and integral expressions of $p$-Bessel matrix function are given in Section 3. Results of Sections 2 and 3 are used in Section 4 and 5 to study the properties of the new class of $p$-modified Bessel and $p$-Tricomi matrix functions.

Here, we discuss the properties of the new class of $p$-hypergeometric, $p$-Bessel, $p$-modified Bessel and $p$-Tricomi matrix functions and fix the notation in order to make the section self-consistent. The following specialized versions of the definitions and results of the previous works, which are useful in our study, are rephrased.

Throughout this paper, for a matrix $A$ in $\mathbb{C}^{N \times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of $A$. Its two-norm is denoted by $\|A\|$, and is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where $\|x\|_2 = (x^T x)^{1/2}$ denotes the usual Euclidean norm of a vector $x$ in $\mathbb{C}^N$. We say that a matrix $A$ in $\mathbb{C}^{N \times N}$ is a positive stable matrix if $\Re(\mu) > 0$ for all $\mu \in \sigma(A)$ (see [14]). In this expression, $\Re(z)$ is the real part of the complex number $z$. The matrices $I$ and $O$ will denote the identity matrix and the null matrix in $\mathbb{C}^{N \times N}$, respectively.

For the purpose of this work, we denote by $\mu(A)$ the logarithmic norm of $A$, which is defined by [6, 7]

$$\mu(A) = \max \left\{ z; \text{z eigenvalue of } \frac{A + A^T}{2} \right\} \quad (1.1)$$

where $A^T$ denotes the conjugate transpose. We denote by the number $\tilde{\mu}(A)$

$$\tilde{\mu}(A) = \min \left\{ z; \text{z eigenvalue of } \frac{A + A^T}{2} \right\}. \quad (1.2)$$
From [7], it follows that \( \|e^{At}\| \leq e^{\mu(A)} \) for \( t > 0 \). Hence, \( t^A = \exp(A \ln t) \) satisfies

\[
\|t^A\| = \left\{
\begin{array}{ll}
t^{\mu(A)} & \text{if } t \geq 1, \\
t^{\tilde{\mu}(A)} & \text{if } 0 \leq t \leq 1.
\end{array}
\right.
\] (1.3)

In [5], if \( f(z) \) and \( g(z) \) are holomorphic functions of the complex variable \( z \), which are defined in an open set \( \Omega \) of the complex plane, and \( A, B \) are matrices in \( \mathbb{C}^{N \times N} \) with \( \sigma(A) \subseteq \Omega \) and \( \sigma(B) \subseteq \Omega \), such that \( AB = BA \), then

\[
f(A)g(B) = g(B)f(A).
\] (1.4)

Let \( P \) be a positive stable matrix in \( \mathbb{C}^{N \times N} \), then Gamma matrix function \( \Gamma(P) \) is defined by [14]

\[
\Gamma(P) = \int_0^\infty e^{-tP-I} dt; \quad t^{P-I} = \exp\left((P-I) \ln t\right).
\] (1.5)

If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) such that \( A+nl \) is an invertible matrix for all integers \( n \geq 0 \), then \( \Gamma(A) \) is an invertible matrix, and Pochhammer symbol (shifted factorial) is defined by (see [18])

\[
(A)_n = A(A+I)(A+2I)\ldots(A+(n-1)I)
= \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1, \quad (A)_0 = I.
\] (1.6)

For the purpose of this our present work, we recall here the following some concepts and properties of the matrix functional calculus. \( H \) is the \( N \)-dimensional Jordan block defined by

\[
H = \begin{pmatrix}
n & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & n & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & n & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & n & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & n
\end{pmatrix} \in \mathbb{C}^{N \times N}.
\] (1.7)

Thus, if \( H \) is a Jordan block of the form (1.7), \( z > 0 \), we can write the image by means of the matrix functional calculus acting on the matrix \( H \) and the function of \( n, H_n(z) \), one can obtain Bessel matrix function of the first kind of order \( H \) as follows:

\[
J_H(z) = \left( \frac{z}{2} \right)^H \Gamma^{-1}(A+I) \mathbf{F}_1\left(-;H+I;\frac{z^2}{4}\right); \quad |z| < \infty; \quad |\arg(z)| < \pi
= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \Gamma^{-1}(H+(k+1)I) \left( \frac{z}{2} \right)^{H+2kI}.
\] (1.8)
for $|z| < \infty, |\arg(z)| < \pi$, where $\lambda \in \sigma(H)$ is not a negative integer (see [12]). Let us take $A \in \mathbb{C}^{N \times N}$ satisfying the condition

$$\lambda \text{ is not a negative integer for every } \lambda \in \sigma(A). \quad (1.9)$$

In [13], the Bessel matrix function $J_A(z)$ of the first kind of order $A$ was defined as follows:

$$J_A(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma^{-1}(A + (k + 1)I) \left( \frac{z}{2} \right)^{A+2kI}$$

$$= \left( \frac{z}{2} \right)^A \Gamma^{-1}(A + I) \, {}_0F_1\left(-; A + I; -\frac{z^2}{4}\right). \quad (1.10)$$

Now, we consider the general case. Let $A$ be a matrix satisfying the condition (1.9) and $H = \text{diag}(H_1, \ldots, H_k)$ be the Jordan canonical form of $A$, where $H_i$ is a Jordan block defined in the following form,

$$H_i = \begin{pmatrix}
  n_i & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
  0 & n_i & 1 & \ldots & 0 & 0 & 0 & 0 \\
  0 & 0 & n_i & \ldots & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & \ldots & 0 & 0 & n_i & 1 \\
  0 & 0 & 0 & \ldots & 0 & 0 & 0 & n_i \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \end{pmatrix} \in \mathbb{C}^{N_i \times N_i}, \quad (1.11)$$

for $N_i \geq 1$, $N_1 + N_2 + \ldots + N_k = N$ and $H_i = (n_i)$ if $H_i$ is a Jordan block of size $1 \times 1$ for $n_i$ is not negative integer for $1 \leq i \leq k$. Bessel matrix function of the first kind can be written as the following

$$J(t, H) = [\text{diag}_{1 \leq i \leq k}(J_{H_i}(t))]. \quad (1.12)$$

Furthermore, if $P$ be an invertible matrix in $\mathbb{C}^{N \times N}$ such that

$$H = \text{diag}(H_1, H_2, \ldots, H_k) = PAP^{-1}, \quad (1.13)$$

then we have the same Bessel matrix function of the first kind of order $A$ in (1.10) (see [13]).

Let $C$ be a matrix in $\mathbb{C}^{N \times N}$ such that $C + nI$ is an invertible matrix for every integer $n \geq 0$ [22, 23]. Then the hypergeometric matrix function can be defined by the matrix power series in the form

$$_0F_1(-; C; z) = \sum_{n=0}^{\infty} \frac{[(C)_n]^{-1} z^n}{n!}.$$

Let $P$ and $Q$ be positive stable matrices in $\mathbb{C}^{N \times N}$, then Beta matrix function $B(P, Q)$ is defined by [14]

$$B(P, Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} dt. \quad (1.15)$$
If $P$, $Q$ and $P+Q$ are positive stable matrices in $\mathbb{C}^{N \times N}$ for which $PQ = QP$, and $P + nI$, $Q + nI$ and $P + Q + nI$ are invertible matrices for $n \geq 0$ [15], then

$$B(P, Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P + Q). \quad (1.16)$$

2. Properties of $p$-Hypergeometric Matrix Function

Let $H$ be a Jordan block as in (1.7). The $p$-hypergeometric matrix function of the complex variable $z$ is defined by the power series in the form

$$\mathbf{p}F_1\left(-; H; z\right) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} z^k [(H)_{k}]^{-1} = \sum_{k=0}^{\infty} U_k z^k; p \geq 1 \quad (2.1)$$

where $U_k = \frac{1}{(pk)!} [(H)_{k}]^{-1}$, $(H)_{k}$ is defined by (1.6) and $p \geq 1$. If $k > \|H\|$, then by perturbation lemma [5], we can write

$$\|(H/k + I)^{-1}\| \leq \frac{k}{k - \|H\|}, \quad (2.2)$$

$$(H + kI)^{-1} = k(H/k + I)^{-1}, \quad (2.3)$$

for $k > \|H\|$. The convergence properties of $p$-hypergeometric matrix function, we use the ratio test with (2.1), (2.2) and (2.3) as in the form

$$\lim_{k \to \infty} \left| \frac{U_{k+1}}{U_k} \right|^\frac{z^{k+1}}{z^k} = \lim_{k \to \infty} \left| \frac{(pk)![(H/k + I)^{-1}]^{-1}}{k(p(k + 1))!} \right|^\frac{z^{k+1}}{z^k}$$

$$\leq \lim_{k \to \infty} \left| \frac{z}{k^{p+1}(p + \frac{p-1}{k})(p + \frac{p-2}{k}) \ldots (p + \frac{1}{k})} \right|$$

$$\leq \lim_{k \to \infty} \frac{1}{k^{p+1}(p + \frac{p-1}{k})(p + \frac{p-2}{k}) \ldots (p + \frac{1}{k})(1 - \|H/k\|)} |z| = 0$$

for all $z$. Thus, the power series (2.1) is convergent for all complex number $z$. Hence, the $p$-hypergeometric matrix function is an entire function.

Now, we append this section by introducing the differential operator $\theta = z \frac{d}{dz}$ to the entire function in successive manner as the following

$$\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p})(\theta - \frac{3}{p}) \ldots (\theta - \frac{p-1}{p}) \mathbf{p}F_1\left(-; H; z\right)$$

$$= \frac{1}{p^p} \sum_{k=1}^{\infty} \frac{pk(pk - 1) \ldots (pk - p + 1)}{(pk)!} z^k [(H)_{k}]^{-1}$$

$$= \frac{1}{p^p} \sum_{k=0}^{\infty} \frac{1}{(pk)!} z^{k+1} [(H + kI)^{-1}] [(H)_{k}]^{-1} = \frac{z}{p^p} H^{-1} \mathbf{p}F_1\left(-; H + I; z\right)$$
i.e, equivalently
\[
\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p}) \cdots (\theta - \frac{p-1}{p}) F_0^p(-; H; z) - \frac{z}{p^p} H^{-1} F_0^p(-; H + I; z) = 0. \tag{2.4}
\]

Similarly to (2.4), we can write
\[
\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p}) \cdots (\theta - \frac{p-1}{p}) F_0^p(-; H; z),
\]
then
\[
\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p}) \cdots (\theta - \frac{p-1}{p})(\theta I + H - I) F_0^p(-; H; z)
\]
\[
= \frac{1}{p^p} \sum_{k=1}^{\infty} \frac{(k I + H - I)}{(pk-p)!} z^k [ (H)_k ]^{-1} = \frac{1}{p^p} \sum_{k=0}^{\infty} \frac{1}{(pk)!} z^{k+1} [ (H)_k ]^{-1}
\]
\[
= \frac{z}{p^p} F_0^p(-; H; z).
\]

These results are summarized in the following.

**Theorem 2.1.** Let \( H \) be a Jordan block as in (1.7), then the \( p \)-hypergeometric matrix function \( p F_1^0(-; H; z) \) is a solution of the matrix differential equation of order \( p + 1 \)
\[
\left[ \theta(\theta - \frac{1}{p})(\theta - \frac{2}{p}) \cdots (\theta - \frac{p-1}{p})(\theta I + H - I) - \frac{z}{p^p} \right] F_0^p(-; H; z) = 0. \tag{2.5}
\]

**Remark 2.2.** Putting \( p = 1 \), we get the hypergeometric matrix function \( 0 F_1 \)
\[
0 F_1(-; H; z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k [ (H)_k ]^{-1}.
\]

The previous properties can be generalized as the following: Let \( H_i \) is defined as in (1.7) and \( H = \text{diag}(H_1, H_2, \ldots, H_k) \) be a matrix in \( \mathbb{C}^{N \times N} \). Here \( N_i \) is not negative integer for \( 1 \leq i \leq k \). The matrix \( H = \text{diag}(H_1, H_2, \ldots, H_k) \), the properties (2.1–2.7) can be easily provided.

Also, if \( P \) be an invertible matrix in \( \mathbb{C}^{N \times N} \) such that \( H = \text{diag}(H_1, H_2, \ldots, H_k) = P C P^{-1} \), we can give the following theorem for the matrix \( C \).

**Theorem 2.3.** Let \( C \) be a matrix in \( \mathbb{C}^{N \times N} \) such that the matrix \( C + n I \) is an invertible matrix for every integer \( n \geq 0 \). Then we have
\[
\left[ \theta(\theta - \frac{1}{p})(\theta - \frac{2}{p}) \cdots (\theta - \frac{p-1}{p})(\theta I + C - I) - \frac{z}{p^p} \right] F_0^p(-; C; z) = 0.
\]

where \( F_0^p(-; C; z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} z^k [ (C)_k ]^{-1}; p \geq 1. \)
Remark 2.4. For \( p \geq 1 \), the \( p \)-hypergeometric matrix function \( \phi \binom{0}{1} \left( -; C; z \right) \) is an entire function.

3. On \( p \)-Bessel Matrix Function

The basic Bessel function is defined when \(|p| < 1\). If \(|p| \geq 1\) we try to define a new class of matrix function, say, \( p \)-Bessel matrix function; \( p \geq 1 \); of the first kind as in the following form

\[
p^J_H(z) = \left( \frac{z}{2} \right)^H \Gamma^{-1}(H + I) \, \phi \binom{0}{1} \left( -; H + I; -\frac{z^2}{4} \right)
= \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \Gamma^{-1}(H + (k + 1)I) \left( \frac{z}{2} \right)^{H + 2kI}
\]

(3.1)

and

\[
\phi \binom{0}{1} \left( -; H + I; -\frac{z^2}{4} \right) = \Gamma(H + I) \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \Gamma^{-1}(H + (k + 1)I) \left( \frac{z}{2} \right)^{2k}
\]

where \( H \) is a Jordan block as in (1.7), and the \( p \)-Bessel matrix function is an entire function.

For the \( p \)-Bessel matrix function which is defined in (3.1), we have:

**Theorem 3.1.** Let \( H \) be a Jordan block as in (1.7) and \( p^J_H(z) \) is defined in (3.1). Then the following differential properties are satisfied

\[
\frac{d}{dz} \left[ z^H \, p^J_H(z) \right] = z^H \, p^J_{H-I}(z),
(1 \frac{d}{dz})^m \left[ z^H \, p^J_H(z) \right] = z^{H-mI} \, p^J_{H-mI}(z),
\]

\[
z \, p^J_H'(z) = z \, p^J_{H-I}(z) - H \, p^J_H(z),
\]

\[
\frac{d}{dz} \left( \frac{d}{dz} - \frac{1}{p} \right) \left( \frac{d}{dz} - \frac{2}{p} \right) \ldots \left( \frac{d}{dz} - \frac{2^{k-1}}{p} \right) \left[ z^{-H} \, p^J_H(z) \right] = -\frac{1}{2} \left( \frac{2}{p} \right)^p z^{-A} \, p^{J_{A+I}}(z).
\]

**Proof.** From (3.1), we have

\[
\frac{d}{dz} \left[ z^H \, p^J_H(z) \right] = \frac{d}{dz} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \Gamma^{-1}(H + (k + 1)I) z^{2k+2kI} \right].
\]

Differentiating term by term of the above equality, we get

\[
\frac{d}{dz} \left[ z^H \, p^J_H(z) \right] = \sum_{k=1}^{\infty} \frac{(-1)^k(2H + 2kI)}{2^{k+2k}(pk)!} \Gamma^{-1}(H + (k + 1)I) z^{2H+(2k-1)I}.
\]
On setting $k = r + 1$ and simplification, gives
\[
\frac{d}{dz} \left[ z^H pJ_H(z) \right] = z^H \sum_{r=0}^{\infty} \frac{(-1)^r}{2^{H+i(2r-1)}(pk)!} \Gamma^{-1}(H - I + (r + 1)I) z^{H+i(2r-1)}I.
\]
This can be written as
\[
\frac{d}{dz} \left[ z^H pJ_H(z) \right] = z^H pJ_{H-1}(z). \tag{3.3}
\]
From (3.3) and mathematical induction, we obtain
\[
\left( \frac{1}{z} \frac{d}{dz} \right)^m \left[ z^H pJ_H(z) \right] = z^H pJ_{H-mI}(z). \tag{3.4}
\]
Development of the left hand side of (3.3), we have the relation
\[
z pJ'_H(z) = z pJ_{H-I}(z) - H pJ_H(z). \tag{3.5}
\]
This relation is called a matrix differential recurrence relation.

Insert the factor $z$ on each side of equation (3.1), and differentiating term by term, we get
\[
\frac{d}{dz} \left( \frac{d}{dz} - \frac{2}{p} \right) \left( \frac{d}{dz} - \frac{2}{p} \right) \cdots \left( \frac{d}{dz} - \frac{p-1}{p} \right) \left[ z^{-H} pJ_H(z) \right]
= \left( \frac{2}{p} \right)^p \sum_{k=1}^{\infty} \frac{(-1)^k(pk)(pk-1) \cdots (pk+p-1)}{2^{H+2kI}(pk)!} \Gamma^{-1}(H + (k+1)I) z^{2k-1} \tag{3.6}
\]
Thus the proof is completed.

Remark 3.2. Taking $p = 1$, we produce the Bessel matrix function $J_H(z)$ (see [2, 3]).

The previous properties can be generalized as the following: Let $H_i$ is defined as in (1.7) and $H = diag(H_1, H_2, \ldots, H_k)$ be a matrix in $\mathbb{C}^{N \times N}$. Here $N_i$ is not negative integer for $1 \leq i \leq k$. The matrix $H = diag(H_1, H_2, \ldots, H_k)$, the properties (3.1-3.6) can be easily provided.

Furthermore, if $P$ be an invertible matrix in $\mathbb{C}^{N \times N}$ such that $H = diag(H_1, H_2, \ldots, H_k) = PAP^{-1}$, then we can give the following theorem for the matrix $A$.

**Theorem 3.3.** Let $A$ be a matrix in $\mathbb{C}^{N \times N}$ satisfy the condition (1.9) and $pJ_A(z)$
is defined in (3.1). Then the following differential properties are satisfied

\[
pJ_A(z) = \left(\frac{z}{2}\right)^A \Gamma^{-1}(A + I) \, _pF_1\left(-; A + I; -\frac{z^2}{4}\right)
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \Gamma^{-1}(A + (k + 1)I) \left(\frac{z}{2}\right)^{A+2kI},
\]

\[
\frac{d}{dz} \left[z^A pJ_A(z)\right] = z^A pJ_{A-1}(z),
\]

\[
\left(\frac{1}{z} \frac{d}{dz}\right)^m \left[z^A pJ_A(z)\right] = z^{A-mI} pJ_{A-mI}(z),
\]

\[
z pJ_A'(z) = z pJ_{A-1}(z) - A pJ_A(z),
\]

\[
\frac{d}{dz} \left(\frac{d}{dz} - \frac{2}{p}\right) \left(\frac{d}{dz} - \frac{2}{p}\right) \cdots \left(\frac{d}{dz} - \frac{2}{p} - \frac{1}{p}\right) \left[z^{-A} pJ_A(z)\right] = -\frac{1}{2} \left(\frac{2}{p}\right)^p z^{-A} pJ_{A+I}(z).
\]

**Remark 3.4.** Putting \(p = 1\), we produce the Bessel matrix function \(J_A(z)\) (see [21]).

Now, the integral representations of the \(p\)-Bessel matrix function defined in (3.1) is given thrash the following theorem.

**Theorem 3.5.** Let \(H\) be a Jordan block matrix as in (1.7) such that \(\tilde{\mu}(H) > -\frac{1}{2}\) and \(\arg z < \pi\). Then the integral representation for \(p\)-Bessel matrix function is given by

\[
pJ_H(z) = \left(\frac{z}{2}\right)^H \frac{\Gamma^{-1}(H + \frac{1}{2}I)}{\sqrt{\pi}} \int_{-1}^{1} (1 - t^2)^{H-\frac{1}{2}} \, _pF_1\left(-; \frac{1}{2}; -\frac{z^2t^2}{4}\right) dt
\]

where \(\, _pF_1\left(-; \frac{1}{2}; -\frac{z^2}{4}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \left(\frac{z^2}{2}\right)^k\).

**Proof.** Using the Gauss’s multiplication theorem

\[
\Gamma(mH) = (2\pi)^{-m} m^{-mH-\frac{1}{2}} \prod_{i=1}^{m} \Gamma(H + \frac{i - 1}{m}I) \forall m = 1, 2, 3, \ldots.
\]

By the Lemma 2 of [15], if \(k > -\frac{1}{2}\), one gets the identity

\[
\Gamma^{-1}(H + (k + 1)I) = \frac{\Gamma^{-1}(H + \frac{1}{2}I)}{\Gamma(k + \frac{1}{2})} \int_{-1}^{1} t^{2k} (1 - t^2)^{H-\frac{1}{2}} dt.
\]
From (3.1), (3.8) and (3.9), we obtain
\[ p_J^H(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \left( \frac{z}{2} \right)^{H+2kI} \frac{\Gamma^{-1}(H + \frac{1}{2})}{\Gamma(k + \frac{1}{2})} \int_{-1}^{1} t^{2k} (1 - t^2)^{-H - \frac{1}{2}} dt \]
\[ = \left( \frac{z}{2} \right)^H \frac{\Gamma^{-1}(H + \frac{1}{2})}{\sqrt{\pi}} \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \left( \frac{1}{2} \right)^{2k} \left( zt^2 \right)^{H - \frac{1}{2}} \left( 1 - t^2 \right)^{-H - \frac{1}{2}} dt. \]

**Theorem 3.6.** Let \( H \) be a Jordan block as in (1.7) such that \( \mu \) is not a negative integer for every \( \mu \in \sigma(H) \). Then the another integral representation for \( p \)-Bessel matrix function is
\[ p_J^H(z) = \left( \frac{z}{2} \right)^H \frac{\Gamma^{-1}(H + \frac{1}{2})}{\sqrt{\pi}} \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \left( \frac{1}{2} \right)^{2k} \left( zt^2 \right)^{H - \frac{1}{2}} \left( 1 - t^2 \right)^{-H - \frac{1}{2}} dt. \]

**Proof.** Another integral representation of \( p_J^H(z) \) can be established starting from the formula
\[ \Gamma^{-1}(H + (k + 1)I) = \frac{1}{2\pi i} \int_c e^{s(H + (k + 1)I)} ds. \] (3.11)
Using (3.1) and (3.11), we have
\[ p_J^H(z) = \left( \frac{z}{2} \right)^H \frac{1}{2\pi i} \int_c e^{s(H + (k + 1)I)} ds \]
\[ = \left( \frac{z}{2} \right)^H \frac{1}{2\pi i} \int_c e^{s(H + (k + 1)I)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \left( \frac{z^2}{4s} \right)^k ds. \]

These integral representations can be generalized through the following results.

**Theorem 3.7.**
(i) Let \( A \) be a matrix in \( \mathbb{C}^{N \times N} \) such that \( \tilde{\mu}(A) > -\frac{1}{2} \) and \( |\arg z| < \pi \). Then we obtain
\[ p_J^A(z) = \left( \frac{z}{2} \right)^A \frac{\Gamma^{-1}(A + \frac{1}{2}I)}{\sqrt{\pi}} \int_{-1}^{1} (1 - t^2)^{A - \frac{1}{2}I} \frac{\Gamma^{-1}(A + \frac{1}{2})}{\Gamma(k + \frac{1}{2})} \int_{-1}^{1} t^{2k} (1 - t^2)^{-A - \frac{1}{2}} dt. \]
(ii) If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) satisfy the condition (1.9). Then we obtain
\[ p_J^A(z) = \left( \frac{z}{2} \right)^A \frac{1}{2\pi i} \int_c e^{s(A + I)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(pk)!} \left( \frac{z^2}{4s} \right)^k ds. \]
Here, we derive some integrals expressions involving $p$-Bessel matrix function defined in (3.1). These are contained in the next theorem.

**Theorem 3.8.** If $H, M$ and $H - M$ are Jordan blocks as in (1.7) satisfying the conditions $\tilde{\mu}(H) > -1, \tilde{\eta}(H - M) > -1$ where $\eta = \mu - \nu$, we obtain

$$pJ_H(z) = 2\Gamma^{-1}(H - M)\left(\frac{z}{2}\right)^{H-M} \int_0^1 (1 - t^2)^{H-M-I} t^M M \ pJ_M(zt) dt. \quad (3.12)$$

**Proof.** Consider the integral matrix functional $\int_0^1 (1 - t^2)^{H-M-I} t^M M \ pJ_M(zt) dt$ and substitute the series (3.1) for $pJ_M(zt)$ to obtain

$$\int_0^1 (1 - t^2)^{H-M-I} t^M M \ pJ_M(zt) dt = \sum_{k=0}^{\infty} \frac{(-1)^k (z)^{M+2kI}}{(pk)!} \Gamma^{-1}(M + (k + 1)I) \int_0^1 (1 - t^2)^{H-M-I} t^{2M+(2k+1)} I dt.$$  

Let $u = t^2$ in the last integral, we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k (z)^{M+2kI}}{(pk)!} \Gamma^{-1}(M + (k + 1)I) \int_0^1 (1 - u)^{H-M-I} u^{M+2kI} du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (z)^{M+2kI}}{(pk)!} \frac{1}{2} \Gamma(H - M) \Gamma^{-1}(H + (k + 1)I)$$

$$= \frac{1}{2} \Gamma(H - M)(\frac{z}{2})^M_H \ pJ_H(z)$$

this is the required result. \[\blacksquare\]

One can generalize the above result as follows.

**Theorem 3.9.** If $A, B$ and $A - B$ are matrices in $\mathbb{C}^{N \times N}$ satisfying the conditions, $\tilde{n}(A) > \tilde{n}(B) > -1, \tilde{r}(A - B) > -1$ for all $\eta = n - m$, then we obtain

$$pJ_A(z) = 2\Gamma^{-1}(A - B)\left(\frac{z}{2}\right)^{A-B} \int_0^1 (1 - t^2)^{A-B-I} t^B B \ pJ_B(zt) dt.$$  

This class of integrals forms is considered in the next theorem.

**Theorem 3.10.** If $H, M$ and $H + M$ are Jordan blocks as in (1.7) satisfying the conditions $\tilde{\mu}(H) > -1, \tilde{\eta}(H + M) > 0$ for all $\eta = \mu + \nu, a, b$ are arbitrary
numbers with $\Re(a \pm ib) > 0$ and $HM = MH$, then

\[
\int_0^\infty e^{-az} z^{M-1} p J_H(bz) \, dz = \Gamma(H + M) \Gamma^{-1}(H + I) a^{-M} \left( \frac{b}{2a} \right)^H
\]

\[\sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2}(H + M) \right)_k \left( \frac{1}{2}(H + M + I) \right)_k \left( \frac{b}{a} \right)^{2k}}{(pk)! \left( \frac{b}{a} \right)^{(H + I)_k}} \Gamma(H + M + (2k-1)I)
\]

(3.13)

where

\[
\sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2}(H + M) \right)_k \left( \frac{1}{2}(H + M + I) \right)_k \left( \frac{b}{a} \right)^{2k}}{(pk)! \left( \frac{b}{a} \right)^{(H + I)_k}} \Gamma(H + M + (2k-1)I)
\]

(3.14)

Proof. First assume that $|\frac{b}{a}| < 1$, substitute the series for $p J_A(bz)$ in the left hand side of the integral (3.13) to get

\[
\int_0^\infty e^{-az} z^{M-1} p J_H(bz) \, dz = \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2}(H + M) \right)_k \left( \frac{1}{2}(H + M + I) \right)_k \left( \frac{b}{a} \right)^{2k}}{(pk)! \left( \frac{b}{a} \right)^{(H + I)_k}} \Gamma(H + M + (2k-1)I)
\]

(3.15)

Applying the duplication formula for the Gamma matrix function, we have

\[
\Gamma(H + M) = \frac{2^{H+M-I}}{\sqrt{\pi}} \Gamma\left( \frac{1}{2}(H + M) \right) \Gamma\left( \frac{1}{2}(H + M + I) \right),
\]

(3.14)

and

\[
\Gamma\left( \frac{1}{2}(H + M) + kI \right) = \left( \frac{1}{2}(H + M) \right)_k \Gamma\left( \frac{1}{2}(H + M) \right),
\]

(3.15)
Using the equations (3.14) and (3.15), we get
\[
\int_0^\infty e^{-az} z^{M-I} p_J(bz) \, dz
= \Gamma(H+M) \Gamma^{-1}(H+I) \times \sum_{k=0}^\infty \left( -1 \right)^k \frac{b^2 (H+M)_k (\frac{1}{2} (H+M+I)_k)}{2^H a^{H+2k} (pk)!} \cdot
\]
Hence, the proof is completed.

Furthermore, if \( P \) is an invertible matrix in \( \mathbb{C}^{N \times N} \) such that \( H = \text{diag}(H_1, H_2, \ldots, H_k) = P A P^{-1} \), then one can get the following theorem for the matrix \( A \).

**Theorem 3.11.** If \( A, B \) and \( A+B \) are matrices in \( \mathbb{C}^{N \times N} \) satisfying the conditions
\[
\tilde{n}(A) > -1, \quad \tilde{n}(A+B) > 0 \] for all \( r = n + m \), \( a, b \) are arbitrary numbers with \( \Re(a \pm ib) > 0 \) and \( AB = BA \), then
\[
\int_0^\infty e^{-az} z^{B-I} p_J(bz) \, dz
= \Gamma(A + B) \Gamma^{-1}(A + I) a^{-B} \left( \frac{b}{2a} \right)^A p_F \left( \frac{1}{2} (A + B), \frac{1}{2} (A + B + I); A + I; -\frac{b^2}{a^2} \right) : \left| \frac{b}{a} \right| < 1.
\]

When \( M = H + I \) or \( H + 2I \), the \( p_F \) in (3.13) reduces to a \( p_F \), which can be summed by the binomial theorem for \( \left| \frac{b}{a} \right| < 1 \).

**Corollary 3.12.** If \( H \) is a Jordan block as in (1.7) such that \( \tilde{\mu}(H) > -\frac{1}{2} \) and \( a, b \) are arbitrary numbers with \( \Re(a \pm ib) > 0 \), then
\[
\int_0^\infty e^{-az} z^{H} p_J(bz) \, dz = \frac{(2b)^H \Gamma(H + \frac{1}{2} I)}{\sqrt{\pi a^2 H + I}} p_F \left( H + \frac{1}{2} I, -; -; -\frac{b^2}{a^2} \right) \quad (3.16)
\]
where
\[
p_F \left( H + \frac{1}{2} I; -; -\frac{b^2}{a^2} \right) = \sum_{k=0}^\infty \frac{(-1)^k (H + \frac{1}{2} I)_k}{(pk)!} \left( \frac{b}{a} \right)^{2k} : \left| \frac{b}{a} \right| < 1.
\]

**Proof.** From (3.1) and \( \left| \frac{b}{a} \right| < 1 \), in the left hand side of the integral (3.16) can be
\[
\int_0^\infty e^{-az} z^H p J_H(bz)\,dz = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^{-1}(H+(k+1)I) \left(\frac{b}{2}\right)^{H+2kI}}{(pk)!} \int_0^\infty e^{-az} z^{H+2kI} \,dz.
\]

On the other hand, with the help of Gamma matrix function in (1.5), we get
\[
\int_0^\infty e^{-az} z^{2H+2kI} \,dz = \frac{1}{a^{2H+(2k+1)I}} \int_0^\infty e^{-t^2} t^{2H+2kI} \,dt = \frac{1}{a^{2H+(2k+1)I}} \Gamma(2H+(2k+1)I).
\]

Using (3.18) and the duplication formula for the Gamma matrix function, we obtain
\[
\int_0^\infty e^{-az} z^H p J_H(bz)\,dz = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^{-1}(H+(k+1)I) \left(\frac{b}{2}\right)^{H+2kI}}{(pk)!} \frac{\Gamma(2H+(2k+1)I)}{a^{2H+(2k+1)I}}.
\]

Thus, the proof is established.

**Corollary 3.13.** If \(H\) is a Jordan block as in (1.7) satisfying the condition \(\tilde{\mu}(H) > -\frac{3}{2}\) and \(a, b\) are arbitrary numbers with \(\Re(a \pm ib) > 0\), then
\[
\int_0^\infty e^{-az} z^{H+I} p J_H(bz)\,dz = \frac{2(2b^2 \Gamma(H + \frac{3}{2} I))}{a^{2H+2I}} \frac{\Gamma_0 \left( H + \frac{3}{2} I, -; -; - \frac{b^2}{a^2} \right)}{\sqrt{\pi a^{2H+2I}}}
\]

where
\[
\frac{\Gamma_0 \left( H + \frac{3}{2} I, -; -; - \frac{b^2}{a^2} \right)}{\sqrt{\pi a^{2H+2I}}} = \sum_{k=0}^{\infty} \frac{(-1)^k \left( H + \frac{3}{2} I \right)_k \left( \frac{b}{a} \right)^{2k}}{(pk)!} \left| \frac{b}{a} \right| < 1.
\]

**Proof.** The result (3.18) follows immediately form (3.16) if we differentiate both sides with respect to \(a\). Thus, the proof is completed.

The next corollary gives another integral for \(p\)-Bessel matrix function.
Corollary 3.14. If $H$ is a Jordan block as in (1.7) satisfying the condition $\tilde{\mu}(H) > -1$, and $a$ and $b$ are arbitrary numbers with $\Re(a \pm ib) > 0$ and $\left| \frac{b}{a} \right| < 1$, then

$$
\int_0^\infty e^{-az} \rho J_H(bz)dz = \frac{1}{a} \left( \frac{b}{2a} \right)^H \frac{\rho}{2} F_1 \left( \frac{1}{2}; \frac{1}{2}(H + I), \frac{1}{2}(H + 2I); H + I; -\frac{b^2}{a^2} \right).
$$

(3.19)

Proof. From the definition of $\rho J_H(z)$, we have

$$
\int_0^\infty e^{-az} \rho J_H(bz)dz = \int_0^\infty e^{-az} \sum_{k=0}^\infty \frac{(-1)^k \Gamma^{-1}(H + (k + 1)I)}{(pk)!} \left( \frac{b}{2} \right)^{H+2kI} dz
$$

$$
= \sum_{k=0}^\infty \frac{(-1)^k \Gamma^{-1}(H + (k + 1)I)}{(pk)!} \left( \frac{b}{2} \right)^{H+2kI} \int_0^\infty e^{-az} z^{H+2kI} dz.
$$

Using the Gamma matrix function, we get the integral on the right hand side equals

$$
\int_0^\infty e^{-az} z^{H+2kI} dz = \frac{\Gamma(H + (2k+1)I)}{a^{H+(2k+1)I}}
$$

and

$$
\Gamma(H + (2k+1)I) = \frac{2^{H+2kI}}{\sqrt{\pi}} \Gamma((k + \frac{1}{2})I + \frac{1}{2}H) \Gamma((k + 1)I + \frac{1}{2}H).
$$

Evaluating the inner integral, we get

$$
\int_0^\infty e^{-az} \rho J_H(bz)dz
$$

$$
= \sum_{k=0}^\infty \frac{(-1)^k \Gamma^{-1}(H + (k + 1)I)}{(pk)!} \left( \frac{b}{2} \right)^{H+2kI} \frac{\Gamma(H + (2k+1)I)}{a^{H+(2k+1)I}}
$$

$$
= \frac{b^n}{\sqrt{\pi} a^{H+I}} \sum_{k=0}^\infty \frac{\Gamma((k+\frac{1}{2})I + \frac{1}{2}H) \Gamma((k+1)I + \frac{1}{2}H) \Gamma^{-1}(H + (k+1)I)}{(pk)!} \left( \frac{b^2}{a^2} \right)^k
$$

$$
= \frac{1}{a} \left( \frac{b}{2a} \right)^H \sum_{k=0}^\infty \frac{\left( \frac{1}{2}(H + I) \right)_k \left( \frac{1}{2}(H + 2I) \right)_k \Gamma(H + I)_k^{-1}}{(pk)!} \left( -\frac{b^2}{a^2} \right)^k.
$$

Thus, the result is established.

Corollary 3.15. If $H$ is a Jordan block as in (1.7) satisfying the condition $\tilde{\mu}(H) >$
Corollary 3.16. (iii) If 

\[
\int_0^\infty e^{-az}z^{-1} pJ_H(bz)dz
\]

\[
\begin{align*}
\nonumber &= H^{-1}\left( \frac{b}{2a} \right)^H \Gamma \left( \frac{1}{2}, \frac{1}{2}(H + I) ; H + I, -\frac{b^2}{a^2} \right). \\
\nonumber &\quad (3.20)
\end{align*}
\]

Proof. The proof of the corollary is very similar to corollary 3.14.

Now, let \( P \) be an invertible matrix in \( \mathbb{C}^{N \times N} \) such that \( H = \text{diag}(H_1, H_2, \ldots, H_k) = PAP^{-1} \), we give a generalization for the above results in the following corollary for the matrix \( A \).

Corollary 3.16.

(i) If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) such that \( \tilde{n}(A) > -\frac{1}{2} \) and \( a, b \) are arbitrary numbers with \( \Re(a \pm ib) > 0 \), then we have

\[
\int_0^\infty e^{-az}z^{-1} pJ_A(bz)dz = \frac{(2b)^A \Gamma(A + \frac{1}{2}I)}{\sqrt{\pi a^2 A + I}} \, _pF_0 \left( A + \frac{1}{2}I, -; -\frac{b^2}{a^2} \right),
\]

for \( \left| \frac{b}{a} \right| < 1 \).

(ii) If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) such that \( \tilde{n}(A) > -\frac{3}{2} \), \( a, b \) are arbitrary numbers with \( \Re(a \pm ib) > 0 \) and \( \left| \frac{b}{a} \right| < 1 \), then we have

\[
\int_0^\infty e^{-az}z^{A + I} pJ_A(bz)dz = \frac{2(2b)^A \Gamma(A + \frac{1}{2}I)}{\sqrt{\pi a^2 A + 2I}} \, _pF_0 \left( A + \frac{3}{2}I, -; -\frac{b^2}{a^2} \right).
\]

(iii) If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) satisfying the condition \( \tilde{n}(A) > -1 \), and \( a, b \) are arbitrary numbers with \( \Re(a \pm ib) > 0 \) and \( \left| \frac{b}{a} \right| < 1 \), then we obtain

\[
\int_0^\infty e^{-az} pJ_A(bz)dz = \frac{1}{a} \left( \frac{b}{2a} \right)^A \, _pF_1 \left( \frac{1}{2}(A + I), \frac{1}{2}(A + 2I); A + I, -\frac{b^2}{a^2} \right).
\]

(iv) If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) satisfying the condition \( \tilde{n}(A) > 0 \), \( a \) and \( b \) are arbitrary numbers with \( \Re(a \pm ib) > 0 \) and \( \left| \frac{b}{a} \right| < 1 \), then we obtain

\[
\int_0^\infty e^{-az}z^{-1} pJ_A(bz)dz = A^{-1} \left( \frac{b}{2a} \right)^A \, _pF_1 \left( \frac{1}{2}A, \frac{1}{2}(A + I); A + I, -\frac{b^2}{a^2} \right).
\]
The next theorem gives another infinite integral of a \( p \)-Bessel matrix function due to Hankel integral.

**Theorem 3.17.** For \( H, M \) and \( H + M \) are Jordan block as in (1.7) satisfying the conditions \( \tilde{\mu}(H) > -1 \), \( \tilde{r}(H + M) > 0 \) for all \( r = \mu + \nu \), \( a \) and \( b \) are arbitrary numbers with \( \Re(a^2) > 0 \), then we have

\[
\int_0^\infty e^{-a^2z^2} z^{M-I} pJ_H(bz)dz = \frac{1}{2}a^{-H} \Gamma\left(\frac{1}{2}(H + M)\right) \Gamma^{-1}(H + I) \\
\times \left( \frac{b}{2a} \right)^H \left( \frac{1}{2}(H + M); H + I; -\frac{b^2}{4a^2} \right),
\]

where

\[
\left( \frac{1}{2}(H + M); H + I; -\frac{b^2}{4a^2} \right) = \sum_{k=0}^\infty \frac{(-1)^k (\frac{1}{2}(H + M))_k (H + I)_k^{-1}}{(pk)!} \left( \frac{b}{2a} \right)^{2k}.
\]

**Proof.** The integral can be evaluated using term by term integration as follows

\[
\int_0^\infty e^{-a^2z^2} z^{M-I} pJ_H(bz)dz = \sum_{k=0}^\infty \frac{(-1)^k \Gamma^{-1}(H + (k + 1)I) \left( \frac{b}{2} \right)^{H+2kI}}{(pk)!} \int_0^\infty e^{-a^2z^2} z^{H+M+(2k-1)I}dz
\]

and using the Gamma matrix function

\[
\int_0^\infty e^{-a^2z^2} z^{H+M+(2k-1)I}dz = \frac{\Gamma\left(\frac{1}{2}(H + M) + kI\right)}{2a^{H+M+2kI}}.
\]

Thus, the result is established.

**Corollary 3.18.** If \( H \) is a Jordan block as in (1.7) satisfying the condition \( \tilde{\mu}(H) > -1 \) and \( a \) is an arbitrary number with \( \Re(a^2) > 0 \), then we obtain

\[
\int_0^\infty e^{-a^2z^2} z^{H+I} pJ_H(bz)dz = \frac{b^H}{(2a^2)^{HI+I}} \left( \frac{b}{2a} \right)^{-} - \frac{b^2}{4a^2}
\]

where

\[
\left( \frac{b}{2a} \right)^{-} - \frac{b^2}{4a^2} = \sum_{k=0}^\infty \frac{(-1)^k (\frac{1}{2}(H + M))_k (H + I)_k^{-1}}{(pk)!} \left( \frac{b}{2a} \right)^{2k}.
\]
Proof. From (3.1), we have
\[
\int_0^\infty e^{-a^2z^2} z^{H+I} p_J(bz) dz = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^{-1}(H + k + 1)}{(pk)!} \left( \frac{b}{2} \right)^{H+2kI} \int_0^\infty e^{-a^2z^2} z^{2H+(2k+1)I} dz.
\]
Since the integral on the right hand side equals to
\[
\int_0^\infty e^{-a^2z^2} z^{2H+(2k+1)I} dz = \frac{\Gamma(H + (k + 1)I)}{2a^{2H+(2k+2)I}}.
\]
Interchanged the order of the integral and summation, we get
\[
\int_0^\infty e^{-a^2z^2} z^{H+I} p_J(bz) dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{2(pk)!a^{2H+(2k+2)I}} \left( \frac{b}{2} \right)^{H+2kI}.
\]
Thus, we have the result.

Generalization for the above properties are given in the following theorem and corollary.

**Theorem 3.19.** For \(A, B\) and \(A + B\) are matrices in \(\mathbb{C}^{N \times N}\) satisfying the condition \(\tilde{n}(A) > -1, \tilde{r}(A + B) > 0\) for all \(r = n + m\) and \(a\) is an arbitrary number with \(\Re(a^2) > 0\), then the \(p\)-Bessel matrix function satisfies the following representation
\[
\int_0^\infty e^{-a^2z^2} z^B p_J(bz) dz = \frac{a^{-A} \Gamma(\frac{1}{2} (A + B)) \Gamma^{-1}(A + I)}{2a^2} \left( \frac{b}{2a} \right)^{A} \Phi(1, A + B; A + I; -\frac{b^2}{4a^2}).
\]

**Corollary 3.20.** If \(A\) is a matrix in \(\mathbb{C}^{N \times N}\) satisfying the condition \(\tilde{n}(A) > -1\) and \(a\) is an arbitrary number with \(\Re(a^2) > 0\), then we obtain
\[
\int_0^\infty e^{-a^2z^2} z^A p_J(bz) dz = \frac{b^A}{(2a^2)^A} \Phi(0, -; -; -\frac{b^2}{4a^2}).
\]

4. On \(p\)-modified Bessel Matrix Function

The purpose of this section is to introduce and study a new class of matrix function, namely the \(p\)-modified Bessel matrix function and derive some of their
If $H$ is a Jordan block as in (1.7) satisfying the conditions $\tilde{\mu}(H) > -\frac{1}{2}$ and $|\arg(z)| < \pi$. Then the $p$-modified Bessel matrix function $pI_H(z)$ can be defined as in the following:

$$pI_H(z) = i^{-H} pJ_H(iz) = (\frac{z}{2})^H \frac{\Gamma^{-1}(H + \frac{1}{2}I)}{\sqrt{\pi}} \int_{-\infty}^{\infty} (1 - t^2)^{H - \frac{1}{2}I} _0F_1 \left( -; \frac{1}{2}; \frac{z^2 t^2}{4} \right) dt$$

and

$$pI_H(z) = \left( \frac{z}{2} \right)^H \frac{\Gamma^{-1}(H + I)}{\sqrt{\pi}} \int_{-\infty}^{\infty} (1 - t^2)^{H - \frac{1}{2}I} _0F_1 \left( -; H + I; \frac{z^2}{4} \right)$$

As in the same preceding manner we can find the recurrence relations of $pI_H(z)$ as follows:

$$\frac{d}{dz} \left[ z^H pI_H(z) \right] = z^H pI_{H-1}(z),$$

$$(1/z) \frac{d}{dz} \left[ z^H pI_H(z) \right] = z pI_{H-1}(z) - H pI_H(z),$$

and

$$\frac{d}{dz} \left[ \frac{1}{z} \frac{d}{dz} \cdots \frac{d}{dz} \left[ z^H pI_H(z) \right] \right] = z^{H-mI} pI_{H-mI}(z)$$

These relations can be generalized as in the following: Let $H_i$ be defined by (1.7) and $H = \text{diag}(H_1, H_2, \ldots, H_k)$ be a matrix in $\mathbb{C}^{N \times N}$. Here $N_i$ is not negative integer and zero for $1 \leq i \leq k$. For matrix $H = \text{diag}(H_1, H_2, \ldots, H_k)$, the properties (4.1-4.6) can be easily provided.

Also, if $P$ be an invertible matrix in $\mathbb{C}^{N \times N}$ such that $H = \text{diag}(H_1, H_2, \ldots, H_k) = PAP^{-1}$, we can the following theorem for the matrix $A$. 


Theorem 4.1. If $A$ is a matrix in $C^{N\times N}$ such that (1.9), we obtain

$$
\frac{d}{dz} \left[ z^A p I_A(z) \right] = z^A p I_{A-I}(z),
$$

$$
z \frac{d}{dz} p I_A(z) = z p I_{A-I}(z) - A p I_A(z),
$$

$$
\left( \frac{1}{z} \frac{d}{dz} \right)^m \left[ z^A p I_A(z) \right] = z^{A-m} p I_{A-mI}(z)
$$

and

$$
\frac{d}{dz} \left( \frac{d}{dz} - \frac{2}{p} \right) \cdots \left( \frac{d}{dz} - \frac{p-1}{p} \right) \left[ z^{-A} p I_A(z) \right] = -\left( \frac{2}{p} \right)^p z^{-A} p I_{A+I}(z).
$$

Remark 4.2. For $p = 1$, reduces to definition for the modified Bessel matrix function $I_A(z)$, see [21].

5. On $p$-Tricomi Matrix Function

The purpose of this section is to introduce and study a new class of $p$-Tricomi matrix function of complex variable $z$ which can be stated in the following form:

$$
p C_H(z) = \sum_{k \geq 0} \frac{(-1)^k \Gamma^{-1}(H + (k + 1)) z^k}{(pk)!} (5.1)
$$

where $H$ is a matrix in $C^{N\times N}$ such that $z$ is not a negative integer for every $z \in \sigma(H)$, this is known as $p$-Tricomi matrix function and is characterized by the following link with $p$-Bessel matrix function of complex variables:

$$
p C_H(z) = z^{-H} p J_H(2\sqrt{z}). (5.2)
$$

With the same way, we can deduce recurrence relations for $p C_H(z)$ as in the following:

$$
\frac{d}{dz} \left[ z^H p C_H(z) \right] = z^H p C_{H-I}(z), (5.3)
$$

$$
z \frac{d}{dz} p C_H(z) = z p C_{H-I}(z) - H p C_H(z), (5.4)
$$

$$
\left( \frac{1}{z} \frac{d}{dz} \right)^m \left[ z^H p C_H(z) \right] = z^{H-m} p C_{H-mI}(z) (5.5)
$$

and

$$
\frac{d}{dz} \left( \frac{d}{dz} - \frac{1}{p} \right) \cdots \left( \frac{d}{dz} - \frac{p-1}{p} \right) \left[ z^{-H} p C_H(z) \right] = -\left( \frac{1}{p^p} \right)^p z^{-H} p C_{H+I}(z). (5.6)
$$
Theorem 5.1. Let $H$ be a Jordan block as in (1.7), then the $pC_H(z)$ is a solution of the matrix differential equation of order $p + 1$

$$\left[\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p})\ldots(\theta - \frac{p-1}{p})(\theta I + H) + \frac{z}{p^p}\right]pC_H(z) = 0. \quad (5.7)$$

Proof. From (5.1) and the differential operator $\theta$, we get

$$\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p})(\theta - \frac{3}{p})\ldots(\theta - \frac{p-1}{p})(\theta I + H) pC_H(z) = \frac{1}{p^p} \sum_{k=0}^{\infty} \frac{1}{(pk)!} z^{k+1} (H)^{-1} = -\frac{z}{p^p} pC_H(z).$$

Remark 5.2. Putting $p = 1$, we get the Tricomi matrix differential equation of second order

$$z^2 \frac{d^2}{dz^2} C_H(z) + (H + I) \frac{d}{dz} C_H(z) + C_H(z) = 0. \quad (5.8)$$

Theorem 5.3. The $p$-Tricomi matrix function satisfies the matrix recurrence relations

$$\frac{d}{dz} \left[z^A pC_A(z)\right] = z^A pC_{A-I}(z),$$

$$z \frac{d}{dz} pC_A(z) = z pC_{A-I}(z) - A pC_A(z),$$

$$\frac{d}{dz} \left(\frac{d}{dz} - \frac{1}{p}\right)\left(\frac{d}{dz} - \frac{2}{p}\right)\ldots\left(\frac{d}{dz} - \frac{p-1}{p}\right) \left[z^{-A} pC_A(z)\right]$$

$$= -\left(\frac{1}{p^p}\right) z^{-A} pC_{A+I}(z)$$

and

$$\left(1 \frac{d}{dz}\right)^m \left[z^A pC_A(z)\right] = z^{A-mI} pC_{A-mI}(z)$$

where $A$ is a matrix in $\mathbb{C}^{N \times N}$, $P$ be an invertible matrix in $\mathbb{C}^{N \times N}$ such that $H = \text{diag}(H_1, H_2, \ldots, H_k) = PAP^{-1}$ satisfy $z$ is not a negative integer for every $z \in \sigma(H)$.

Theorem 5.4. Let $A$ be a matrix in $\mathbb{C}^{N \times N}$ satisfying the condition (1.9), then the $p$-Tricomi matrix function is a solution of the matrix differential equation

$$\left[\theta(\theta - \frac{1}{p})(\theta - \frac{2}{p})\ldots(\theta - \frac{p-1}{p})(\theta I + A) + \frac{z}{p^p}\right]pC_A(z) = 0. \quad (5.9)$$
Remark 5.5. Putting $p = 1$, we get the Tricomi matrix differential equation of second order
\[ z^2 \frac{d^2}{dz^2} C_A(z) + (A + I) \frac{d}{dz} C_A(z) + C_A(z) = 0. \] (5.10)

In a similar manner as in the proof of Theorem 3.5, we derive in the following theorem.

**Theorem 5.6.** Let $H$ be a Jordan block as in (1.7) such that $\bar{\mu}(H) > -\frac{1}{2}$, the integral representation of $p$-Tricomi matrix function is given as
\[ p C_H(z) = \frac{\Gamma^{-1}(H + \frac{1}{2}I)}{\sqrt{\pi}} \int_{-1}^{1} (1 - t^2)^{H - \frac{1}{2}} e^{0}_{\frac{1}{2}z} F_1 \left( -; \frac{1}{2}; -zt^2 \right) dt. \] (5.11)

The following theorem can be proved using the same technique as in Theorem 3.7.

**Theorem 5.7.** Let $H$ be a Jordan block as in (1.7) satisfy the condition $\mu$ is not a negative integer for every $\mu \in \sigma(H)$. The $p$-Tricomi matrix function holds the following integral representation:
\[ p C_H(z) = \frac{1}{2 \pi i} \int_{c} e^{s} s^{-(H+I)} e^{0}_{\frac{1}{2}z} F_{0} \left( -; -z \frac{s}{2} \right) ds. \] (5.12)

The previous results can be generalized as in the following. Let $H_i$ is defined by (1.7) and $H = \text{diag}(H_1, H_2, \ldots, H_k)$ be matrix in $\mathbb{C}^{N \times N}$. Here $N_i$ is not negative integer for $1 \leq i \leq k$. For matrix $H = \text{diag}(H_1, H_2, \ldots, H_k)$, the properties (5.1-..5.7) can be easily provided.

Also, if $P$ be an invertible matrix in $\mathbb{C}^{N \times N}$ such that $H = P A P^{-1}$, we give the following theorem.

**Theorem 5.8.**

(i) Let $A$ be a matrix in $\mathbb{C}^{N \times N}$ such that $\bar{\mu}(A) > -\frac{1}{2}$, the integral representation of $p$-Tricomi matrix function is given as
\[ p C_A(z) = \frac{\Gamma^{-1}(A + \frac{1}{2}I)}{\sqrt{\pi}} \int_{-1}^{1} (1 - t^2)^{A - \frac{1}{2}} e^{0}_{\frac{1}{2}z} F_{1} \left( -; \frac{1}{2}; -zt^2 \right) dt. \]

(ii) Let $A$ be a matrix in $\mathbb{C}^{N \times N}$ satisfy the condition (1.9). The $p$-Tricomi matrix function holds the following integral representation:
\[ p C_A(z) = \frac{1}{2 \pi i} \int_{c} e^{s} s^{-(A+I)} e^{0}_{\frac{1}{2}z} F_{0} \left( -; -z \frac{s}{2} \right) ds. \]
Acknowledgement. (1) The Author expresses their sincere appreciation to Prof. Dr. Nassar H. Abdel-All (Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt), Dr. M. S. Metwally, (Department of Mathematics, Faculty of Science (Suez), Suez Canal University, Egypt) and Dr. Mahmoud Tawfik Mohamed, (Department of Mathematics and Science, Faculty of Education (New Valley), Assiut University, New Valley, EL-Kharga 72111, Egypt) for their kinds interests, encouragements, helps, suggestions, comments them and the investigations for this series of papers.

(2) The author would like to thank the anonymous referees for their several useful comments and suggestions towards the improvement of this paper.

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