Abstract- Exponential spline polynomials (E-splines) represent the best smooth transition between continuous and discrete domains. As they are constructed from convolution of exponential segments, there are many degrees of freedom to optimally choose the most convenient E-spline; suitable for a specific application. In this paper, the parameters of these E-splines are optimally chosen, to sharpen the performance of an interpolated high resolution systems HR derived from a given low resolution decimated one whether noisy or noiseless. The proposed technique is based on minimizing the aliasing effects due to the high frequency bands of the HR images. Illustrative examples are given to verify image enhancement of the proposed E-spline scheme, when compared with the existing approaches.

Keywords- Image de-noising, interpolators, E-spline functions

1. INTRODUCTION

During the past decade, there have been an increasing number of papers devoted to the use of polynomial splines in different signal processing applications, [1-3]. B-spline polynomials, is a class of these polynomial splines that find extensive applications in many engineering applications. In [4], a complete analysis for a B-spline Perfect Reconstruction (PR) frame work with a derivation for the scaling and wavelet functions was presented. However, as they are constructed using Haar functions, there is no much degree of freedom to use in optimizing the performance of some signal processing applications like the design of digital interpolators. On the other hand, Exponential splines enjoy a unique feature of being able to convert from analog to digital applications. This is crucial in several signal processing applications such as differential operators, fractional delays, interpolators and sampling rate converters, [5-7]. Moreover, E-splines have many degrees of freedom that have to be optimized in a specific application, as they are constructed from the convolution of exponential segments with different rates. In [8], a preliminary application for the usage of E-splines in image zooming and interpolation was presented. At this point, it is worth mentioning that; digital interpolators are of prime importance in super resolution systems, especially those concerned with constructing a high resolution image HR from a single low resolution one, [9-12].

This paper is concerned with the design of sharper digital interpolators. In this respect; it is proposed to use E-splines in enhancing the performance of HR images generated from LR noisy or noiseless images. The E-spline parameters are optimally chosen to reduce the aliasing effects of the high frequency sub-bands of the high resolution image HR. This aliasing effects are more prominent in increasing the energies of wavelet packet detail coefficients of the HR image. In the noisy case, performance enhancing is performed in two steps. In the first step, the LR image is denoised using the total variation technique described in, [16-18]. In the second step, the coefficient of the E-spline interpolator, are optimally chosen to reduce the contaminated noise.

Illustrative examples are given to verify the ability of E-spline polynomials to significantly enhance image quality.

The paper is organized as follows: in sec. 2, a brief description of E-spline polynomials and its properties, are given. In sec. 3, the proposed technique of generating high resolution image HR from a given down sampled LR image whether noisy or noiseless, is described. Simulation examples are given in sec. 4. Finally, sec. 5 concludes the paper.

2. MATHEMATICAL BACKGROUND

The Exponential \( m^{th} \) order spline polynomial \( B^m_\alpha (t) \), is constructed as \( m \) successive convolution of lower ones, i.e:

\[
B^m_\alpha (t) = B^1_\alpha (t) * B^2_\alpha (t) * ... * B^m_\alpha (t)
\]

where \( B^\alpha_\alpha (t) = e^{\alpha t}, 0 \leq t \leq 1 \). The vector \( \alpha \) can be any positive, negative or even complex conjugate variables.

This leads to a considerable flexibility over Cardinal B-spline polynomials that set \( \alpha = 0 \). \( B^m_\alpha (t) \) is of finite support and equals zeros at \( t \leq 0 \) and \( t \geq m \). It is best used to fit or interpolate continuous signals \( s(t) \), due to its continuity and smoothness. In the discrete case, \( s(n) \) can be expressed using the convolution relation.

\[
s(n) = \sum_{k} c(k) B^m_\alpha (n - k)
\]

The \( c_k \) coefficients are obtained using the concept of inverse filtering described in [4]. An alternate approach is also given to determine these coefficients as the solution of the linear system.
3. SHARPENED INTERPOLATORS

3.1: NOISELESS CASE:

In this case, if a high resolution image \( x(m,n) \) is down-sampled by a ratio \( L \), then the Fourier Transform \( FT \) of \( x(m,n) \) is composed of shifted aliased versions of \( x(n_1, n_2) \). This aliasing is most prominent at the high frequency tails of \( x(n_1, n_2) \). So, to reconstruct a super resolution image \( SR \), from a given low resolution down-sampled image \( LR \); the interpolation algorithm should be tailored to minimize these aliasing effects. As images are mainly of baseband nature; it is clear that the aliasing effects are more prominent in its high frequency sub-bands. Thus, if the interpolated image is decomposed by \( n \)-level wavelet family, then the aliasing effects increase the energies of the detail sub-bands \( HH_1, HH_2, ..., HH_n \) of the image's \( n \)-level wavelet decomposition. The following table illustrates this property, for the case of \( 3 \)-level 'bior4.4' wavelet family. \( S_1, Y_p, Y_s \) are the original as well as the ordinary B-spline least squares interpolated images [4], respectively.

<table>
<thead>
<tr>
<th>( L )</th>
<th>( Y_{LS} )</th>
<th>( Y_p )</th>
<th>( Y_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0517, 0.0483, 0.1027</td>
<td>0.0422, 0.1169, 0.1885</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0010, 0.0031, 0.1007</td>
<td>0.0001, 0.0036, 0.1979</td>
<td></td>
</tr>
</tbody>
</table>

This table indicates that, to sharpen the interpolated image, the parameters \( \alpha \) of the E-spline interpolator should be chosen to minimize the high frequency detail energies. The proposed algorithm achieves this minimization and proceeds as follows:

1- Initially, the \( LR \) image, \( (X_0) \), is interpolated to the desired \( HR \) scale using cubic B-spline interpolator, i.e. set \( \alpha = 0 \) in Eqn. (3), (4).

2- Decompose the interpolated image \( Y_p \) using \( n \)-level wavelet decomposition. Let \( V_0 \) be a vector containing the detail energies of \( HH_1, HH_2, ..., HH_n \), respectively.

3- For an arbitrary choice of \( \alpha \) parameters, compute the vector \( V \) containing the sub-band energies of \( HH_1, HH_2, ..., HH_n \) detail sub-bands.

4- Using any standard unconstrained minimization program, determine the optimum \( \alpha \) that minimizes \( V/V_0 \). \( Y_{op} \) denotes the optimized E-spline interpolator while keeping its approximation coefficients \( LL_n \) the same as that of the B-spline case.

5- Having minimized the aliasing effects, attention is now focused on sharpening the approximation coefficient \( LL_n \) of \( Y_{op} \), i.e. \( Y_{op}^{LL_n} \). This is achieved through filtering \( Y_{op}^{LL_n} \) by a laplacian filter \( H \) to construct a sharpened approximation coefficients \( Y_{sh}^{LL_n} \), as

\[
y_{sh}^{LL_n} = Y_{op}^{LL_n} + \lambda \text{conv} \left( Y_{op}^{LL_n}, H \right)
\]

\( \lambda \) is optimally chosen to maximize the ratio of the high frequency sub-band energy of \( Y_s^{LL_n} \) over \( Y_{op}^{LL_n} \). \( Y_s \) is the enhanced image after sharpening \( Y_{op}^{LL_n} \).
Thus, the proposed algorithm has two jobs. That is minimizing the aliasing effects, as well as sharpening the approximation 

1.2: NOISY CASE

In case of noisy LR image, one has to denoise it before applying the interpolation techniques described by Eqn. (3,4). In [15], the classical denoising described, is based on thresholding the details coefficients of the noisy image wavelet packet decomposition. The thresholding level $T$, is estimated in terms of the noise variance of the high frequency sub-bands. In [16], a total variation denoising technique is described. It is based on wavelet packet decomposition of the noisy image by an $n$-level wavelet family. Next, the detail wavelet packet coefficients are optimally thresholded to minimize the total variation of the reconstructed image. Now, to obtain an HR denoised image, the E-spline $L$-interpolator has to be modified to reduce the noise associated with the $c$’s evaluation of Eqn.(3-a). The modifications are summarized as follows:

1. **Denoise the LR image** $X_{dn}$ using the method of [15] or [16].
2. **Obtain the classical B-spline $L$ interpolator**. Denote the interpolated image by $Y_{0,n}$
3. For a specified $\hat{\alpha}, \hat{\lambda}_0$, compute $c$’s of Eqn.(3), as well as the interpolated image $Y_{1,n}$ as
   \[ c = (B + \lambda_0I)^{-1} X_{dn}, \quad Y_{1,n} = B^l c \quad (6) \]
   $I$ is the identity matrix. Obtain the optimum $\hat{\alpha}, \hat{\lambda}_0$ that minimize the total variation of $Y_{1,n}$
4. Further sharpening is possible through processing $Y_{1,n}$ by a Gaussian-Laplacian LoG filter with variance $\sigma^2$, to yield an output $Y_{g,n}$. Let the sharpened output be $Y_{s,n}=Y_{1,n}+\lambda_1 Y_{g,n}$, $\sigma^2$ & $\lambda_1$ are optimally chosen to minimize $E_{s,n}/E_{1,n}$, where $E_{1,n}$ & $E_{s,n}$ denote the detail energies of 1-level wavelet decomposition of $Y_{1,n}$ & $Y_{s,n}$, respectively.

4. Simulation Results

The following tables compare the PSNR improvement achieved by the proposed noiseless enhancing technique, with other techniques. The LR decimated image is initially interpolated using B-spline interpolation. The interpolated image $Y_{p}$, is further decomposed by a 3 levels ‘bior4.4’ wavelet family, to generate the $V_o$. Next, the LR image is interpolated using E-spline interpolator, whose parameters are optimally chosen to minimize aliasing, i.e. the ratio of $V_e/V_o$, $V_e$ is the E-spline detail sub-band energy vector. Finally, $Y_p$, is $Y_{op}$ with sharpened $LL_3$ coefficients: The tables compare the PSNR improvements for some images interpolated from their decimated versions, for different decimation ratios $L$.

**Table 2: PSNR Comparisons, $L=2$**

<table>
<thead>
<tr>
<th></th>
<th>Cameraman 256x256</th>
<th>Lena 256x256</th>
<th>Old Man 1074x810</th>
<th>Girl 274x184</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-spl. $Y_{op}$</td>
<td>24.91</td>
<td>28.91</td>
<td>33.18</td>
<td>33.53</td>
</tr>
<tr>
<td>E-spl. $Y_{op}$</td>
<td>25.22</td>
<td>28.91</td>
<td>33.89</td>
<td>34.18</td>
</tr>
<tr>
<td>Proposed $Y_p$</td>
<td>25.47</td>
<td>29.24</td>
<td>34.26</td>
<td>34.58</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Cameraman 256x256</th>
<th>Lena 256x256</th>
<th>Old Man 1074x810</th>
<th>Girl 274x184</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-spl. $Y_{op}$</td>
<td>20.51</td>
<td>22.95</td>
<td>30.44</td>
<td>26.63</td>
</tr>
<tr>
<td>E-spl. $Y_{op}$</td>
<td>20.55</td>
<td>23.60</td>
<td>30.60</td>
<td>27.02</td>
</tr>
<tr>
<td>Proposed $Y_p$</td>
<td>21.31</td>
<td>23.78</td>
<td>30.72</td>
<td>27.30</td>
</tr>
</tbody>
</table>

Figures (1-3), show the HR images, constructed from the corresponding LR ones, for different $L$.

The proposed technique is also applied to LR images contaminated with Gaussian as well as Salt and Pepper noise with different variances and concentrations. Initially, the LR noisy image is first denoised using the classical technique of [16], where thresholding is applied to 2-level ‘bior4.4’ wavelet family decompositions. The denoised image is then interpolated using classical B-spline interpolation. Comparison is also carried out when E-spline interpolators are used in place of B-spline interpolators. In this case, the $\hat{\alpha}$ are adjusted to minimize the sub-band detail energies of the reconstructed image. Figures (4-6), show the PSNR performance of some of these results, for the case $L=2$. Note the poor performance of the classical denoising technique of [15] for Salt & Pepper case, as it was basically derived for Gaussian-type noises.

5. CONCLUSION

As a result of the extra degrees of freedom possessed by E-splines, they enjoy a higher level of energy concentration than their B-spline counterpart. Hence, when its parameters are correctly chosen, they can provide enhanced performance in different image denoising as well as zooming applications. Simulation results have verified, they improved denoising performance and its ability of constructing interpolators with an even sharper image than those available in the existing methods. This is a crucial problem in SR system design.

6. REFERENCES


Fig. 2: HR of Girl image, PSNR = 34.58 dBs.

Fig. 2: HR of Lena image, PSNR = 23.78 dBs.

Fig. 3: HR of Old man image, PSNR = 27.3 dBs.
Fig. 4: Noise performance of Cameraman image

Fig. 5: Noise performance of Lena image

Fig. 5: Noise performance of Old Man image