E-spline in Image De-noising Applications

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ABSTRACT

B-splines caught interest of many engineering applications due to their merits of being flexible and provide a large degree of differentiability and cost/quality trade off relationship. However they have less impact with continuous time applications as they are constructed from piecewise polynomials. On the other hand, Exponential spline polynomials (E-splines) represent the best smooth transition between continuous and discrete domains as they are made of exponential segments. In this paper we present a technique for utilizing E-splines in image de-noising applications. This technique is based upon sub-band decomposition of the image through an E-spline based perfect reconstruction (PR) system. Different thresholdings are applied on the decomposition layers for de-noising purposes. Due to the selective nature of E-spline based decomposition, the performance of our E-spline based de-noising technique outperforms all other literature techniques.

Key words: Splines, De-noising, Perfect Construction B-spline wavelet family

1. Introduction

During the past decade, there have been an increasing number of papers devoted to the use of polynomial splines in different signal processing applications [1-3]. These polynomial splines have been found to be useful in many image processing applications (zooming, interpolation, and least square fitting) for either uniform or non-uniform grids. Basis of these (piecewise) splines (known as B-splines) represented the best cost-quality trade off among different linear applications. However due to the limited application of B-splines with non-band limited signals, there has been little application with polynomial splines in continuous time applications.

On the other hand, continues-time signal processing splines became very promising when dealing with non-band limited functions. This is due to the uniform grid based analysis that was given in [4-5] for cardinal exponential splines. Exponential splines enjoy a unique feature of being able to convert from analog to digital applications in a mathematically efficient and completely error free process. This is crucial and advantageous in several signal processing applications such as differential operators, fractional delays, interpolators and sampling rate converters. In [6], a complete analysis for a B-spline Perfect Reconstruction PR framework with a derivation for the scaling and wavelet functions was presented. In [7], a preliminary application for the usage of Exponential splines (E-splines) in image zooming and interpolation was presented. In this paper we propose a novel framework for an E-spline based PR wavelet system on a uniform grid. The proposed system, which is more suitable for non-band-limited signal applications and continuous time signal and system theories, is utilized in image de-noising applications. In our proposed technique, we decompose the image through E-spline based wavelet decomposition and threshold some bands based upon a certain measure for de-noising measures.

We applied the proposed decomposition system on different de-noising applications (images) and compared its performance with other literature adopted techniques for image de-noising. Mathematical analysis for E-splines is in section 2, application of the proposed analysis in image de-noising is presented in section 3, different simulation and testing results are in section 4, conclusions are in section 5.
2. Background

For a signal $s(t)$ that is digitally sampled as, $s(t) = \sum_k a_k \delta(t - t_k)$ where $a_k = e^{\alpha t}$ and $a = (a_1, a_2, ..., a_N)$ for the duration of $t$. This means that in between samples, the signal is represented by continuous function, where,

$$s(t) = \sum_{k} a_k \rho_{\bar{\alpha}}(t - t_k)$$

As $\rho_{\bar{\alpha}}(p) = \frac{1}{\Pi_{k=1}^{N} (p - a_k)}$, then $s(p) = \frac{\sum a_k e^{-lp}}{\Pi_{k=1}^{N} (p - a_k)}$.

Hence, $\lim_{p \to \infty} p^n s(p) = \lim_{p \to \infty} \frac{\sum a_k e^{-lp}}{\Pi_{k=1}^{N} (p - a_k)} = \frac{1}{p^n}$

$s(t)$ has continuous derivatives up to the $(n-2)$ derivatives. At this point it is worth mentioning that, the exponential B-spline has a finite duration,, where $\beta_{\alpha}(t) = \rho_{\bar{\alpha}}(t) - e^{\alpha t} \rho_{\bar{\alpha}}(t - 1)$

As $\beta_{\alpha}(t)$ is compactly supported, it is absolutely sum able, irrespective whether $\alpha$ is real, (negative or positive), or complex. The exponent $\alpha$, provides an extra parameter that improve the performance of classical B-spline polynomials and makes it more smooth. Moreover, it can be easily shown that, an $m^{th}$ order Exponential B-spline function $\beta_{\bar{\alpha}}^m(t)$, can be generated from the $m$ times convolution of $1^{st}$ order B-spline function, i.e.

$$\beta_{\bar{\alpha}}^m(t) = \beta_{\alpha_1}(t)^* \beta_{\alpha_2}(t)^* ... \beta_{\alpha_n}(t)$$

Finally, the FT of Exponential B-spline function, can be derived as follows

$$\beta_{\bar{\alpha}}^m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \beta_{\bar{\alpha}}^m(j\omega) e^{j\omega t} d\omega$$

$$r_{\bar{\alpha}}^m(k) = \int \beta_{\bar{\alpha}}^m(t) \beta_{\bar{\alpha}}^m(t - k) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int \beta_{\bar{\alpha}}^m(j\omega) e^{j\omega t} \beta_{\bar{\alpha}}^m(j\omega) e^{j\omega (t - k)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \beta_{\bar{\alpha}}^m(j\omega) e^{j\omega k} d\omega$$

This means that, (using Fourier series expansion)

$$\sum_{n=-\infty}^{\infty} |\beta_{\bar{\alpha}}^m(\omega + 2m\pi)|^2 = \sum_{k=-\infty}^{\infty} r_{\bar{\alpha}}^m(k) e^{j\omega k}$$

3. Perfect Reconstruction of E-spline Wavelet

In order to construct a perfect reconstruction Exponential B-spline (PR) system, it is necessary to construct the Exponential B-spline scaling and wavelet function, together with their corresponding dual scaling and wavelet...
functions. In this respect, the Exponential B-spline scaling function which is taken to be $\beta^m_{\alpha}(t)$ must satisfy the 2-scale recurrence relation,

$$\beta^m_{\alpha}(t) = \sum_{k=-m}^{m} p(k) \beta^m_{\alpha}(2t-k)$$

In the frequency domain, this means that

$$\beta^m_{\alpha}(j\omega) = \frac{1}{2} \sum_{k=-m}^{m} p(k) z^{-k} \beta^m_{\alpha}(j\frac{\omega}{2}) ,$$

i.e. $\beta^m_{\alpha}(j\omega) = P(z) \beta^m_{\alpha}(j\frac{\omega}{2})$.

$$P(z) = \frac{1}{2} \sum_{k=-m}^{m} p(k) z^{-k}$$

Fig. 1, shows how $P(z)$ is computed. Indeed, $P(z)$ can represent an exponential B-spline low pass filter that concentrates samples prior to any up or down sampling.

The wavelet Exponential B-spline function $\psi^m_{\alpha}(t)$, is constructed to satisfy the orthogonality relation:

$$\frac{1}{\omega} \int_{-\infty}^{\infty} \psi^m_{\alpha}(t) \beta^m_{\alpha}(t-1) dt = 0$$

i.e. $\psi^m_{\alpha}(t) = \sum_{k=-m}^{m} q(k) \beta^m_{\alpha}(2t-k)$

In the frequency domain, this means that,

$$\psi^m_{\alpha}(j\omega) = Q(z) \beta^m_{\alpha}(j\frac{\omega}{2}) ,$$

$$Q(z) = \frac{1}{2} \sum_{k=-m}^{m} q(k) z^{-k}$$

$Q(z)$ can also represent the complementary high pass filter to $P(z)$. Fig 2, shows how the wavelet scaling function $\psi^m_{\alpha}(t)$ is constructed. Fig (3-a), shows the performance of $\alpha = [0.1 \ 0.2 \ 0.3 \ 0.4]$ for $\beta^m_{\alpha}(t)$ & $\psi^m_{\alpha}(t)$ and $m=4$

Using Eq. (4), and the recurrence relation between $P(z)$ & $Q(z)$, the following relation is satisfied,

$$P(z)P(z^{-1})R(z) + P(-z)P(-z^{-1})R(-z) = R(z^2)$$

$$Q(z)P(z^{-1})R(z) + Q(-z)P(-z^{-1})R(-z) = 0$$

i.e. $Q(z) = z^{-m+1}P(-z^{-1})R(-z)$

To complete the construction of the Exponential B-spline wavelet family, the Exponential dual scaling and wavelet B-spline functions have to be constructed. The Exponential B-spline dual scaling function $\beta^{\perp m}_{\alpha}(t)$, is constructed to satisfy the orthogonality relation

$$\beta^{\perp m}_{\alpha}(t) = \sum_{k=-m}^{m} w(k) \beta^{m}_{\alpha}(t-k)$$

under the condition

$$\int_{-\infty}^{\infty} \beta^{\perp m}_{\alpha}(t) \left( \beta^{m}_{\alpha}(t-1) \right)^{\ast} dt = \delta(l)$$

i.e. $\beta^{\perp m}_{\alpha}(j\omega) = W(z^2) \beta^{m}_{\alpha}(j\omega)$.

$$W(z^2) = \sum_{k=-m}^{m} w(k) e^{-im}$$

$W(z^2)$ can be shown to satisfy $W(z^2) = \frac{1}{R(z^2)}$
Thus, the \( w(k) \) of Eqn.(8) is obtained as the solution of a set of linear equations describing the orthogonality relation of Eqn. (8), at integer shifts of time. Fig. (3-b), compares the performance of the scaling Exponential B-spline function \( \beta^m_a(t) \), with the dual Exponential B-spline function \( \beta^\perp_a(t) \) for \( \vec{a}=[0.1 \ 0.2 \ 0.3 \ 0.4] \).

One can easily show that, \( \beta^\perp_a(t) \) satisfies the following 2-scale exponential dual recurrence relation

\[
\beta^\perp_a(t) = \sum_{k=-\infty}^{\infty} a(k) \beta^\perp_a(2t-k)
\]

\[
\beta^\perp_a(j\omega) = A(z) \beta^\perp_a(\frac{j\omega}{2}), \quad A(z) = \frac{1}{2} \sum_{k=-\infty}^{\infty} a(k)
\]

Hence, \( i.e. \)

\[
A(z) = \frac{R(z)}{R(z^2)} P(-z) \equiv W(z^2) R(z) P(z)
\]

Finally, the Exponential B-spline dual wavelet function \( \psi^\perp_a(t) \), is constructed to satisfy the orthogonality relation, as well as the 2-scale dual wavelet recurrence relation, \( \text{(which can be proved using Eq.(4-8)} \)

\[
\int_{-\infty}^{\infty} \psi^\perp_a(t) \beta^\perp_a(t-l) \, dt = 0
\]

\[
\psi^\perp_a(t) = \sum_{k=-\infty}^{\infty} b(k) \beta^\perp_a(2t-k)
\]

\( i.e. \)

\[
\psi^\perp_a(j\omega) = B(z) \beta^\perp_a(\frac{j\omega}{2}), \quad B(z) = \frac{1}{2} \sum_{k=-\infty}^{\infty} b(k) z^{-k} = \frac{z^{2m-1}}{R(z^2)} P(-z^{-1})
\]

Using Eq.(7-10), it can be verified that

\[
P(z)P(z^{-1})R(z)+P(-z)P(-z^{-1})R(-z) = R(z^2)
\]

\[
.: \quad P(z)A(z^{-1})+Q(z)B(z^{-1})=1
\]

\[
P(-z)A(z^{-1})+Q(-z)B(z^{-1})=0
\]

Fig(4), shows the complete Exponential B-spline wavelet family.

4. Application in Image De-noising

Wavelet decomposition amounts to representing signals by few nonzero coefficients. In case of a noisy data \( x \), that represents a signal \( f \) corrupted with uncorrelated zero-mean noise \( w \), i.e. \( x=f+w \), these coefficients are given by

\[
\langle x, \psi_{j,m} \rangle = \langle f, \psi_{j,m} \rangle + \langle w, \psi_{j,m} \rangle, \quad \text{where} \quad \langle \cdot, \cdot \rangle \text{ is the inner product and} \quad \psi_{j,m} \text{ represents the jth-scale wavelet basis used.}
\]

As physical signals like speech and images, are nominally treated as baseband signals, the wavelet coefficients at fine scales, are mainly due to noise. In [8] the noise variance is computed from the median \( M \) of the \( \langle x, \psi_{j,m} \rangle \) band at the finest \( j \) scale. It is shown that the variance \( \sigma^2 \) is given by \( \sigma = E(M)^2 / 0.6745 \).

Now, thresholding the wavelet coefficients, amounts to keeping coefficients greater than a specific threshold level \( T \), while setting to zero all coefficients that are less than \( T \). In [8], it has been shown that, if the noise is Gaussian then, \( f \) is estimated from \( x \), with high probability, if the threshold level of the wavelet decomposition, is chosen as

\[
T = \sigma \log_2(\sqrt{2N}), \quad \text{where} \quad N \text{ is the length of the finest wavelet packet} \quad \langle x, \psi_{j,m} \rangle. \quad \text{Alternatively, one can precisely estimate the noise variance, by estimating the probability density distribution of the finest scale wavelet packet, through smoothing its histogram amplitude distribution, using B-spline wavelet [6], which we will refer to it in the figures as PDF based.}
\]

Recently [9-11], efficient de-noising is performed by total variation minimization. The total variation function of a vector \( X \), is

\[
\sum_{n} |X(n+1) - X(n)|
\]
For a matrix $Y$, the total variation amounts to evaluating the derivatives along the horizontal and vertical axis. Denote these derivatives by $Y_x^d(m,n), Y_y^d(m,n)$. The total variation along the $x$ and $y$ axis, is defined as the $L_1$ first norm of $Y_x^d(m,n), Y_y^d(m,n)$, respectively. Denote these norms by $A_x$ & $A_y$. In this paper, we take $A_x^2 + A_y^2$ as a measure of the total variation of the noisy image. In this section, we propose to choose the $\alpha$'s of the E-spline wavelet, as well as the threshold level $T$, as follows:

1. For a prescribed decomposition level and initial choices of $\alpha$'s and $T$, construct E-spline wavelet decomposition of the noisy image. Threshold the finest scale wavelet packet.
2. Reconstruct the image, using the E-spline synthesis bank. Evaluate the total variation function $\zeta$ of the de-noised image.
3. Using any unconstrained optimization algorithm, find $\alpha$'s and $T$, that minimize $\zeta$.

Figures (5, 6), compare the de-noising performance of the Cameraman image, using these 3 techniques, namely the Median-based, PDF-based and the proposed E-spline technique, using different decomposition levels and different images. These results verify the improved performance of the proposed E-spline de-noising.

5. Discussion

We note here that E-splines enjoys a higher level of energy concentration than their B-spline counterpart, hence they provide enhanced performance in different image de-noising applications, as shown in Fig. 5 and 6. Our proposed Exponential spline based Perfect Reconstruction framework can analyze different signals in a more noise free environment, as it consists of exponential segments that are connected together in a smooth manner. Our proposed system can represent the link between discrete and continuous time signal processing, it can also be utilized for efficient multi-resolution signal interpolation and extrapolation. E-splines can also connect impulse responses of analog filters to their digital counterpart through basic analog filter poles and zeros. This would not only allow us to obtain exact discrete implementation of analog filters, but it can also bridge the gap to the world of non band limited signal processing applications. E-splines also inherit more applications in the area of sensors and Finite Rate of Innovation (FRI) sampling kernels [13], as they construct rational kernels with stable rational Fourier Transform. These rational kernels include many of nowadays linear differential acquisition devices.

6. Conclusion

In this paper, a complete analysis was presented for an E-spline based decomposition subband coding system in a PR framework; the proposed system derives the analysis and synthesis Low Pass and High Pass filters, and verifies the PR status by measuring aliasing and impulse response of the overall system. The proposed system has the advantage that it is based on Exponential splines, rather than our earlier work in [6], that was based on B-splines. The flexibility of choosing its parameters means that, E-spline wavelets can possess a high level of energy concentration. This makes our system more suitable for continuous signals and continuous time domain applications, and subsequently can yield better image compression. Future directions of research would include finding more faster and reliable optimization methods for the E-spline parameter selection such as in [12]. This work is funded from the Alexander von Humboldt foundation, Germany.

7. References


Fig:1 E-Spline Low Pass filter
Fig:2 E-Spline High Pass filter

Fig. (3-a)
Fig. (3-b)

Fig. 3 Comparison between E-Spline Scaling (Low Pass) and Wavelet (High Pass) functions

Fig. 4: Exponential B-spline Perfect Construction (PR) wavelet family

2 Decomposition Levels

3 Decomposition Levels

Fig. 5 De-noising performance Cameraman Image. Gaussian Noise Case
Fig. 6 De-noising performance Cameraman Image. Salt and Pepper Noise Case