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Pre-Separation Axioms in Fuzzifying Topology *

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Abstract: In the present paper we introduce and study pre-T_0-, pre-R_0-, pre-R_1-, pre-T_2(pre-Hausdorff)-, pre-T_3(pre-regularity)-, pre-T_4(pre-normality)-, pre-strong T_3- and pre-strong T_4-separation axioms in fuzzifying topology and give some of their characterisations as well as the relations of these axioms and other separation axioms in fuzzifying topology introduced by Shen[7].

Key words: Fuzzy Logic; Fuzzifying Topology; Fuzzifying Pre-Open Sets; Fuzzifying Separation Axioms

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1 Introduction

Ying[5,6] introduced and elementally developed so called fuzzifying topology with the semantic method of continuous valued Logic. Shen[7] introduced and studied T_0-, T_1-, T_2(Hausdorff)-, T_3(regularity)-, T_4(normality)- separation axioms in fuzzifying topology. In [3] the concepts of the family of fuzzifying pre-open sets, fuzzifying pre- neighbourhood structure of a point and fuzzifying pre-closure are introduced and studied. It is worth to mention that pre-separation axioms are introduced and studied in fuzzy topology [1] in [8]. In the present paper we introduce and study pre-T_0-, pre-R_0-, pre-T_1-, pre-R_1-, pre-T_2(pre-Hausdorff)-, pre-T_3(pre-regularity)-, pre-T_4(pre-normality)-, pre-strong T_3- and pre-strong T_4-separation axioms in fuzzifying topology.

2 Preliminaries

We present the fuzzy logical and corresponding set theoretical notations due to Ying[5,6].

For any formulae φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0,1]$. We write $\models\varphi$ if $[\varphi]=1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) $[\alpha]=\alpha(\alpha\in[0,1])$; $[\varphi\wedge\psi]:=\min([\varphi],[\psi])$; $[\varphi\rightarrow\psi]:=\min(1,1-[\varphi]+[\psi])$.

(2) Let $\widetilde{A}\in\mathscr{F}(X)$, then $[x\in\widetilde{A}]:=\widetilde{A}(x)$.

(3) Let X be the universe of discourse, $[\forall\, x\varphi(x)]:=\inf_{x\in X}[\varphi(x)]$.

In addition the following derived formulae are given,

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(1) $[\neg\varphi]:=[\varphi\rightarrow 0]:=1-[\varphi]$.

(2) $[\varphi\vee\psi]:=[\neg(\neg\varphi\wedge\neg\psi)]:=\max([\varphi],[\psi])$.

(3) $[\varphi\leftrightarrow\psi]:=[\varphi\rightarrow\psi]\wedge[\psi\rightarrow\varphi]$.

(4) $[\varphi\dot{\wedge}\psi]:=[\neg(\varphi\rightarrow\neg\psi)]:=\max(0,[\varphi]+[\psi]-1)$.

(5) $[\varphi\dot{\vee}\psi]:=[\neg(\neg\varphi\dot{\wedge}\neg\psi)]:=[\neg\varphi\rightarrow\psi]:=\min(1,[\varphi]+[\psi])$.

(6) $[\exists\, x\varphi(x)]:=[\neg\forall\, x\neg\varphi(x)]:=\sup\limits_{x\in X}[\varphi(x)]$.

(7) Let $\tilde{A},\tilde{B}\in\mathscr{F}(X)$, $[\tilde{A}\subseteq\tilde{B}]:=[\forall\, x(x\in\tilde{A}\rightarrow x\in\tilde{B})]:=\inf\limits_{x\in X}\min(1,1+\tilde{A}(x)+\tilde{B}(x))$, $[\tilde{A}\equiv\tilde{B}]:=[[\tilde{A}\subseteq\tilde{B}]\wedge[\tilde{B}\subseteq\tilde{A}]]$, $[\tilde{A}\dot{\equiv}\tilde{B}]:=[[\tilde{A}\subseteq\tilde{B}]\dot{\wedge}[\tilde{B}\subseteq\tilde{A}]]$, where $\mathscr{F}(X)$ the set of all fuzzy subsets of X.

Often we do not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying did.

We give now the following definitions and results in fuzzifying topology which are used in the sequel.

Definition 2.1[5] Let X be a universe of discourse, and $\tau\in\mathscr{F}(P(X))$ satisfy the following conditions:

(1) $\tau(X)=1$, $\tau(\varnothing)=1$;

(2) $\forall$ A,B, $\tau(A\cap B)\geqslant\tau(A)\wedge\tau(B)$;

(3) $\forall\ \{A_\lambda|\lambda\in\Lambda\}$, $\tau(\bigcup\limits_{\lambda\in\Lambda}A_\lambda\geqslant\bigwedge\limits_{\lambda\in\Lambda}\tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X,τ) is a fuzzifying topological space.

Definition 2.2[5] Let (X,τ) be a fuzzifying topological space. $\forall\ \tilde{A}\in\mathscr{F}(X)$, the closure of $\tilde{A}$ is denoted by $\bar{\tilde{A}}$ and defined as follows: $\bar{\tilde{A}}=\sup\limits_{x\in X}(\tilde{A}(x)\wedge\bar{\tilde{A}}_{\tilde{A}(x)})$, for each $x\in X$, where $\tilde{A}_\alpha=\{x|\tilde{A}(x)\geqslant\alpha\}$.

Definition 2.3[4] Let (X,τ) be a fuzzifying topological space. $\forall\ \tilde{A}\in\mathscr{F}(X)$, the interior of $\tilde{A}$ is denoted by $(\tilde{A})^\circ$ and defined as follows: $(\tilde{A})^\circ(x)=1-\overline{(1-\tilde{A})}(x)$, for each $x\in X$.

Definition 2.4[3] (1) The family of fuzzifying pre-open sets is denoted by $\tau_P\in\mathscr{F}(P(X))$ and defined as $A\in\tau_P:=\forall\ x(x\in A\rightarrow x\in A^{-\circ})$.

(2) The family of fuzzifying pre-closed sets is denoted by $F_P\in\mathscr{F}(P(X))$ and defined as $A\in F_P:=X\sim A\in\tau_P$, where $X\sim A$ is the complement of A.

(3) Let $x\in X$. The fuzzifying pre-neighborhood system of x is denoted by $N_x^P\in\mathscr{F}(P(X))$ and defined as follows: $A\in N_x^P:=\exists\ B(x\in B\subseteq A\rightarrow B\in\tau_P)$.

(4) The fuzzifying pre-closure of A is denoted and defined as follows: $Cl_P(A)(x)=1-N_x^P(X\sim A)$.

Theorem 2.1[3] Let (X,τ) be a fuzzifying topological space. Then,

(1) $\models\tau\subseteq\tau_P$;

(2) $\models F\subseteq F_P$.

Theorem 2.2[3] The mapping $N^P:X\rightarrow\mathscr{F}^N(P(X))$, $x\rightarrow N_x^P$, where $\mathscr{F}^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$ has the following properties:

(1) $\models A\in N_x^P\rightarrow x\in A$;

(2) $\models A\subseteq B\to(A\in N_x^P\to B\in N_x^P)$;

(3) $\models A\in N_x^P\to\exists H(H\in N_x^P\wedge H\subseteq A\wedge\forall y(y\in H\to H\in N_y^P))$.

Theorem 2.3 $\tau_P(A)=\inf_{x\in A}N_x^P(A)$.

Remark 2.1 For simplicity we put the following notations:

$$K(x,y):=\exists A((A\in N_x\wedge y\notin A)\vee(A\in N_y\wedge x\notin A))$$
$$H(x,y):=\exists B\exists C((B\in N_x\wedge y\notin B)\wedge(C\in N_y\wedge x\notin C))$$
$$M(x,y):=\exists B\exists C(B\in N_x\wedge C\in N_y\wedge B\cap C=\varnothing)$$
$$V(x,D):=\exists A\exists B(A\in N_x\wedge B\in\tau\wedge D\subseteq B\wedge A\cap B=\varnothing)$$
$$W(A,B):=\exists G\exists H(G\in\tau\wedge H\in\tau\wedge(A\subseteq G)\wedge(B\subseteq H)\wedge G\cap H=\varnothing)$$

Definition 2.5[7] Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i\in\mathscr{F}(\Omega)$, $i=1,2,3,4$, are defined as follows:

$$(X,\tau)\in T_0:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to K(x,y))$$
$$(X,\tau)\in T_1:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to H(x,y))$$
$$(X,\tau)\in T_2:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to M(x,y))$$
$$(X,\tau)\in T_3:=\forall x\forall D((x\in X\wedge D\in F\wedge x\notin D)\to V(x,D))$$
$$(X,\tau)\in T_4:=\forall A\forall B((A\in F\wedge B\in F\wedge A\cap B\neq\varnothing)\to W(A,B))$$

3 Fuzzifying pre-separation axioms

Remark **3.1** For simplicity we put the following notations:

$$K_P(x,y):=\exists A((A\in N_x^P\wedge y\notin A)\vee(A\in N_y^P\wedge x\notin A))$$
$$H_P(x,y):=\exists B\exists C((B\in N_x^P\wedge y\notin B)\wedge(C\in N_y^P\wedge x\notin C))$$
$$M_P(x,y):=\exists B\exists C(B\in N_x^P\wedge C\in N_y^P\wedge B\cap C=\varnothing)$$
$$V_P(x,D):=\exists A\exists B(A\in N_x^P\wedge B\in\tau_P\wedge D\subseteq B\wedge A\cap B=\varnothing)$$
$$W_P(A,B):=\exists G\exists H(G\in\tau_P\wedge H\in\tau_P\wedge(A\subseteq G)\wedge(B\subseteq H)\wedge G\cap H=\varnothing)$$

Definition 3.1 Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates pre-T_i(T_i^P for short)$\in\mathscr{F}(\Omega)$, $i=0,1,2,3,4$, pre-strong-T_i(T_i^{PS} for short)$\in\mathscr{F}(\Omega)$, $i=3,4$ and pre-R_i(R_i^P for short)$\in\mathscr{F}(\Omega)$, $i=0,1$ are defined as follows:

(1) $(X,\tau)\in T_0^P:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to K_P(x,y))$;

(2) $(X,\tau)\in T_1^P:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to H_P(x,y))$;

(3) $(X,\tau)\in T_2^P:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to M_P(x,y))$;

(4) $(X,\tau)\in T_3^P:=\forall x\forall D((x\in X\wedge D\in F\wedge x\notin D)\to V_P(x,D))$;

(5) $(X,\tau)\in T_4^P:=\forall A\forall B((A\in F\wedge B\in F\wedge A\cap B\neq\varnothing)\to W_P(A,B))$;

(6) $(X,\tau)\in T_3^{PS}:=\forall x\forall D((x\in X\wedge D\in F_P\wedge x\notin D)\to V(x,D))$;

(7) $(X,\tau)\in T_4^{PS}:=\forall A\forall B((A\in F_P\wedge B\in F_P\wedge A\cap B=\varnothing)\to W(A,B))$;

(8) $(X,\tau)\in R_0^P:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to(K_P(x,y)\to H_P(x,y)))$;

(9) $(X,\tau)\in R_1^P:=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\to(K_P(x,y)\to M_P(x,y)))$。

Lemma 3.1 (1) $\models K(x,y)\to K_P(x,y)$;

(2) $\models H(x,y)\to H_P(x,y)$;

(3) $\models M(x,y)\to M_P(x,y)$;

(4) $\models V(x,D) \to V_P(x,D)$;

(5) $\models W(A,B) \to W_P(A,B)$。

Proof From Theorem 2.1(1), $\models \tau \subseteq \tau_P$ and so one can deduce that $N_x(A) \leqslant N_x^P(A)$ for any $A \in P(X)$, the proof is immediate.

Theorem 3.1 (1) $\models (X,\tau) \in T_i \to (X,\tau) \in T_i^P$, where $i=0,1,2,3,4$。

(2) $\models (X,\tau) \in T_i^{PS} \to (X,\tau) \in T_i$, where $i=3,4$。

Proof (1) It is obtained from Lemma 3.1.

(2) It follows from Theorem 2.1(2).

Lemma 3.2 (1) $\models M_P(x,y) \to H_P(x,y)$;

(2) $\models H_P(x,y) \to K_P(x,y)$;

(3) $\models M_P(x,y) \to K_P(x,y)$。

Proof (1) First, if $N_x^P(B)=0$ or $N_y^P(C)=0$, then $[M_P(x,y)] \leqslant [H_P(x,y)]$.

Second, If $N_x^P(B)>0$ and $N_y^P(C)>0$, then

$$[M_P(x,y)] = \sup_{B\cap C=\varnothing} \min(N_x^P(B), N_y^P(C)) \leqslant \sup_{y\notin B, x\notin C} \min(N_x^P(B), N_y^P(C)) = [H_P(x,y)]$$

$$[K_P(x,y)] = \max(\sup_{y\notin C} N_x^P(B), \sup_{x\notin B} N_y^P(C)) \geqslant \min(\sup_{y\notin C} N_x^P(B), \sup_{x\notin B} N_y^P(C)) = [H_P(x,y)]$$

(3) It is obtained from (1) and (2).

Theorem 3.2 (1) $\models (X,\tau) \in T_1^P \to (X,\tau) \in T_0^P$;

(2) $\models (X,\tau) \in T_2^P \to (X,\tau) \in T_1^P$.

Proof The proof of (1) and (2) are obtained from Lemma 3.2(2) and (1) respectively.

Corollary 3.1 $\models (X,\tau) \in T_2^P \to (X,\tau) \in T_0^P$.

Theorem 3.3 $\models (X,\tau) \in T_0^P \leftrightarrow \forall x \forall y((x\in X \wedge y \in X \wedge x \neq y) \to (\neg(x \in Cl_P(\{y\}))) \vee (\neg(y \in Cl_P(\{x\}))))$.

Proof $[(X,\tau) \in T_0^P]$

$$\begin{aligned}
&= \inf_{x\neq y} \max(\sup_{y\notin A} N_x^P(A), \sup_{x\notin A} N_y^P(A)) \\
&= \inf_{x\neq y} \max(N_x^P(X\sim\{y\}), N_y^P(X\sim\{x\})) \\
&= \inf_{x\neq y} \max(1-Cl_P(\{y\})(x), 1-Cl_P(\{x\})(y)) \\
&= \inf_{x\neq y} (\neg(Cl_P(\{y\}))(x) \vee \neg(Cl_P(\{x\}))(y)) \\
&= [\forall x \forall y((x\in X \wedge y\in X \wedge x\neq y) \to (\neg(x\in Cl_P(\{y\}))) \vee (\neg(y \in Cl_P(\{x\}))))]
\end{aligned}$$

Theorem 3.4 Let (X,τ) be a fuzzifying topological space. Then $\models (X,\tau) \in T_1^P \leftrightarrow \forall x(\{x\} \in F_P)$.

Proof For any x_1, x_2, $x_1 \neq x_2$, we have $[\forall x(\{x\} \in F_P)] = \inf_{x\in X} F_P(\{x\}) = \inf_{x\in X} \tau_P(X\sim\{x\}) = \inf_{x\in X} \inf_{y\in X\sim\{x\}} N_y^P(X\sim\{x\}) \leqslant \inf_{y\in X\sim\{x_2\}} N_y^P(X\sim\{x_2\}) \leqslant N_{x_1}^P(X\sim\{x_2\}) = \sup_{x_2\notin A} N_{x_1}^P(A)$. Similarly we have, $[\forall x(\{x\}\in F_P)] \leqslant \sup_{x_1\notin B} N_{x_2}^P(B)$. Then,

$$\begin{aligned}
[\forall x(\{x\} \in F_P)] &\leqslant \inf_{x_1\neq x_2} \min(\sup_{x_2\notin A} N_{x_1}^P(A), \sup_{x_1\notin B} N_{x_2}^P(B)) \\
&= \inf_{x_1\neq x_2} \sup_{x_1\notin B, x_2\notin A} \min(N_{x_1}^P(A), N_{x_2}^P(B)) \\
&= [(X,\tau) \in T_1^P]
\end{aligned}$$

On the other hand,

$$\begin{aligned}[(X,\tau)\in T_1^P] &= \inf_{x_1\neq x_2}\min(\sup_{x_2\notin A}N_{x_1}^P(A),\sup_{x_1\notin B}N_{x_2}^P(B))\\ &= \inf_{x_1\neq x_2}\min(N_{x_1}^P(X\sim\{x_2\}),N_{x_2}^P(X\sim\{x_1\}))\\ &\leqslant \inf_{x_1\neq x_2}N_{x_1}^P(X\sim\{x_2\})\\ &= \inf_{x_2\in X}\inf_{x_1\in X-\{x_2\}}N_{x_1}^P(X\sim\{x_2\})\\ &= \inf_{x_2\in X}\tau_P(X\sim\{x_2\})\\ &= \inf_{x\in X}\tau_P(X\sim\{x\})\\ &= [\forall x(\{x\}\in F_P)]\end{aligned}$$

Thus, $[(X,\tau)\in T_1^P]=[\forall x(\{x\}\in F_P)]$.

Definition 3.2 The pre-local base $P\beta_x$ of x is a function from $P(X)$ into I such that the following conditions are satisfied:

(1) $\models P\beta_x\subseteq N_x^P$;

(2) $\models A\in N_x^P\rightarrow\exists B(B\in P\beta_x\wedge x\in B\subseteq A)$.

Lemma 3.3 $\models A\in N_x^P\leftrightarrow\exists B(B\in P\beta_x\wedge x\in B\subseteq A)$.

Proof From the condition (1) in Definition 3.2 and Theorem 2.2(2) then $N_x^P(A)\geqslant N_x^P(B)\geqslant P\beta_x(B)$ for each $B\subseteq X$ such that $x\in B\subseteq A$. So, $N_x^P(A)\geqslant\sup_{x\in B\subseteq A}P\beta_x(B)$. From condition (2) in Definition 2.3, $N_x^P(A)\leqslant\sup_{x\in B\subseteq A}P\beta_x(B)$. Hence, $N_x^P(A)=\sup_{x\in B\subseteq A}P\beta_x(B)$.

Theorem 3.5 If $P\beta_x$ is a pre-local basis of x, then

$$\models(X,\tau)\in T_2^P\leftrightarrow\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\rightarrow(\exists B(B\in P\beta_x\wedge y\notin Cl_P(\{x\}))))$$

Proof $[\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\rightarrow(\exists B(B\in P\beta_x\wedge y\notin Cl_P(\{B\}))))]$

$$\begin{aligned}&=\inf_{x\neq y}\sup_{B\in P(X)}\min(P\beta_x(B),N_y^P(X\sim B))\\ &=\inf_{x\neq y}\sup_{B\cap C=\varnothing}\sup_{y\in C\subseteq X\sim B}\min(P\beta_x(B),P\beta_y(C))\\ &=\inf_{x\neq y}\sup_{B\cap C=\varnothing}\sup_{x\in D\subseteq B,y\in E\subseteq C}\min(P\beta_x(D),P\beta_y(E))\\ &=\inf_{x\neq y}\sup_{B\cap C=\varnothing}\min(\sup_{y\in D\subseteq B}(P\beta_x(D),\sup_{y\in E\subseteq C}P\beta_y(E))\\ &=\inf_{x\neq y}\sup_{B\cap C=\varnothing}\min(N_x^P(B),N_y^P(C))\\ &=[(X,\tau)\in R_2^P]\end{aligned}$$

Theorem 3.6 $\models(X,\tau)\in R_1^P\rightarrow(X,\tau)\in T_0^P$.

Proof From Lemma 3.2(1) the proof is immediate.

Theorem 3.7 (1) $\models(X,\tau)\in T_1^P\rightarrow(X,\tau)\in R_0^P$;

(2) $\models(X,\tau)\in T_1^P\rightarrow((X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P$;

(3) If $T_0^P(X,\tau)=1$, then $\models(X,\tau)\in T_1^P\leftrightarrow((X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P)$.

Proof (1) $T_1^P(X,\tau)$

$$\begin{aligned}&=\inf_{x\neq y}[H_P(x,y)]\\ &\leqslant\inf_{x\neq y}\min(1,1-[K_P(x,y)]+[H_P(x,y)])\\ &=\inf_{x\neq y}[K_P(x,y)\rightarrow H_P(x,y)]\\ &=R_0^P(X,\tau).\end{aligned}$$

(2) It is obtained from (1) and from Theorem 3.2(1).

(3) Since $T_0^P(X,\tau)=1$, then for every $x,y\in X$ such that $x\neq y$ we have $K_P(x,y)=1$. Now,

$$\begin{aligned}&[(X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P]\\ &=[(X,\tau)\in R_0^P]\\ &=\inf_{x\neq y}\min(1,1-[K_P(x,y)]+[H_P(x,y)])\\ &=\inf_{x\neq y}[H_P(x,y)]\\ &=T_1^P(X,\tau)\end{aligned}$$

Theorem 3.8 (1) $\models((X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P)\to(X,\tau)\in T_1^P$;

(2) If $T_0^P(X,\tau)=1$, then $\models((X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P)\leftrightarrow(X,\tau)\in T_1^P$.

Proof (1) $[(X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P]$

$$\begin{aligned}&=\max(0,R_0^P(X,\tau)+T_0^P(X,\tau)-1)\\ &=\max(0,\inf_{x\neq y}\min(1,1-[K_P(x,y)]+[H_P(x,y)])+\inf_{x\neq y}[K_P(x,y)]-1)\\ &\leqslant\max(0,\inf_{x\neq y}(\min(1,1-[K_P(x,y)]+[H_P(x,y)])+[K_P(x,y)]-1))\\ &=\inf_{x\neq y}[H_P(x,y)]\\ &=T_1^P(X,\tau).\end{aligned}$$

(2) $[(X,\tau)\in R_0^P\wedge(X,\tau)\in T_0^P]$

$$\begin{aligned}&=[(X,\tau)\in R_0^P]\\ &=\inf_{x\neq y}\min(1,1-[K_P(x,y)]+[H_P(x,y)])\\ &=\inf_{x\neq y}[H_P(x,y)]\\ &=T_1^P(X,\tau),\end{aligned}$$

because $T_0^P(X,\tau)=1$, we have for each $x,y\in X$ such that $x\neq y$, $[K_P(x,y)]=1$.

Theorem 3.9 (1) $\models(X,\tau)\in T_0^P\to((X,\tau)\in R_0^P\to(X,\tau)\in T_1^P)$;

(2) $\models(X,\tau)\in R_0^P\to((X,\tau)\in T_0^P\to(X,\tau)\in T_1^P)$.

Proof From Theorems 3.7(1) and Theorems 3.8(1), we have

$$\begin{aligned}&[(X,\tau)\in T_0^P\to((X,\tau)\in R_0^P\to(X,\tau)\in T_1^P)]\\ &=\min(1,1-[(X,\tau)\in T_0^P]+\min(1,1-[(X,\tau)\in R_0^P]+[(X,\tau)\in T_1^P]))\\ &=\min(1,1-[(X,\tau)\in T_0^P]+1-[(X,\tau)\in R_0^P]+[(X,\tau)\in T_1^P])\\ &=\min(1,1-([(X,\tau)\in T_0^P]+[(X,\tau)\in R_0^P]-1)+[(X,\tau)\in T_1^P])\\ &=1\end{aligned}$$

(2) The proof is similar to (1).

By a similar procedure to Theorems 3.7, Theorems 3.8, and Theorems 3.9 we have the following theorems respectively.

Theorem 3.10 (1) $\models(X,\tau)\in T_2^P\to(X,\tau)\in R_1^P$;

(2) $\models(X,\tau)\in T_2^P\to((X,\tau)\in R_1^P\wedge(X,\tau)\in T_0^P)$;

(3) If $T_0^P(X,\tau)=1$, then $\models(X,\tau)\in T_2^P\leftrightarrow((X,\tau)\in R_1^P\wedge(X,\tau)\in T_0^P)$.

Theorem 3.11 (1) $\models((X,\tau)\in R_1^P\wedge(X,\tau)\in T_0^P)\to(X,\tau)\in T_2^P$;

(2) If $PT_0(X,\tau)=1$, then $\models((X,\tau)\in R_1^P\wedge(X,\tau)\in T_0^P)\leftrightarrow(X,\tau)\in T_2^P$.

Theorem 3.12 (1) $\models(X,\tau)\in T_0^P\to((X,\tau)\in R_1^P\to(X,\tau)\in T_2^P)$;

(2) $\models(X,\tau)\in R_1^P\to((X,\tau)\in T_0^P\to(X,\tau)\in T_2^P)$.

Lemma 3.4 (1) If $[D\subseteq B]=1$, then $\sup_{A\cap B=\varnothing} N_x^P(A)=\sup_{A\cap B=\varnothing, D\subseteq B} N_x^P(A)$;

(2) $\sup_{A\cap B=\varnothing}\inf_{y\in D} N_y^P(X\sim A)=\sup_{A\cap B=\varnothing, D\subseteq B}\tau_P(B)$.

Proof (1) Since $[D\subseteq B]=1$, then

$$\sup_{A\cap B=\varnothing} N_x^P(A) = \sup_{A\cap B=\varnothing} N_x^P(A) \wedge [D\subseteq B] = \sup_{A\cap B=\varnothing, D\subseteq B} N_x^P(A)$$

(2) Let $[y\in D]=1$. Then,

$$\begin{aligned}
&\sup_{A\cap B=\varnothing, D\subseteq B}\tau_P(B)\\
&= \sup_{A\cap B=\varnothing, D\subseteq B}\tau_P(B) \wedge [y\in D]\\
&= \sup_{y\in D\subseteq B\subseteq X\sim A}\tau_P(B)\\
&= \sup_{y\in B\subseteq X\sim A}\tau_P(B)\\
&= N_y^P(X\sim A)\\
&= \inf_{y\in D} N_y^P(X\sim A)\\
&= \sup_{A\cap B=\varnothing}\inf_{y\in D} N_y^P(X\sim A)
\end{aligned}$$

Definition 3.4 $PT_3^{(1)}(X,\tau) := \forall x \forall y((x\in X\wedge D\in F\wedge x\notin D)\to(\exists A(A\in N_x^P\wedge(Cl_P(A)\cap D\equiv\varnothing))))$.

Theorem 3.13 $\models (X,\tau)\in T_3^P \leftrightarrow (X,\tau)\in PT_3^{(1)}$.

Proof Now,

$$\begin{aligned}
PT_3^{(1)}(X,\tau) &= \inf_{x\notin D}\min(1, 1-\tau(X\sim D)+\sup_{A\in P(X)}\min(N_x^P(A), \inf_{y\in D}(1-Cl_P(A)(y))))\\
&= \inf_{x\notin D}\min(1, 1-\tau(X\sim D)+\sup_{A\in P(X)}\min(N_x^P(A), \inf_{y\in D} N_y^P(X\sim A)))
\end{aligned}$$

and

$$T_3^P(X,\tau) = \inf_{x\notin D}\min(1, 1-\tau(X\sim D)+\sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B)))$$

So, the result holds if we prove that

$$\sup_{A\in P(X)}\min(N_x^P(A), \inf_{y\in D} N_y^P(X\sim A)) = \sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B))) \qquad (*)$$

It is clear that, in the left site of (*) when $A\cap D\neq\varnothing$ and then there exists $y\in X$ such that $y\in D$ and $y\notin X\sim A$. So, $\inf_{y\in D} N_y^P(X\sim A)=0$ and thus (*) become $\sup_{A\in P(X), A\cap B=\varnothing}\min(N_x^P(A), \inf_{y\in D} N_y^P(X\sim A))=\sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B)))$, which is obtained from Lemma 3.4.

Definition 3.5 $PT_3^{(2)}(X,\tau) := \forall x \forall B((x\in B\wedge B\in\tau)\to(\exists A(A\in N_x^P\wedge Cl_P(A)\subseteq B)))$.

Theorem 3.14 $\models (X,\tau)\in T_3^P \leftrightarrow (X,\tau)\in PT_3^{(2)}$.

Proof From Theorem 3.13, we have

$$T_3^P(X,\tau) = \inf_{x\notin D}\min(1, 1-\tau(X\sim D)+\sup_{A\in P(X)}\min(N_x^P(A), \inf_{y\in D} N_y^P(X\sim A)))$$

Now, if we put $B=X\sim D$, then

$$\begin{aligned}
PT_3^{(2)}(X,\tau) &= \inf_{x\in B}\min(1, 1-\tau(B)+\sup_{A\in P(X)}\min(N_x^P(A), \inf_{y\in X\sim B} N_y^P(X\sim A)))\\
&= \inf_{x\notin D}\min(1, 1-\tau(X\sim D)+\sup_{A\in P(X)}\min(N_x^P(A), \inf_{y\in D} N_y^P(X\sim A)))\\
&= T_3^P(X,\tau)
\end{aligned}$$

Definition 3.6 Let (X,τ) be any fuzzifying topological space and let

(1) $PST_3^{(1)}(X,\tau) := \forall x \forall D((x\in X\wedge D\in F_P\wedge x\notin D)\to(\exists A(A\in N_x\wedge Cl(A)\cap D\equiv\varnothing)))$;

(2) $PST_3^{(2)}(X,\tau) := \forall x \forall B((x\in B\wedge B\in\tau)\to(\exists A(A\in N_x^P\wedge Cl_P(A)\subseteq B)))$;

(3) $PT_4^{(1)}(X,\tau) := \forall A \forall B((A\in\tau \wedge B\in F \wedge A\cap B\equiv\varnothing)\to(\exists G(G\in\tau\wedge A\subseteq G\wedge Cl_P(G)\cap B\equiv\varnothing)))$;

(4) $PT_4^{(2)}(X,\tau) := \forall A \forall B((A\in F \wedge B\in\tau \wedge A\subseteq B)\to(\exists G(G\in\tau\wedge A\subseteq G\wedge Cl_P(G)\subseteq B)))$;

(5) $PST_4^{(1)}(X,\tau) := \forall A \forall B((A\in\tau \wedge B\in F_P \wedge A\cap B\equiv\varnothing)\to(\exists G(G\in\tau\wedge A\subseteq G\wedge Cl(G)\cap B\equiv\varnothing)))$;

(6) $PST_4^{(2)}(X,\tau) := \forall A \forall B((A\in F \wedge B\in\tau_P \wedge A\subseteq B)\to(\exists G(G\in\tau\wedge A\subseteq G\wedge Cl(G)\subseteq B)))$.

By a similar proof of Theorems 3.13 and Theorems 3.14 we have the following theorem.

Theorem 3.15

(1) $\models (X,\tau)\in T_3^{PS}\leftrightarrow (X,\tau)\in PST_3^{(1)}$;

(2) $\models (X,\tau)\in T_3^{PS}\leftrightarrow (X,\tau)\in PST_3^{(2)}$;

(3) $\models (X,\tau)\in T_4^{P}\leftrightarrow (X,\tau)\in PT_4^{(1)}$;

(4) $\models (X,\tau)\in T_4^{P}\leftrightarrow (X,\tau)\in PT_4^{(2)}$;

(5) $\models (X,\tau)\in T_4^{PS}\leftrightarrow (X,\tau)\in PST_4^{(1)}$;

(6) $\models (X,\tau)\in T_4^{PS}\leftrightarrow (X,\tau)\in PST_4^{(2)}$.

4 Relation among separation axioms

Theorem 4.1 $\models ((X,\tau)\in T_3^P \wedge (X,\tau)\in T_1) \to (X,\tau)\in T_2^P$.

Proof From Theorem 2.2[6] we have, $T_1(X,\tau)=\inf_{z\in X}\tau(X\sim\{z\})$. So,

$$\begin{aligned}
&T_3^P(X,\tau)+T_1(X,\tau)\\
&=\inf_{x\notin D}\min(1,1-\tau(X\sim D)+\sup_{A\cap B=\varnothing,D\subseteq B}\min(N_x^P(A),\tau_P(B)))+\inf_{z\in X}\tau(X\sim\{z\})\\
&\leqslant \inf_{x\in X,x\neq y}\inf_{y\in X}\min(1,1-\tau(X\sim\{y\})+\sup_{A\cap B=\varnothing}\min(N_x^P(A),N_y^P(B)))+\inf_{z\in X}\tau(X\sim\{z\})\\
&=\inf_{x\in X,x\neq y}(\inf_{y\in X}\min(1,1-\tau(X\sim\{y\})+\sup_{A\cap B=\varnothing}\min(N_x^P(A),N_y^P(B)))+\inf_{z\in X}\tau(X\sim\{z\}))\\
&\leqslant \inf_{x\in X,x\neq y}\inf_{y\in X}(\min(1,1-\tau(X\sim\{y\})+\sup_{A\cap B=\varnothing}\min(N_x^P(A),N_y^P(B)))+\tau(X\sim\{z\}))\\
&\leqslant \inf_{x\neq y}(1+\sup_{A\cap B=\varnothing}\min(N_x^P(A),N_y^P(B)))\\
&\leqslant 1+\inf_{x\neq y}\sup_{A\cap B=\varnothing}\min(N_x^P(A),N_y^P(B))\\
&=1+T_2^P(X,\tau)
\end{aligned}$$

namely, $T_2^P(X,\tau)\geqslant T_3^P(X,\tau)+T_1(X,\tau)-1$. Thus, $T_2^P(X,\tau)\geqslant \max(0,T_3^P(X,\tau)+T_1(X,\tau)-1)$.

Theorem 4.2 $\models (X,\tau)\in T_4^P \wedge (X,\tau)\in T_1\to (X,\tau)\in T_3^P$.

Proof It is equivalent to prove that $T_3^P(X,\tau)\geqslant T_4^P(X,\tau)+T_1(X,\tau)-1$. In fact,

$$\begin{aligned}
&T_4^P(X,\tau)+T_1(X,\tau)\\
&=\inf_{E\cap D=\varnothing}\min(1,1-\min(\tau(X\sim E),\tau(X\sim D))\\
&\quad+\sup_{A\cap B=\varnothing,E\subseteq A,D\subseteq B}\min(\tau_P(A),\tau_P(B)))+\inf_{z\in X}\tau(X\sim\{z\})\\
&\leqslant \inf_{x\notin D}\min(1,1-\min(\tau(X\sim\{x\}),\tau(X\sim D))\\
&\quad+\sup_{A\cap B=\varnothing,D\subseteq B}\min(N_x^P(A),\tau_P(B)))+\inf_{z\in X}\tau(X\sim\{z\})\\
&\leqslant \inf_{x\notin D}\min(1,\max(1-\tau(X\sim D)+\sup_{A\cap B=\varnothing,D\subseteq B}\min(N_x^P(A),\tau_P(B)),\\
&\quad 1-\tau(X\sim\{x\})+\sup_{A\cap B=\varnothing,D\subseteq B}(N_x^P(A),\tau_P(B)))+\inf_{z\in X}\tau(X\sim\{z\})\\
&=\inf_{x\notin D}\max(\min(1,1-\tau(X\sim D))+\sup_{A\cap B=\varnothing,D\subseteq B}\min(N_x^P(A),\tau_P(B)),
\end{aligned}$$

$$
\begin{aligned}
&\quad \min(1, 1-\tau(X\sim\{x\}) + \sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B)))) + \inf_{z\in X}\tau(X\sim\{z\}) \\
&\leqslant \inf_{x\notin D}\max(\min(1, 1-\tau(X\sim D) + \sup_{A\cap B=\varnothing, D\subseteq B}(N_x^P(A), \tau_P(B)) + \tau(X\sim\{x\}), \\
&\quad \min(1, 1-\tau(X\sim\{x\}) + \sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B)))) + \tau(X\sim\{x\}) \\
&\leqslant \inf_{x\notin D}(\min(1, 1-\tau(X\sim D) + \sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B))) + 1) \\
&\leqslant \inf_{x\notin D}(1-\tau(X\sim D) + \sup_{A\cap B=\varnothing, D\subseteq B}\min(N_x^P(A), \tau_P(B))) + 1 \\
&= T_3^P(X, \tau) + 1
\end{aligned}
$$

By a similar procedures of Theorems 4.1 and Theorems 4.2 we have the following theorems respectively.

Theorem 4.3 $\models((X,\tau)\in T_3^{PS}\wedge(X,\tau)\in T_1^P)\rightarrow(X,\tau)\in T_2$.

Theorem 4.4 $\models((X,\tau)\in T_4^{PS}\wedge(X,\tau)\in T_1^P)\rightarrow(X,\tau)\in T_3^{PS}$.

From the above discussion one can have the following diagram

$$
\begin{array}{ccccccccc}
 & & & & T_3^{PS}\wedge T_1^P & \leftarrow & T_4^{PS}\wedge T_1^P & & \\
 & & & & & & \downarrow & & \\
 & & & & \downarrow & & T_3^{PS} & \leftarrow & T_4^{PS} \\
 & & & & & & \downarrow & & \downarrow \\
T_0 & \leftarrow & T_1 & \leftarrow & T_2 & \leftarrow & T_3 & \leftarrow & T_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
PT_0 & \leftarrow & PT_1 & \leftarrow & PT_2 & \leftarrow & PT_3 & \leftarrow & PT_4
\end{array}
$$

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