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FUZZY γ -CONVERGENCE AND FUZZY $c\gamma$-CONVERGENCE OF NETS AND FILTERS IN FUZZIFYING TOPOLOGY

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Abstract: In this paper, the theory of γ-convergence and $c\gamma$-convergence on nets and filters is established in fuzzifying topology. Some important and interesting results in fuzzifying topology are obtained by means of the theory.

Keywords: Fuzzy logic; fuzzifying topology; convergence, fuzzifying γ-open sets and fuzzifying $c\gamma$-open sets. 2000 Mathematics Subject Classification: 54A40

Introduction

In [4-6], Ying elementally established the notion of fuzzifying topology with the semantic method of continuous valued logic. He discussed the neighborrhood structure of a point and the convergence of nets and filters in this new framework. Also, he presented the concepts of interior, closure and continuity and their fundamental properties in fuzzifying topology. In [2] the concept of fuzzy γ-open sets and fuzzy γ-continuity were introduced and studied in fuzzifying topology. In [3], the concepts of fuzzy $c\gamma$-open sets and fuzzy $c\gamma$-continuity in fuzzifying topology were presented and by making use of these concepts, some decomposition of fuzzifying continuity were introduced.

The main purpose in the present paper is to introduce the theory of γ-convergence and $c\gamma$-convergence on nets and filters in fuzzifying topology. Furthermore, we provide some interesting characterizations concerning γ-continuity, $c\gamma$-continuity and γ-Hausdorff spaces by making use of the γ-convergence and $c\gamma$-convergence theory of nets in fuzzifying topology.

Preliminaries

We present the fuzzy logical and corresponding set theoretical notations [4,5] since we need them in this paper.

For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0,1]. We write $\models \varphi$ if $[\varphi]=1$ for any interpretation. Also, $\Im(X)$ is the family of all fuzzy sets in X. The truth valuation rules for primary fuzzy logical formulae and corresponding set theoretical notations are:

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(a) (i) $[\alpha] = \alpha(\alpha \in [0,1])$;
(ii) $[\varphi \wedge \psi] = \min([\varphi],[\psi])$;
(iii) $[\varphi \to \psi] = \min(1, 1-[\varphi]+[\psi])$.
(b) If $\widetilde{A} \in \Im(X), [x \in \widetilde{A}] := \widetilde{A}(x)$.
(c) If X is the universe of discourse, then. $[\forall x\varphi(x)] := \inf_{x\in X} [\varphi(x)]$.

In addition the truth valuation rules for derived formulae are:
(a) $[\neg\varphi] := [\varphi \to 0] = 1-[\varphi]$;
(b) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi],[\psi])$;
(c) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;
(d) $[\exists x\varphi(x)] := [\neg\forall x\neg\varphi(x)] := \sup_{x\in X} [\varphi(x)]$;
(e) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then
(i) $[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] = \inf_{x\in X} \min(1, 1-\widetilde{A}(x)+\widetilde{B}(x))$,
(ii) $[\widetilde{A} \equiv \widetilde{B}] := [\widetilde{A} \subseteq \widetilde{B}] \wedge [\widetilde{B} \subseteq \widetilde{A}]$.

We give now the following definitions and results in fuzzifying topology which are used in the sequel.

Definition 2.1 [4]. Let X be a universe of discourse, $\tau \in \Im(P(X))$ satisfies the following conditions:
(a) $\tau(X)=1, \tau(\phi)=1$;
(b) for any $A, B, \tau(A\cap B) \ge \tau(A)\wedge\tau(B)$;
(c) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \ge \bigwedge_{\lambda\in\Lambda} \tau(A_\lambda)$.
Then τ is called a fuzzifying topology and (X,τ) is a fuzzifying topological space.

Definition 2.2 [4]. The family of all fuzzifying closed sets, denoted by $F \in \Im(P(X))$, is defined as follows: $A \in F := X - A \in \tau$, where $X-A$ is the complement of A.

Definition 2.3 [4]. The fuzzifying Neighbourhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as $N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$.

Definition 2.4 [4, Lemma 5.2]. The closure $\overline{A}$ of A is defined as $\overline{A}(x) = 1 - N_x(X-A)$. In Theorem 5.3, Ying proved that the closure $^{-}: P(X) \to \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [6]) because its extension
$^{-}: \Im(X) \to \Im(X), \overline{\widetilde{A}} = \bigcup_{\alpha\in[0,1]} \alpha\overline{\widetilde{A}_\alpha}, \widetilde{A} \in \Im(X)$,
where $\widetilde{A}_\alpha = \{x : \widetilde{A}(x) \ge \alpha\}$ is the α -cut of A and $\alpha\widetilde{A}(x) = \alpha \wedge \widetilde{A}(x)$ satisfies the following Kuratowski closure axioms:
(a) $\models \overline{\phi} = \phi$;
(b) for any $\widetilde{A} \in \Im(X), \models \widetilde{A} \subseteq \overline{\widetilde{A}}$;
(c) for any $\widetilde{A}, \widetilde{B} \in \Im(X)$, $\models \overline{\widetilde{A} \cup \widetilde{B}} \equiv \overline{\widetilde{A}} \cup \overline{\widetilde{B}}$;
(d) for any $\widetilde{A}, \widetilde{B} \in \Im(X)$, $\models \overline{\left(\overline{\widetilde{A}}\right)} \subseteq \widetilde{A}$.

Definition 2.5 [4]. For any $A \subseteq X$, the fuzzy set of interior points of A is called the interior of A, and given as $A^\circ(x) := N_x(A)$. From Lemma 3.1 [4] and the definitions of $N_x(A)$ and A°, we have $\tau(A) = \inf_{x\in A} A^\circ(x)$.

Definition 2.6 [2]. Let (X,τ) be a fuzzifying topological space.
(a) The family of all fuzzifying γ-open sets, denoted by $\tau_\gamma \in \Im(P(X))$, is defined as $A \in \tau_\gamma := \forall x(x \in A \to x \in A^{\circ-} \cup A^{-\circ})$, i.e., $\tau_\gamma(i) = \inf_{x\in A} \max(A^{\circ-}(x), A^{-\circ}(x))$.
(b) The family of all fuzzifying γ-closed sets, denoted by $F_\gamma \in \Im(P(X))$, is defined as $A \in F_\gamma := X - A \in \tau_\gamma$.
(c) The fuzzifying γ-neighborhood system of a point $x \in X$ is denoted by $N_x^\gamma \in \Im(P(X))$ and defined as $N_x^\gamma(A) = \sup_{x\in B\subseteq A} \tau_\gamma(B)$.
(d) For any $A \subseteq X$ the γ-interior of A, denoted by $Int_\gamma \in \Im(P(X))$, is defined as $Int_\gamma(A) := N_x^\gamma(A)$.
(e) The fuzzifying γ-derived set of $A \in P(X)$, denoted by $d_\gamma \in \Im(X)$, is

defined as $d_\gamma(A)(x) = 1 - N_x^\gamma((X-A)\cup\{x\})$.
(f) The fuzzifying γ-closure of a set $A \in P(X)$, denoted by $Cl_\gamma \in \Im(X)$, is defined as $Cl_\gamma(A)(x) = 1 - N_x^\gamma(X-A)$.
(g) Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and $f \in Y^X$. A unary fuzzy predicate $C_\gamma \in \Im(Y^X)$, called fuzzifying γ-continuity, is given as
$C_\gamma(f) \coloneqq \forall B(B \in \sigma \to f^{-1}(B) \in \tau_\gamma)$.

Definition 2.7 [3]. Let (X,τ) be a fuzzifying topological space.
(a) The family of all fuzzifying $c\gamma$-open sets, denoted by $\tau_{c\gamma} \in \Im(P(X))$, is defined as
$A \in \tau_{c\gamma} \coloneqq \forall x(x \in A \cap (A^{\circ -} \cup A^{- \circ}) \to x \in A^\circ)$, i.e.,
$\tau_\gamma(A) = \inf_{x\in A} (1 - \max(A^{\circ -}(x), A^{- \circ}(x)) + A^\circ(x))$.
(b) The family of all fuzzifying $c\gamma$-closed sets, denoted by $F_{c\gamma} \in \Im(P(X))$, is defined as $A \in F_{c\gamma} \coloneqq X - A \in \tau_{c\gamma}$.
(c) The fuzzifying $c\gamma$-neighborhood system of a point $x \in X$ is denoted by $N_x^{c\gamma} \in \Im(P(X))$ and defined as
$N_x^{c\gamma}(A) = \sup_{x\in B\subseteq A} \tau_{c\gamma}(B)$.
(d) For any $A \subseteq X$ the $c\gamma$-interior of A, denoted by $Int_{c\gamma} \in \Im(P(X))$, is defined as
$Int_{c\gamma}(A) \coloneqq N_x^{c\gamma}(A)$.
From Theorem 4.1 of [3] and the definitions of $N_x^{c\gamma}(A)$ and $Int_{c\gamma}(A)$ we have
$\tau_{N^{c\gamma}}(A) = \inf_{x\in A} N_x^{c\gamma}(A)$.
(e) The fuzzifying $c\gamma$-derived set of $A \in P(X)$, denoted by $d_{c\gamma} \in \Im(X)$, is defined as $d_{c\gamma}(A)(x) = 1 - N_x^{c\gamma}((X-A)\cup\{x\})$.
(f) The fuzzifying $c\gamma$-closure of a set $A \in P(X)$, denoted by $Cl_{c\gamma} \in \Im(X)$, is defined as $Cl_{c\gamma}(A)(x) = 1 - N_x^{c\gamma}(X-A)$.
(g) Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and $f \in Y^X$. A unary fuzzy predicate $C_{c\gamma} \in \Im(Y^X)$, called fuzzifying $c\gamma$-continuity, is given as
$C_{c\gamma}(f) \coloneqq \forall B(B \in \sigma \to f^{-1}(B) \in \tau_{c\gamma})$.

Theorem 2.1 [2]. Let (X,τ) be a fuzzifying topological space. Then
$\models A \in F_\gamma \leftrightarrow Cl_\gamma(A) \subseteq A$.

Theorem 2.2 [3]. Let (X,τ) be a fuzzifying topological space. Then
$\models A \in F_{N^{C\gamma}} \leftrightarrow Cl_{C\gamma}(A) \subseteq A$.

Theorem 2.3 [3]. Let (X,τ) be a fuzzifying topological space. Then
$\models A \in \tau \to (A \in \tau_\gamma \wedge A \in \tau_{C\gamma})$.

Theorem 2.4. [3]. Let (X,τ) be a fuzzifying topological space. Then
$\models A \in N_x^{C\gamma} \wedge B \in N_x^{C\gamma} \to A \cap B \in N_x^{C\gamma}$.

On γ-convergence and $c\gamma$-convergence

Let (X,τ) be a fuzzifying topological space. The class of all nets in X is denoted by $N(X) = \{S : S : D \to X$, where $(D,\geq)$ is a directed set$\}$. Now, we introduce

Definition 3.1. (a) The binary fuzzy predicates $\triangleright^\gamma, \propto^\gamma$ (resp. $\triangleright^{C\gamma}, \propto^{C\gamma}$) $\in \Im(N(X)\times X)$ are defined as follows:
$S \triangleright^\gamma x \coloneqq \forall A(A \in N_x^\gamma \to S \Subset A), S \propto^\gamma x :$
$= \forall A(A \in N_x^\gamma \to S \sqsubseteq A)$ (resp.
$S \triangleright^{C\gamma} x \coloneqq \forall A(A \in N_x^{C\gamma} \to S \Subset A), S \propto^{C\gamma} x :$
$= \forall A(A \in N_x^{C\gamma} \to S \sqsubseteq A)$,
where $[S \triangleright^\gamma x]$ (resp. $[S \triangleright^{C\gamma} x]$) stands for the degree to which "S γ-convergece to x" (resp. "S $c\gamma$-convergece to x"); $[S \propto^\gamma x]$ (resp. $[S \propto^{c\gamma} x]$) stands for the degree to which "x is a γ-accumulation point of S" (resp. "x is a $c\gamma$-accumulation point of S")

and $\Subset$ (resp. $\sqsubseteq$) is the binary crisp predictatate "almost in" (resp. "often in").

(b) The fuzzy sets $\lim_\gamma S$, adh $_\gamma S$ (resp. $\lim_{C\gamma} S$, adh $_{C\gamma} S) \in \Im(X)$ are defined as $\lim_\gamma S(x) := S \rhd^\gamma x$, adh $_\gamma S(x) := S \propto^\gamma x$ (resp. $\lim_{C\gamma} S(x) := S \rhd^{C\gamma} x$, adh $_{C\gamma} S(x) := S \propto^{C\gamma} x$) and called the γ-limit and γ-adherence (resp. $c\gamma$-limit and $c\gamma$-adher-ence) sets of S respectively.

Theorem 3.1. Let (X, τ) be a fuzzifying topological space. Then for any $x \in X, A \in P(X)$ and $S \in N(X)$.

(a)$\models \exists S(S \subseteq A - \{x\} \wedge S \rhd^\gamma x) \to x \in d_\gamma(A)$;

(b)$\models \exists S(S \subseteq A \wedge S \rhd^\gamma x) \to x \in Cl_\gamma(A)$;

(c)$\models A \in F_\gamma \to \forall S(S \subseteq A \to \lim_\gamma S \subseteq A)$;

(d)$\models \exists T((T < S) \wedge (T \rhd^\gamma x)) \to S \propto^\gamma x$,

where $S \subseteq A, T < S$ stand for "S is all in A", "T is a subnet of S".

Proof. (a) We know that

$[S \rhd^\gamma x] = \inf_{S ? A}(1 - N_x^\gamma(A))$.

$[\exists S(S \subseteq A - \{x\} \wedge S \rhd^\gamma x)] = \sup_{S \in N(X), S \subseteq A - \{x\}} [S \rhd^\gamma x]$

$= \sup_{S \in N(X), S \subseteq A - \{x\}} \inf_{B \in P(X) S ? B} (1 - N_x^\gamma(B)).$

Now, for any $S \in N(X)$ such that $S \subseteq A - \{x\}$ one can deduce that $S \notin (X - A) \cup \{x\}$ because if $S \Subset (X - A) \cup \{x\}$, then there exist $m \in D$ and $n \in D$ such that $n \geq m$ and $S(n) \in (X - A) \cup \{x\}$ and so $S(n) \notin X - ((X - A) \cup \{x\}) = A - \{x\}$.

Thus, $S \Subset A - \{x\}$.

So,

$\sup_{S \in N(X), S \subseteq A - \{x\}} \inf_{B \in P(X) S ? B} (1 - N_x^\gamma(B)) \leq \sup_{S \in N(X), S \subseteq A - \{x\}} (1 - N_x^\gamma((X - A) \cup \{x\}))$

$= 1 - N_x^\gamma((X - A) \cup \{x\}) = [x \in d_\gamma(A)]$.

(b) If $x \in A$, then the statement is obtained. If $x \notin A$, then from (a) above and Theorem 2.3 we have

$[Cl_\gamma(A)(x)] = [d_\gamma(A)(x)] \geq [\exists S(S \subseteq A - \{x\} \wedge S \rhd^\gamma x)]$

$= [\exists S(S \subseteq A \wedge S \rhd^\gamma x)]$.

(c) From Theorem 2.1 we have

$\models A \in F_\gamma \leftrightarrow \forall x(x \in Cl_\gamma(A) \to x \in A)$

$\leftrightarrow \forall x(x \in X - A \to x \in X - Cl_\gamma(A))$.

By some calculations, we have

$[\forall S(S \subseteq A \to \lim_\gamma S \subseteq A)]$

$= \inf_{S \subseteq A} \inf_{x \in X - A} (1 - \inf_{S \not\Subset B} (1 - N_x^\gamma(B)))$

$= \inf_{S \subseteq A} \inf_{x \in X - A} \sup_{S ? B} N_x^\gamma(B)$.

Therefore from (b) we have

$F_\gamma(A) = \inf_{x \in X - A} (1 - Cl_\gamma(A)(x))$

$\leq \inf_{x \in X - A} (1 - \sup_{S \subseteq A} \inf_{S \not\Subset B} (1 - N_x^\gamma(B)))$

$= \inf_{x \in X - A} \inf_{S \subseteq A} \sup_{S \not\Subset B} N_x^\gamma(B)$.

(d) Set ℜ_{S}={A:S⊑A}, β_{T}={A:T⋐A} Then for any $T < S$ (for the definition of the subnet see [1]), one can deduce that $\Re_S \subseteq \beta_T$ as follows.

Suppose $T = S \circ K$. If S⊑A, then there exists $m \in D$ such that $S(n) \notin A$ when $n \geq m$.

Now, we will show that T⋐A. If not, then there exists $p \in E$ such that $T(q) \in A$ when $q \geq p$. Moreover, there exists $n_1 \in E$ such that $K(n_1) \geq m$ because $T < S$, and there exists $n_2 \in E$ such that $n_2 \geq n_1, p$ because $(E, \geq)$ is directed.

In this way, $K(n_2) \geq K(n_1) \geq m, S \circ K(n_2) \notin A$ and $S(K(n_2)) = T(n_2) \in A$. This is a contradiction. Therefore,

$[\exists T((T < S) \wedge (T \rhd^\gamma x))]$

$= \sup_{T < S} \inf_{T \not\Subset A} (1 - N_x^\gamma(A)) = \sup_{T < S} \inf_{A \in \beta_T} (1 - N_x^\gamma(A))$

$\leq \inf_{A \in \Re_S} (1 - N_x^\gamma(A)) = \inf_{S \not\sqsubseteq A} (1 - N_x^\gamma(A)) = [S \propto^\gamma x]$.

Theorem 3.2. Let (X, τ) be a fuzzifying topological space. Then for any $x \in X, A \in P(X)$ and $S \in N(X)$.

(a) $\models \exists S(S \subseteq A - \{x\} \wedge S \rhd^{C\gamma} x) \leftrightarrow x \in d_{C\gamma}(A)$;

(b) $\models \exists S(S \subseteq A \wedge S \rhd^{C\gamma} x) \leftrightarrow x \in Cl_{C\gamma}(A)$;

(c)

$\vDash A \in F_{N^{C\gamma}} \to \forall S(S \subseteq A \to \lim_{C\gamma} S \subseteq A)$;
(d) $\vDash \exists T((T < S) \wedge (T \rhd^{C\gamma} x)) \leftrightarrow S \propto^{C\gamma} x$.

Proof. (a) First, from Theorem 3.1 (a), we have
$[\exists S(S \subseteq A - \{x\} \wedge S \rhd^{C\gamma} x)] \le [x \in d_{C\gamma}(A)]$.
Second, we prove
$[x \in d_{C\gamma}(A)] \le [\exists S(S \subseteq A - \{x\} \wedge S \rhd^{C\gamma} x)]$.
If $[x \in d_{C\gamma}(A)] = 0$, the result holds.
If $[x \in d_{C\gamma}(A)] > 0$, then for any $B \in (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}$, we have $B \cap (A - \{x\}) \neq \phi$, where $(N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]} = \{B : N_x^{C\gamma}(B) > 1 - d_{C\gamma}(A)(x)\}$ is the strong $[1 - d_{C\gamma}(A)(x)]$-cut of $N_x^{C\gamma}$ because
$[B \in N_x^{C\gamma} \to B \cap (A - \{x\}) \neq \phi] \ge d_{C\gamma}(A)(x)$.
So, we can assign $x_B \in B \cap (A - \{x\})$ for any $B \in (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}$.
From Theorem 2.4, one can show that $((N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}, \subseteq)$ is a directed set. Now, we consider the net
$S^* : (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]} \to A - \{x\}$ defined as $S^*(B) = x_B$ for every $B \in (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}$.
Then,
$[\exists S(S \subseteq A - \{x\} \wedge S \rhd^{C\gamma} x)] \ge \inf_{S^* ? B}(1 - N_x^{C\gamma}(B))$.
It is clear that $1 - N_x^{C\gamma}(B) \ge d_{C\gamma}(A)(x)$ in the case that $B \notin (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}$.
Now, suppose that $B \cap (A - \{x\}) \neq \phi$ and $B \in (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}$. For any $C \subseteq B$ we have $x_C \in C \subseteq B$. So, $S^* \subseteq B$, i.e., if $S^* \subseteq B$ and $B \in (N_x^{C\gamma})_{[1-d_{C\gamma}(A)(x)]}$, then $B \cap (A - \{x\}) = \phi$, and furthermore
$B \subseteq (X - A) \cup \{x\}$,
$1 - N_x^{C\gamma}(B) \ge 1 - N_x^{C\gamma}((X - A) \cup \{x\}) = d_{C\gamma}(A)(x)$.
In other words,
$[\exists S(S \subseteq A - \{x\} \wedge S \rhd^{C\gamma} x)] \ge \inf_{S^* \notin B}(1 - N_x^{C\gamma}(B)) \ge d_{C\gamma}(A)(x)$.

(b) If $x \in A$, then $Cl_{C\gamma}(A) = 1$ and one can deduce that $\sup_{S \subseteq A}[S \rhd^{C\gamma} x] = 1$ as follows.
Consider the net $S^* : H \to X$ defined by $S^*(n) = x$ for each $n \in H$, where H is any directed set. Then, $S^* \subseteq A$ and for any $B \subseteq X$ with $x \notin B$, $1 - N_x^{C\gamma}(B) = 1 - 0 = 1$ and for any $B \subseteq X$ with $x \in B$, then $S^* \Subset B$ and so
$[S^* \rhd^{C\gamma} x] = \inf_{S^* \notin B}(1 - N_x^{C\gamma}(B)) = 1$.
So, the result is obtained. If $x \notin A$, then
$Cl_{C\gamma}(A) = d_{C\gamma}(A)(x) = [\exists S(S \subseteq A - \{x\} \wedge S \rhd^{C\gamma} x)]$
$= [\exists S(S \subseteq A \wedge S \rhd^{C\gamma} x)]$.
(c) From Theorem 2.2 and (b) above, we have

$$
\begin{aligned}
F_{N^{C\gamma}}(A) &= [Cl_{C\gamma}(A) \subseteq A] \\
&= [X - A \subseteq X - Cl_{C\gamma}(A)] \\
&= \inf_{x \in X-A}(1 - Cl_{C\gamma}(A)(x)) \\
&= \inf_{x \in X-A}(1 - \sup_{S \subseteq A} \inf_{S \notin B}(1 - N_x^{C\gamma}(B))) \\
&= \inf_{x \in X-A} \inf_{S \subseteq A} \sup_{S \notin B} N_x^{C\gamma}(B) \\
&= [\forall S(S \subseteq A \to \lim_{C\gamma} S \subseteq A)].
\end{aligned}
$$

(d) From Theorem 3.1 (d), we have $[\exists T((T < S) \wedge (T \rhd^{C\gamma} x))] \le [S \propto^{C\gamma} x]$. Thus we want to prove that $[\exists T((T < S) \wedge (T \rhd^{C\gamma} x))] \ge [S \propto^{C\gamma} x]$. Let $[S \propto^{C\gamma} x] = t$. If $t = 0$, then the result holds. If $t > 0$, then $X \in (N_x^{C\gamma})_{[1-t]}$, i.e., $(N_x^{C\gamma})_{[1-t]} \neq \phi$. From Theorem 2.4, the Intersection of any two elements in $(N_x^{C\gamma})_{[1-t]}$ is still in it. For each $A \in (N_x^{C\gamma})_{[1-t]}$, i.e., $1 - N_x^{C\gamma}(A) < t$, we have $S \not\Subset A$. Then from Lemma 2.5 [1] there exists $T < S$ such that $T \notin A$ for each $A \in (N_x^{C\gamma})_{[1-t]}$. Thus,
$\beta_T = \{A : T \not\Subset A\} \subseteq P(X) - (N_x^{C\gamma})_{[1-t]}$
because the implication
$A \in (N_x^{C\gamma})_{[1-t]} \Rightarrow T \Subset A$

is equivalent to the implication $A \in \beta_T \Rightarrow A \in P(X) - (N_x^{C\gamma})_{[1-t]}$. We know that $\Re_S = \{A : S \Subset A\} \subseteq P(X) - (N_x^{C\gamma})_{[1-t]}$ because if $A \in \Re_S$, then $S \Subset A$ $1 - N_x^{C\gamma}(A) \geq t$, i.e., $N_x^{C\gamma}(A) \leq 1 - t$. Hence, $A \in P(X) - (N_x^{C\gamma})_{[1-t]}$.

So,

So, $t = [S \propto^{C\gamma} x] = \inf_{A \in \Re_S} (1 - N_x^{C\gamma}(B))$

$= \inf_{A \in P(X) - (N_x^{C\gamma})_{[1-t]}} (1 - N_x^{C\gamma}(B))$.

Since $\Re_S \subseteq \beta_T$, then $\inf_{A \in \beta_T} (1 - N_x^{C\gamma}(B))$

$= \inf_{A \in \Re_S} (1 - N_x^{C\gamma}(B)) \wedge \inf_{A \in \beta_T - \Re_S} (1 - N_x^{C\gamma}(B)) = t$.

Thus, $[\exists T((T < S) \wedge (T \rhd^{C\gamma} x))]$

$= \sup_{T<S} \inf_{A \in \beta_T} (1 - N_x^{C\gamma}(B)) \geq t = [S \propto^{C\gamma} x]$.

Theorem 3.3. If S is a universal net, then

(a) $\models \lim_\gamma S \equiv \text{adh}_\gamma S$;

(b) $\models \lim_{C\gamma} S \equiv \text{adh}_{C\gamma} S$.

Proof. For any net $S \in N(X)$ and any $A \in P(X)$, one can obtain that $S \not\Subset A \Rightarrow S \not\sqsubseteq A$. Suppose S is a universal net in X and S⊑A. Then S⋐X-A. Now, one can deduce that if $S \Subset X - A$, then $S \not\Subset A$ because $S \Subset X - A$ if and only if there exists $m_1 \in D$ such that for every $n \in D, n \geq m_1, S(n) \notin A$ if and only if $S \not\Subset A$. Hence for any universal net S in X,

(a) $\lim_\gamma S(x) = \inf_{S?A}(1 - N_x^\gamma(A)) = \inf_{S?A}(1 - N_x^\gamma(A)) = \text{adh}_\gamma S(x)$.

(b) $\lim_{C\gamma} S(x) = \inf_{S?A}(1 - N_x^{C\gamma}(A)) = \inf_{S?A}(1 - N_x^{C\gamma}(A)) = \text{adh}_{C\gamma} S(x)$.

Theorem 3.4. $\models S \rhd^\gamma x \leftrightarrow \forall A(x \in A \in \tau_\gamma \rightarrow S \Subset A)$.

Proof. Since $B \subseteq A$ and $S \not\Subset A$, we have $S \not\Subset B$ and hence

$[S \rhd^\gamma x] = \inf_{S \not\Subset A} (1 - N_x^\gamma(A)) = 1 - \sup_{S \not\Subset A} N_x^\gamma(A)$

$= 1 - \sup_{S \not\Subset A} \sup_{x \in B \subseteq A} \tau_\gamma(B) \geq 1 - \sup_{S \not\Subset B, x \in B} \tau_\gamma(B)$

$= \inf_{S \not\Subset B, x \in B} (1 - \tau_\gamma(B))$

$= [\forall B(x \in B \wedge S \not\Subset B \rightarrow B \in 1 - \tau_\gamma))]$

$= [\forall B(x \in B \wedge B \in \tau_\gamma \rightarrow S \Subset B))]$

$= [\forall B(x \in B \in \tau_\gamma \rightarrow S \Subset B))]$

$= [\forall A(x \in A \in \tau_\gamma \rightarrow S \Subset A)]$.

Conversely, since $N_x^\gamma(A) \geq \tau_\gamma(A)$, then $[\forall A(x \in A \in \tau_\gamma \rightarrow S \Subset A)]$

$= \inf_{S \not\Subset A, x \in A} (1 - \tau_\gamma(A)) \geq \inf_{S \not\Subset A} (1 - N_x^\gamma(A)) = [S \rhd^\gamma x]$.

Theorem 3.5. $\models S \rhd^{C\gamma} x \leftrightarrow \forall A(x \in A \in \tau_{C\gamma} \rightarrow S \Subset A)$.

Proof. The proof is similar to the proof of Theorem 3.4.

Theorem 3.6. Let D and E_m be directed sets for each $m \in D$. Consider the directed set $H = D \times \prod_{m \in D} E_m$. If $\overline{S} = \{\overline{s}(m) : m \in D\} \in N(X), S^{(m)} = \{s(m,n), n \in E_m\} \in N(X)$ and $S \circ R(m, f) = \{S(m, f(m), (m, f) \in H\} \in N(X)$, then

(a) $\models \forall m((m \in D) \rightarrow (S^{(m)} \rhd^\gamma \overline{S}(m)) \wedge (\overline{S} \rhd^\gamma x) \rightarrow S \circ R \rhd^\gamma x)$;

(b) $\models \forall m((m \in D) \rightarrow (S^{(m)} \rhd^{C\gamma} \overline{S}(m)) \wedge (\overline{S} \rhd^{C\gamma} x) \rightarrow S \circ R \rhd^{C\gamma} x)$.

Proof. From Theorem 3.4 we have $[\forall m((m \in D) \rightarrow (S^{(m)} \rhd^\gamma \overline{S}(m)) \wedge (\overline{S} \rhd^\gamma x)]$

$= (1 - \sup_{m \in D} \sup_{S^{(m)} \not\Subset A, \overline{S}(m) \in A} \tau_\gamma(A)) \wedge (1 - \sup_{\overline{S} \not\Subset A, x \in A} \tau_\gamma(A))$

$= 1 - ((\sup_{m \in D} \sup_{S^{(m)} \not\Subset A, \overline{S}(m) \in A} \tau_\gamma(A)) \vee (\sup_{\overline{S} \not\Subset A, x \in A} \tau_\gamma(A)))$.

Also, $[S \circ R \rhd^\gamma x] = 1 - \sup_{S \circ R \not\Subset A, x \in A} \tau_\gamma(A)$.

Therefore the proof is obtained if we show that

$$1-(\sup_{m\in D}\ \sup_{S^{(m)}\not\Subset A,\overline{S}(m)\in A}\tau_\gamma(A))\ \vee$$

$$(\sup_{\overline{S}\not\Subset A,x\in A}\tau_\gamma(A))\leq 1-\sup_{S\circ R\not\Subset A,x\in A}\tau_\gamma(A),\text{ i.e.,}$$

$$(\sup_{m\in D}\ \sup_{S^{(m)}\not\Subset A,\overline{S}(m)\in A}\tau_\gamma(A))\ \vee$$

$$(\sup_{\overline{S}\not\Subset A,x\in A}\tau_\gamma(A))\geq\sup_{S\circ R\not\Subset A,x\in A}\tau_\gamma(A).$$

Suppose $\sup_{S\circ R\not\Subset A,x\in A}\tau_\gamma(A)>t.$

Then there exists $A_\circ$ such that $x\in A_\circ, S\circ R\not\Subset A$ and $\tau(A_\circ)>t$. Hence, for any $(m,f)\in H$

Case 1. If there exists $m_\circ\in D$ such that $\overline{S}(m_\circ)\in A_\circ$, then

$$(\sup_{m\in D}\ \sup_{S^{(m)}\not\Subset A,\overline{S}(m)\in A}\tau_\gamma(A))\vee(\sup_{\overline{S}\not\Subset A,x\in A}\tau_\gamma(A))$$

$$\geq\sup_{S^{(m_\circ)}\not\Subset A,\overline{S}(m_\circ)\in A}\tau_\gamma(A)\geq\tau_\gamma(A_\circ)>t.$$

Case 2. If $\overline{S}(m)\notin A_\circ$ for any $m\in D$, then $\overline{S}\not\Subset A_\circ$ and furthermore $\sup_{\overline{S}\not\Subset A,x\in A}\tau_\gamma(A))$

$\geq\tau_\gamma(A_\circ)>t.$

$$\sup_{\overline{S}\not\Subset A,x\in A}\tau_\gamma(A))\geq\tau_\gamma(A_\circ)>t.$$

Hence, we always have

$$(\sup_{m\in D}\ \sup_{S^{(m)}\not\Subset A,\overline{S}(m)\in A}\tau_\gamma(A))\vee(\sup_{\overline{S}\not\Subset A,x\in A}\tau_\gamma(A))>t.$$

On filter convergence in fuzzifying topology

Definition 4.1. Let $F(X)$ be the set of all filters on X .

(a) The binary fuzzy predicates $\rhd^\gamma,\propto^\gamma$ (resp. $\rhd^{C\gamma},\propto^{C\gamma})\in\Im(F(X)\times X)$ are defined as

$K\rhd^\gamma x:=\forall A(A\in N_x^\gamma\to A\in K),$

$K\propto^\gamma x:=\forall A(A\in K\to x\in Cl_\gamma(A))$ (resp

$K\rhd^{C\gamma}x:=\forall A(A\in N_x^{C\gamma}\to A\in K),$

$K\propto^{C\gamma}x:=\forall A(A\in K\to x\in Cl_{C\gamma}(A))).$

(b) The fuzzy sets $\lim_\gamma K$, $\text{adh}_\gamma K$ (resp. $\lim_{C\gamma}K$, $\text{adh}_{C\gamma}K)\in\Im(X)$ are defined as $\lim_\gamma K(x):=K\rhd^\gamma x$, $\text{adh}_\gamma K(x):=K\propto^\gamma x$ (resp. $\lim_{C\gamma}K(x):=K\rhd^{C\gamma}x$, $\text{adh}_{C\gamma}K(x):=K\propto^{C\gamma}x$) and called the γ-limit and γ-adherence (resp. $c\gamma$-limit and $c\gamma$-adherence) sets of K, respectively.

(c) If $K\in\Im(X)$, then $S^K:D\to X$ is a net on X corresponding to K defined by $(x,A)\mapsto x$, where $D=\{(x,A):x\in A\in K\}$ and the order of D is defined as follows: $(x,A)\geq(y,B)$ if and only if $A\subseteq B$.

(d) If $S\in N(X)$, then $K^S=\{A:S\Subset A\}$ is a filter on X corresponding to S.

Theorem 4.1.

(a) $\models\lim_\gamma K^S\equiv\lim_\gamma S$; (b) $\text{adh}_\gamma K^S\equiv\text{adh}_\gamma S$.

(c) $\models\lim_\gamma S^K\equiv\lim_\gamma K$; (d) $\text{adh}_\gamma S^K\equiv\text{adh}_\gamma K$.

Proof.

(a) $\lim_\gamma K^S(x)=\inf_{A\notin K^S}(1-N_x^\gamma(A))$

$=\inf_{S\not\Subset A}(1-N_x^\gamma(A))=\lim_\gamma S(x).$

(b) $\text{adh}_\gamma K^S(x)=\inf_{A\in K^S}(Cl_\gamma(A))(x)$

$=\inf_{S\Subset A}(1-N_x^\gamma(X-A))=\inf_{S\not\Subset X-A}(1-N_x^\gamma(X-A))=\text{adh}_\gamma S(x).$

(c) First we prove that $S^K\Subset A$ if and only if $A\in K$. If $A\in K$, then $A\neq\phi$ and so there exists at least an element $x\in A$. So for $(x,A)\in D$ and any $(y,B)\in D$ such that $(y,B)\geq(x,A),B\subseteq A$ and so $S^K(y,B)=y\in B\subseteq A$. Thus $S^K\Subset A$. Conversely suppose $S^K\Subset A$. Then there exists $(y,B)\in D$ such that $(z,C)\geq(y,B)$ and we have $S^K(z,C)\in A$. So for every $z\in B,(z,B)\geq(y,B)$ and $S^K(z,B)=z\in A$ implies $B\subseteq A$. Then, $A\in K$. Thus $A\notin K\Leftrightarrow S^K\not\Subset A$. Now,

$\lim_\gamma S^K(x) = [S^K \rhd^\gamma x] = \inf_{S \notin A} (1 - N_x^\gamma(A))$
$= \inf_{A \notin K} (1 - N_x^\gamma(A)) = \lim_\gamma K(x).$

(d) First we prove that $X - A \in K \Leftrightarrow S^K \not\sqsubseteq A.$ Suppose $S^K \not\sqsubseteq A.$ Then, there exists $(z, B) \in D$ such that $S^K(z, C) \notin A$ for every $(y, C) \in D$ with $(y, C) \geq (z, B).$ Now, for every $x \in B, (x, B) \geq (z, B)$ and $S^K(x, B) = x \notin A,$ i.e., $B \cap A = \phi$ so $B \subseteq X - A$ and then $X - A \in K.$ Conversely, suppose $X - A \in K$ then $X - A \neq \phi$ and thus it contains at least an element x. Now, for any $(z, C) \in D$ such that $(z, C) \geq (x, X - A)$ we have $S^K(z, C) = z \notin A.$ Hence, $S^K \not\sqsubseteq A.$ So, adh ${}_\gamma S^K(x) = [S^K \propto^\gamma x] = \inf_{S^K \not\sqsubseteq A} (1 - N_x^\gamma(A))$
$= \inf_{X - A \in K} Cl_\gamma(X - A)(x) = \inf_{B \in K} Cl_\gamma(B)(x) =$ adh ${}_\gamma K(x).$

Theorem 4.2.
(a) $\models \lim_{C\gamma} K^S \equiv \lim_{C\gamma} S;$
(b) adh ${}_{C\gamma} K^S \equiv$ adh ${}_{C\gamma} S.$
(c) $\models \lim_{C\gamma} S^K \equiv \lim_{C\gamma} K;$
(d) adh ${}_{C\gamma} S^K \equiv$ adh ${}_{C\gamma} K.$

Proof. The proof is similar to that of Theorem 4.1.

Some applications

Definition 5.1 [2]. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predicates $\gamma_k \in \Im(Y^X)$, where $k = 1, \ldots, 5,$ as

(a) $f \in \gamma_1 = \forall B(B \in F^Y \to f^{-1}(B) \in F_\gamma^X),$ where F^Y is the family of fuzzifying closed subsets Y, and F_γ^X is the family of fuzzifying $\gamma-$ closed subsets X;

(b) $f \in \gamma_2 = \forall x \forall u(u \in N_{f(x)} \to f^{-1}(u) \in N_x^\gamma),$ where N is the fuzzifying neighbourhood system of Y and N^γ is the fuzzifying γ-neighborhood systems of X ;

(c) $f \in \gamma_3 = \forall x \forall u(u \in N_{f(x)} \to \exists v(f(v) \subseteq u \to v \in N_x^\gamma));$

(d) $f \in \gamma_4 = \forall A\left(f\left(cl_\gamma^X(A)\right) \subseteq cl_Y\left(f(A)\right)\right);$

(e) $f \in \gamma_5 = \forall B(cl_\gamma^X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))).$

Theorem 5.1 [2].
$\models f \in C_\gamma \leftrightarrow f \in \gamma_k, \; k = 1, \ldots, 5.$

Definition 5.2 [3]. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predictates $\theta_k \in \Im(Y^X),$ where $k = 1, \ldots, 5,$ as follows:

(a) $f \in \theta_1 = \forall B\left(B \in F^Y \to f^{-1}(B) \in F_{C\gamma}^X\right),$ where F^Y is the family of fuzzifying closed subsets Y, and $F_{C\gamma}^X$ is the family of fuzzifying $c\gamma-$closed subsets X ;

(b) $f \in \theta_2 = \forall x \forall u\left(u \in N_{f(x)} \to f^{-1}(u) \in N_x^{C\gamma}\right),$ where N is the fuzzifying neighbourhood system of Y and $N^{C\gamma}$ is the fuzzifying $c\gamma$-neighborhood systems of X;

(c) $f \in \theta_3 = \forall x \forall u\left(u \in N_{f(x)} \to \exists v(f(v) \subseteq u \to v \in N_x^{C\gamma})\right);$

(d) $f \in \theta_4 = \forall A\left(f\left(cl_{C\gamma}^X(A)\right) \subseteq cl_Y\left(f(A)\right)\right);$

(e) $f \in \theta_5 = \forall B\left(cl_{C\gamma}^X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))\right).$

Theorem 5.2 [3].
(a) $\models f \in C_{C\gamma} \leftrightarrow f \in \theta_1;$
(b) $\models f \in C_{C\gamma} \to f \in \theta_2;$
(c) $\models f \in \theta_2 \leftrightarrow f \in \theta_J;$ where $j = 3, 4, 5.$

Definition 5.3. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predicates $\gamma_6, \; \theta_6 \in \Im(Y^X)$ as follows:
(a)

$f \in \gamma_6 := \forall x \forall S(S \in N(X) \wedge (S \rhd^{\gamma} x) \to (f \circ S \rhd f(x)));$

(b)

$f \in \theta_6 := \forall x \forall S(S \in N(X) \wedge (S \rhd^{C\gamma} x) \to (f \circ S \rhd f(x))).$

Theorem 5.3.

(a) $\models f \in C_\gamma \to f \in \gamma_6;$

(b) $\models f \in C_{C\gamma} \leftrightarrow f \in \theta_6.$

Proof. (a) From Theorem 5.1 we have $\models f \in C_\gamma \leftrightarrow f \in \gamma_2$. then, the result holds if we proved that $\models f \in \gamma_2 \to f \in \gamma_6$. If $[f \circ S \rhd f(x)] \geq [S \rhd^{\gamma} x]$, then the result holds. If $[f \circ S \rhd f(x)] < [S \rhd^{\gamma} x]$, then one can deduce that for each $x \in X$ and $S \in N(X)$,

$\min(1, 1 - [S \rhd^{\gamma} x] + [f \circ S \rhd f(x)]) \geq \gamma_2(f).$

It is shown as follows:

Since $f \circ S \notin B$ implies $S \notin f^{-1}(ii)$, then

$$[S \rhd^{\gamma} x] - [f \circ S \rhd f(x)]$$
$$= \inf_{A \subseteq X, S \notin A} (1 - N_x^{\gamma}(A)) - \inf_{B \subseteq Y, f \circ S \notin B} (1 - N_{f(x)}(B))$$
$$\leq \inf_{B \subseteq Y, f \circ S \notin B} (1 - N_x^{\gamma}(f^{-1}(B))) - \inf_{B \subseteq Y, f \circ S \notin B} (1 - N_{f(x)}(B)$$
$$\leq \sup_{B \subseteq Y, f \circ S \notin B} N_{f(x)}(B) - \sup_{B \subseteq Y, f \circ S \notin B} N_x^{\gamma}(f^{-1}(B))$$
$$\leq \sup_{B \subseteq Y, f \circ S \notin B} (N_{f(x)}(B) - N_x^{\gamma}(f^{-1}(B))).$$

Then

$$1 - [S \rhd^{\gamma} x] + [f \circ S \rhd f(x)]$$
$$\geq \inf_{B \subseteq Y, f \circ S \notin B} (1 - N_{f(x)}(B) + N_x^{\gamma}(f^{-1}(B))).$$

Thus

$$\min(1, 1 - [S \rhd^{\gamma} x] + [f \circ S \rhd f(x)])$$
$$\geq \inf_{B \subseteq Y, f \circ S \notin B} \min(1, 1 - N_{f(x)}(B) + N_x^{\gamma}(f^{-1}(B)))$$
$$\geq \inf_{u \subseteq Y} \min(1, 1 - N_{f(x)}(u) + N_x^{\gamma}(f^{-1}(u))) = \gamma_2(f).$$

Hence,

$$\gamma_6(f) = \inf_{x \in X} \inf_{S \in N(X)} \min(1, 1 - [S \rhd^{\gamma} x] + [f \circ S \rhd f(x)]) \geq \gamma_2(f).$$

(b) Similarly to (a) above we can prove that $C_{C\gamma}(f) \leq \theta_6(f)$. So, we want to prove that $C_{C\gamma}(f) \geq \theta_6(f)$. From Theorem 5.2 (c) we have $C_{C_\gamma}(f) = \theta_4(f)$. Then the result holds if we proved that $\theta_4(f) \geq \theta_6(f)$. Since

$$\theta_4(f) = \inf_{A \subseteq X} \inf_{y \in Y} \min\left(1, 1 - f\left(cl_{C\gamma}^X(A)\right)(y) + cl_Y(f(A))(y)\right),$$

then the result holds if we proved that for every $A \subseteq X$ and every $y \in Y, \min\left(1, 1 - f\left(cl_{C\gamma}^X(A)\right)(y) + cl_Y(f(A))(y)\right) \geq \theta_6(f).$

If $f\left(cl_{C\gamma}^X(A)\right)(y) \leq cl_Y(f(A))(y)$, the result holds. Suppose that $f\left(cl_{C\gamma}^X(A)\right)(y) > cl_Y(f(A))(y)$, then from Theorem 3.2 (b) we have

$$f\left(cl_{C\gamma}^X(A)\right)(y) - cl_Y(f(A))(y)$$
$$= \sup_{f(x)=y} \left(cl_{C\gamma}^X(A)\right)(x) - cl_Y(f(A))(y)$$
$$= \sup_{f(x)=y} \sup_{S \subseteq A} [S \rhd^{C\gamma} x] - \sup_{T \subseteq f(A)} [T \rhd y]$$
$$\leq \sup_{f(x)=y} \sup_{S \subseteq A} [S \rhd^{C\gamma} x] - \sup_{f(S) \subseteq f(A)} [f \circ S \rhd y]$$
$$\leq \sup_{f(x)=y} \sup_{S \subseteq A} [S \rhd^{C\gamma} x] - \sup_{f(x)=y} \sup_{S \subseteq A} [f \circ S \rhd y]$$
$$\leq \sup_{f(x)=y} \sup_{S \subseteq A} ([S \rhd^{C\gamma} x] - [f \circ S \rhd y]).$$

So,

$$\min\left(1, 1 - f\left(cl_{C\gamma}^X(A)\right)(y) + cl_Y(f(A))(y)\right)$$
$$\geq \inf_{f(x)=y} \inf_{S \subseteq A} \min(1, 1 - [S \rhd^{C\gamma} x] + [f \circ S \rhd y])$$
$$\geq \inf_{x \in X} \inf_{S \in N(X)} \min(1, 1 - [S \rhd^{C\gamma} x] + [f \circ S \rhd f(x)]) \geq \theta_6(f).$$

Theorem 5.4. We always have

$\models T_2^{\gamma}(X) := \forall S \forall x \forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^{\gamma} x) \wedge (S \rhd^{\gamma} y) \to x = y),$

where T_2^{γ} is the unary fuzzy predicate "Hausdorrf (T_2^{γ}) separable" on the class of all fuzzifying topological spaces, and defined as

$T_2^\gamma(X) := \forall x\forall y((x \in X) \wedge (y \in X)(x \neq y) \rightarrow \exists A\exists B(A \in N_x^\gamma) \wedge (B \in N_y^\gamma) \wedge (A \cap B = \phi))$.

Proof. $T_2^\gamma(X) = \inf_{x\neq y} \sup_{A\cap B=\phi} (N_x^\gamma(A) \wedge N_y^\gamma(B))$,

$[\forall S\forall x\forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^\gamma x) \wedge (S \rhd^\gamma y) \rightarrow x = y)]$

$= \inf_{x\neq y} \inf_{S\subset X} (\sup_{S\notin A} N_x^\gamma(A) \vee \sup_{S\notin B} N_y^\gamma(B))$

$= \inf_{x\neq y} \inf_{S\subset X} \sup_{S\notin A} \sup_{S\notin B} (N_x^\gamma(A) \vee N_y^\gamma(B)))$.

(a) If $A \cap B = \phi$, then for any S, we have $S \notin A$ or $S \notin B$, and

$N_x^\gamma(A) \wedge N_y^\gamma(B) \leq \sup_{S\notin A} N_x^\gamma(A)$,

$N_x^\gamma(A) \wedge N_y^\gamma(B) \leq \sup_{S\notin B} N_y^\gamma(B)$.

So,

$\sup_{A\cap B=\phi} (N_x^\gamma(A) \wedge N_y^\gamma(B))$

$\leq \inf_{S\subset X} (\sup_{S\notin A} N_x^\gamma(A) \vee \sup_{S\notin B} N_y^\gamma(B))$

and furthermore,

$T_2^\gamma(X) \leq [\forall S\forall x\forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^\gamma x) \wedge (S \rhd^\gamma y) \rightarrow x = y)]$.

(b) First, let $x, y \in X$ with $x \neq y$. Suppose $\sup_{A\cap B=\phi} (N_x^\gamma(A) \wedge N_y^\gamma(B)) \leq t$.

Then for $A \cap B = \phi$ we have $N_x^\gamma(A) \leq t$ or $N_y^\gamma(B) \leq t$, i.e., $A \cap B \neq \phi$ in the case of $A \in (N_x^\gamma)_t$ and $B \in (N_y^\gamma)_t$.

Now, define a net $S^* : (N_x^\gamma)_t \times (N_y^\gamma)_t \rightarrow X$ by $(A,B) \mapsto x_{(A,B)} \in A \cap B$. Then for any $A \in (N_x^\gamma)_t, B \in (N_y^\gamma)_t$ we have $S^* \notin A$ and $S^* \notin B$. Therefore, if $S^* \notin A$ and $S^* \notin B$, then $A \notin (N_x^\gamma)_t, B \notin (N_y^\gamma)_t$, i.e., $N_x^\gamma(A) \vee N_y^\gamma(B) < t$. Consequently,

$\sup_{S^*\notin A} \sup_{S^*\notin B} (N_x^\gamma(A) \vee N_y^\gamma(B)) \leq t$.

Moreover,

$\inf_{S\subset X} \sup_{S\notin A} \sup_{S\notin B} (N_x^\gamma(A) \vee N_y^\gamma(B)) \leq t$.

Second, for any positive integer i, there exists (x_i, y_i) with $x_i \neq y_i$ and $\sup_{A\cap B=\phi} (N_x^\gamma(A) \wedge N_y^\gamma(B)) < [T_2^\gamma(X)] + 1/i$,

and hence

$\inf_{S\subset X} \sup_{S\notin A} \sup_{S\notin B} (N_x^\gamma(A) \vee N_y^\gamma(B)) \leq [T_2^\gamma(X)] + 1/i$.

So, we have

$[\forall S\forall x\forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^\gamma x) \wedge (S \rhd^\gamma y) \rightarrow x = y)]$

$= \inf_{x\neq y} \inf_{S\subset X} \sup_{S\notin A} \sup_{S\notin B} (N_x^\gamma(A) \vee N_y^\gamma(B))$

$\leq [T_2^\gamma(X)]$.

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BENZOYLSALIREPOSIDE AN ANTIOXIDANT, LIPOXYGENASE AND CHYMOTRYPSIN INHIBITOR

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Abstract: Two phenolic glycosides, benzoylsalireposide (**1**) and salireposide (**2**) were isolated from *Symplocos racemosa* Roxb, which showed DPPH radical scavenging activity, with the IC_{50} values of 773 ± 11.83 µM and 757 ± 0.374 µM respectively. In addition to this, compound **1** also displayed *in vitro* inhibitory potential against lipoxygenase and chymotrypsin in a concentration-dependent fashion with the IC_{50} values of 75.1 ± 0.5 µM and 65.07 ± 0.10 µM respectively, while **2** was inactive against these enzymes.

***Keywords*:** phenolic glycosides, DPPH scavengers, lipoxygenase, chymotrypsin

Introduction

Symplocos racemosa Roxb. (Lodh) belongs to the family Symplocaceae, which is a unigeneric family of about 290 species. Lodh has a wide range of usage in Ayurveda and Unani medicines. Its bark is described as an emmenagogue tonic for the persons of plethoric constitution and is useful in bowel complaints and ulcers. Its decoction is used as a gargle for giving firmness to bleeding and spongy gums. It cures watery eyes, opthalmia and is good for all diseases of the eye. It also cures diseases of the blood, dysentery, inflammations, vaginal discharges and leprosy. The bark is also prescribed in the treatment snake-bite and scorpion-sting [1].

There is extensive evidence to implicate free radicals in the development of degenerative diseases. It is suggested that free radical damage to cells leads to the pathological changes associated with aging. Free radicals may also be a contributory factor in a progressive decline in the function of the immune system. The consequences of oxidative stress are serious, and in many cases are manifested by increased activities of enzymes involved in oxygen detoxification. Identification of new anti-oxidants remains a highly active research area because anti-oxidants may reduce the risk of various chronic diseases caused by free radicals [2].

Lipoxygenases (EC 1.13.11.12) constitute a family of non-haem iron containing dioxygenases that are widely distributed in animals and plants. In mammalian cells these are key enzymes in the biosynthesis of many bioregulatory compounds such as hydroxyeicosatetraenoic acids (HETEs), leukotrienes, lipoxins and hepoxylines [3]. It has been found that these lipoxygenase products play a role in a variety of disorders such as bronchial asthma, inflammation [4] and tumor angiogenesis [5]. Lipoxygenases are therefore potential target for the rational drug design and discovery of mechanism–based inhibitors for the treatment of bronchial

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asthma, inflammation, cancer and autoimmune diseases.

The physiological role of serine proteases inhibitors has been clearly established, and it has been proposed that they are part of plants' natural defense system against insect predation. They function by inhibiting insect proteinases. Hence these inhibitors have gained attention as possible sources of engineered resistance against pests and pathogens for transgenic plants expressing heterologous inhibitors [6,7,8,9]. Serine proteases such as chymotrypsin and trypsin are involved in the destruction of certain fibrous proteins [10]. Chronic infection by hepatitis C virus can lead to the progressive liver injury, cirrhosis, and liver cancer. A chymotrypsin-like serine protease known as NS3 protease is considered essential for viral replication and has become a target for anti-HCV drugs [11].

Materials and Methods

General

For column chromatography (CC), silica gel (70-230 mesh) and for flash chromatography (FC), silica gel (230-400 mesh) was used. TLC was performed on pre-coated silica gel G-25-UV_{254} plates. Detection was carried out at 254 nm by ceric sulphate reagent. Purity was checked on TLC with different solvent systems using methanol, acetone and $CHCL_3$ giving single spot. The optical rotations were measured on a Jasco-DIP-360 digital polarimeter. The UV and IR spectra were recorded on Hitachi-UV-3200 and Jasco-320-A spectrophotometers, respectively. ^{1}H-NMR, ^{13}C-NMR, COSY, HMQC and HMBC spectra were run on Bruker spectrometers operating at 500, 400 and 300 MHz. The chemical shifts are given in δ in ppm and coupling constants in Hz. EI-MS and FAB-MS spectra were recorded on a JMS-HX-110 spectrometer with a data system.

Plant material

The plant *Symplocos racemosa* Roxb. (Symplocaceae) was collected from Abbottabad, Pakistan in August 2002, and identified by Dr. Manzoor Ahmed at the Department of Botany, Post-Graduate College, Abbottabad, Pakistan. A voucher specimen (no. 6453) has been deposited at the herbarium of this Botany Department.

Extraction and purification

The air-dried ground plant (30 kg) was exhaustively extracted with methanol at room temperature. The extract was evaporated to yield the residue (818 g), which was extracted with hexane, chloroform, ethyl acetate and butanol. The ethyl acetate extract (106.2 g) was subjected to CC over a silica gel column using hexane with gradient of $CHCl_3$ up to 100 % and followed by methanol. Ten fractions (Fr. 1-10) were collected. The Fr. 8 was loaded on silica gel (flash silica 230-400 mesh) and eluted with MeOH:$CHCl_3$ (5:95) to purify compound **1**. The Fr. 9 was subjected to column chromatography and eluted with MeOH: $CHCl_3$ (7: 93) to purify compound **2**.

DPPH free radical scavenging activity

The reaction mixture containing 5 µL of test samples was dissolved in DMSO and 95 µL of DPPH in ethanol. Different concentrations of test sample were taken in the reaction mixture, while the concentration of DPPH was kept as 300 µ?mol.dm^{-3}. These reaction mixtures were taken in 96-well plate microlitre plates (*Molecular Devices, USA*) and incubated at 37^0 C for 30 min. The absorbance was measured at 515 nm. Percent radical scavenging activity by samples was determined in comparison with a DMSO-treated control group. IC_{50} values represent the concentration of sample, which is required to scavenge 50% DPPH free radicals. 3-

t-Butyl-4-hydroxyanisole (BHA) and propyl gallate were used as positive control [12,13].

In vitro lipoxygenase inhibition assay

Lipoxygenase inhibiting activity was measured by modifying the spectrophotometric method developed by *A.L. Tappel* [14]. Lipoxygenase (1.13.11.12) type I-B and linoleic acid was purchased from sigma (St. Louis, MO, USA). All other chemicals were of analytical grade. The reaction mixture contained 165µL (100 mM) sodium phosphate buffer (pH 8.0), 5.0 µL of test-compound solution and 20 µL of lipoxygenase solution. After mixing the contents the mixture was incubated for 10 minutes at 25 ^{0}C. The reaction was then initiated by the addition of 10µL linoleic acid (substrate) solution, with the formation of (9*Z*, 11E)-(13*S*)-13-hydroperoxyoctadeca-9, 11-dienoate. The change of absorbance at 234 nm was followed for 6 minutes. Test compounds and the positive control (Baicalein) were dissolved in MeOH. All the reactions were performed in triplicate in 96-well micro-plate in *SpectraMax* 384 plus (Molecular Devices, USA). The percentage (%) inhibition was calculated as (E – S)/ E x 100, where E is the activity of the enzyme without test compound and S is the activity of enzyme with the test compound. The IC_{50} values were then calculated using the EZ-Fit Enzyme kinetics program (*Perrella Scientific Inc.*, *Amherst*, USA).

In vitro chymotrypsin assay

The α-chymotrypsin inhibitory activity of compounds was performed by the method of Cannell [15]. Chymotrypsin (9 units/ ml of 50 mM Tris-HCl buffer pH 7.6; Sigma Chemical Co. USA) was preincubated with the compounds for 20 min at 25 ^{0}C. A 100 µL aliquot of the substrate solution (N-succinyl-phenylalanine-*p*-nitroanilide, 1 mg/mL of 50 mmol.dm^{-3} Tris-HCl buffer pH 7.6) was added to start the enzyme reaction. The absorbance of released *p*-nitroaniline was continuously monitored at 410 nm until a significant color change was achieved. The final DMSO concentration in the reaction mixture was 7 %.

Results and Discussion

From the ethyl acetate soluble fraction of *Symplocos racemosa* Roxb., two phenolic glycosides namely, benzoylsalireposide (**1**) and salireposide (**2**) [16] were isolated and their structures were established by extensive NMR studies. In a search for new bioactive substances of plant origin, we studied the compound **1** and **2** for their antioxidant activity in the DPPH radical scavenging assay. Both of these compounds scavenge DPPH (1,1-diphenyl-2-picrylhydrazyl radical) moderately as shown in Table 1. In addition to this, benzoylsalireposide (**1**) also showed moderate inhibitory activity against lipoxygenase and chymotrypsin, but **2** was inactive against these enzymes. The IC_{50} values of **1** are shown in Table 2. In terms of structure-activity relationship, the inhibitory potential of **1** against lipoxygenase and chymotrypsin can be attributed to the presence of benzoyl esterifying group on 3-position of glucose moiety, which was absent in **2**. However both the compounds exhibited radical scavenging activity due to presence of phenolic groups in them.

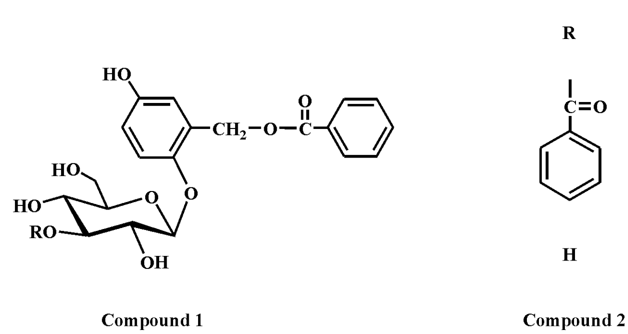

Figure 1. Structures of compounds **1** and **2**

Table 1. Antioxidant Activities of the Compounds **1** and **2** as Compared with the Standard Inhibitors.

Compounds	DPPH Radical Scavenging activity (%) (1000 μmol.dm^{-3})	IC_{50} ± S.E.M.[a]) [μ?mol.dm^{-3}]
1	66.04	773 11.83
2	59.13	757 ± 0.374
Propyl gallate[b]	92.00	30 ± 0.52
3-t-Butyl-4-hydroxy anisole[b]	91.25	44 ± 0.022

[a] Standard error of the mean of three assays.
[b] Standard antioxidants

Table 2. *In vitro* quantitative inhibition of lipoxygenase and chymotrypsin by 1.

Name of Substance	Lipoxygenase IC_{50} ± SEM[a]	Chymotrypsin [μmol.dm^{-3}]
Benzoylsalireposide	75.1 ± 0.5	65.07 ± 0.10
Salireposide	-	-
Positive controls	Baicalein[b] 22.6 ± 0.2	Chymostain[c] 8.23 ± 0.0024

[a] Standard error of the mean of five assays.
[b] positive control for lipoxygenase; [c] for chymotrypsin.

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SMOOTH SOLUTIONS OF DEGENERATE BELLMAN EQUATIONS

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Abstract. The paper studies the smoothness of solutions of the degenerate Hamilton-Jacobi-Bellman (HJB) equation associated with a stochastic control problem. We establish the existence of a classical solution of the degenerate HJB equation associated with this problem by the technique of viscosity solutions, and hence derive an optimal control from the optimality conditions in the HJB equation.

Keywords: Stochastic differential equation, Hamilton-Jacobi-Bellman equation, optimal control problem, viscosity solutions, application to control theory, MSC 2000: 60H10, 70H20, 49J15, 49L25, 58E25

Introduction

We are concerned with the stochastic control problem to minimize the expected cost:

$$J(c) = E[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] \tag{1}$$

over $c \in \mathbf{A}$ subject to the degenerate stochastic differential equation

$$dx_t = [Ax_t + c_t]dt + \sigma x_t dw_t,$$
$$x_0 = x \in \mathbf{R}, \quad t \ge 0, \tag{2}$$

where $\alpha > 0, A, \sigma \neq 0$ are constants, and the function h is assumed to have the following properties:

$$h \ge 0 \; : \text{ convex}; \tag{3}$$

There exists $C > 0$ such that
$$h(x) \le C(1 + |x|^n); \tag{4}$$

and $C_\rho > 0$, for any $\rho > 0$, such that

$$|h(x) - h(y)| \le C_\rho |x - y|^n +$$
$$\rho(1 + |x|^n + |y|^n), \quad \forall x, y \in \mathbf{R}, \tag{5}$$

for a fixed integer $n \ge 2$. Here, w_t is a one-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathbf{F}, P)$ endowed with the natural filtration $\mathbf{F}_t$ generated by $\sigma(w_s, s \le t)$, and $\mathbf{A}$ denotes the class of all $\mathbf{F}_t$ – progressively measurable processes $c = (c_t)$ with $E[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] < \infty$.

This kind of stochastic control problem has been studied by many authors [2,5] for non-degenerate diffusions. We notice that (3), (4) and (5) are fulfilled for $h(x) = |x|^{\overline{n}}$, $\overline{n} \in [\mathbf{2}, \mathbf{n}]$, and also mention [4] for the quadratic case of degenerate diffusions with finite horizon.

The purpose of this paper is to show the existence of a classical solution u of the degenerate Bellman equation of the form:

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + Axu' +$$
$$\min_{a \in \mathbf{R}}(a^2 + au') + h(x) = 0 \text{ in } \mathbf{R}, \tag{6}$$

and to give a synthesis of optimal control. Our method consists in finding the viscosity solution u of (6) [3,5], and then in considering the smoothness of u by its convexity. We show that

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the value function $v_L(x)=\inf_{c\in \mathbf{A}_L} J(c)$ is a viscosity solution of

$$-\beta v+\frac{1}{2}\sigma^2 x^2 v''+Axv'+,$$
$$\min_{|a|\le L}(a^2+av')+h(x)=0 \text{ in } \mathbf{R} \tag{7}$$

for each $L>0$, and that $u(x):=\lim_{L\to\infty} v_L(x)$ is a viscosity solution of (6), where $\mathbf{A}_L=\{c=(c_t)\in \mathbf{A} : |c_t|\le L \text{ for all } t\ge 0\}$. The section on classical solutions is devoted to the study of smoothness of u. Finally, we present an optimal control to the optimization problem (1) and (2).

Viscosity solutions

Here we study the properties of the value function $v_L(x)$, and show that $v_L(x)$ is a viscosity solution of the Bellman equation (20) for any fixed $L>0$, and then v_L converges to a viscosity solution u of the Bellman equation (6). Given a continuous and degenerate elliptic map $H : \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, we recall by [3] the definition of viscosity solutions of

$$H(x,w,w',w'')=0 \text{ in } \mathbf{R}. \tag{8}$$

Definition

$w\in C(\mathbf{R})$ is called a viscosity subsolution (resp., super-solution) of (8) if, whenever for $\varphi\in C^2(\mathbf{R}), w-\varphi$ attains its local maximum (resp., minimum) at $x\in\mathbf{R}$, then

$$H(x,w(x),\varphi'(x),\varphi''(x))\le$$
$$0 \;\; (resp, H(x,w(x),\varphi'(x),\varphi''(x))\ge 0).$$

We also call $w\in C(\mathbf{R})$ a viscosity solution of (8) if it is both viscosity sub- and super-solution of (8). We remark that this definition is equivalent to the following: for $x\in\mathbf{R}$,

$$H(x,w(x),p,q)\le 0 \text{ for } (p,q)\in J^{2,+}w(x)$$
$$(resp., H(x,w(x),p,q)\ge 0 \text{ for } (p,q)\in J^{2,-}w(x)),$$

where $J^{2,+}$ and $J^{2,-}$ are the second-order superjets and subjets defined by

$$J^{2,+}w(x)=\{(p,q)\in\mathbf{R}^2 :$$
$$\limsup_{y\to x}\frac{w(y)-w(x)-p(y-x)-\frac{1}{2}q|y-x|^2}{|y-x|^2}\le 0\},$$

$$J^{2,-}w(x)=\{(p,q)\in\mathbf{R}^2 :$$
$$\liminf_{y\to x}\frac{w(y)-w(x)-p(y-x)-\frac{1}{2}q|y-x|^2}{|y-x|^2}\ge 0\}.$$

In order to obtain the viscosity property of v_L, we assume that there exists $\beta_0\in(0,\beta)$ satisfying

$$-\beta_0+\sigma^2 n(2n-1)+2n|A|<0, \tag{9}$$

and we set $f_k(x)=\gamma+|x|^k$ for any $2\le k\le 2n$ and a constant $\gamma\ge 1$ chosen later.

Lemma 1

Assuming (9), then there exist $\gamma\ge 1$ and $\eta>0$, depending on $L,k,$ such that

$$-\beta_0 f_k+\frac{1}{2}\sigma^2 x^2 f_k''+Axf_k'+$$
$$\max_{|a|\le L}(a^2+af_k')+\eta f_k\le 0 \tag{10}$$

Further

$$E[\int_0^\tau e^{-\beta_0 s}\eta f_k(x_s)ds+e^{-\beta_0\tau}f_k(x_\tau)]\le$$
$$f_k(x) \;\; for \;\; 2\le k\le 2n, \tag{11}$$

$$E[\sup_t e^{-\beta_0 t}f_k(x_t)]<\infty \;\; for \;\; 2\le k\le n, \tag{12}$$

where τ is any stopping time and x_t is the response to $(c_t) \in \mathbf{A}_L$.

Proof

By (9), we choose $\eta \in (0, \beta_0)$ such that

$$-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta < 0, \tag{13}$$

and then $\gamma \geq 1$ such that

$$(-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta)|x|^k + Lk|x|^{k-1} + (L^2 + \eta\gamma - \beta_0\gamma) \leq 0.$$

Then (10) is immediate. By (10) and Ito's formula, we deduce (11). Moreover, by moment inequalities for martingales we get

$$E[\sup_t e^{-\beta_0 t} f_k(x_t)] \leq f_k(x) + E[\sup_t |\int_0^t e^{-\beta_0 s} f_k'(x_s)\sigma x_s dw_s|] \leq f_k(x) + KE[(\int_0^\infty e^{-2\beta_0 s}\sigma^2 |x_s|^{2k} ds)^{1/2}],$$

for some constant $K > 0$. Therefore (12) follows from this relation together with (11).

Theorem 1

We assume (3), (4), (5) and (9). Then

$$v_L \text{ fulfills (3), (4), (5),} \tag{14}$$

and the dynamic programming principle holds, i.e.,

$$v_L(x) = \inf_{c \in \mathbf{A}_L} E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau} v(x_\tau)\right] \tag{15}$$

for any stopping time τ.

Proof

We suppress L of v_L for simplicity. The convexity of v follows from the same line as [5, Chap. 4, Lemma 10.6]. Let x_t^0 be the unique solution of

$$dx_t^0 = Ax_t^0 dt + \sigma x_t^0 dw_t, \quad x_0^0 = x. \tag{16}$$

Then, by (11) and (4)

$$v(x) \leq E[\int_0^\infty e^{-\beta t} h(x_t^0)dt] \leq CE[\int_0^\infty e^{-\beta_0 t} f_n(x_t^0)dt] \leq Cf_n(x)/\eta. \tag{17}$$

For the solution y_t of (2) with $y_0 = y$, it is clear that $x_t - y_t$ fulfills (16) with initial condition $x - y$. We note by (13) with $k = n$ and Ito's formula that

$$E[e^{-\beta_0 t}|x_t^0|^n] \leq |x|^n.$$

Thus by (5) and (11)

$$\begin{aligned}|v(x) - v(y)| &\leq \sup_{c \in \mathbf{A}_L} E[\int_0^\infty e^{-\beta t}|h(x_t) - h(y_t)|dt] \\ &\leq \sup_{c \in \mathbf{A}_L} E[\int_0^\infty e^{-\beta t}\left[C_\rho |x_t - y_t|^n + \rho(1 + |x_t|^n + |y_t|^n)\right]dt] \\ &\leq \sup_{c \in \mathbf{A}_L} \int_0^\infty e^{-\beta t}\left[C_\rho |x - y|^n e^{\beta_0 t} + \rho(h_n(x) + h_n(y))e^{\beta_0 t}\right]dt \\ &\leq \frac{1}{\beta - \beta_0}[C_\rho |x - y|^n + 2\rho\gamma(1 + |x|^n + |y|^n)].\end{aligned} \tag{18}$$

Therefore we get (14).

To prove (15), we denote by $v^r(x)$ the right hand side of (15). By the formal Markov property

$$E[\int_\tau^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt \,|\, \mathbf{F}_\tau] = e^{-\alpha\tau} J_{\tilde{c}}(x_\tau),$$

with $\tilde{c}$ equal to c shifted by τ. Thus

$$J_c(x) = E\left[\int_0^\tau + \int_\tau^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt \right]$$
$$\geq E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau} v_L(x_\tau) \right]$$

It is known in [5,8] that this formal argument can be verified, and we deduce $v_L(x) \geq v^r(x)$.

To prove the reverse inequality, let $\rho > 0$ be arbitrary. We set

$$V_c(x) = E[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] \qquad (19)$$

By the same calculation as (18), there exists $C_\rho > 0$ such that

$$|V_c(x) - V_c(y)| \leq C_\rho |x-y|^n + \rho(1 + |x|^n + |y|^n).$$

Take $0 < \delta < 1$ with $C_\rho \delta^n < \rho$. Then, we have for $|x-y| < \delta$,

$$|v(x) - v(y)| \leq \sup_{c \in \mathbf{A}_L} |V_c(x) - V_c(y)|$$
$$\leq \rho(2 + |x|^n + |y|^n)$$
$$< \Xi_\rho(x) := \rho(2^n + 2) f_n(x).$$

Let $\{S_i\}$ be a sequence of disjoint subsets of $\mathbf{R}$ such that

$$diam(S_i) < \delta \ \ and \ \ \cup_i S_i = \mathbf{R}.$$

For any i, we take $x^{(i)} \in S_i$ and $c^{(i)} \in \mathbf{A}_L$ such that

$$V_{c^{(i)}}(x^{(i)}) \leq \inf_{c \in \mathbf{A}_L} V_c(x^{(i)}) + \rho.$$

Define $c^\tau \in \mathbf{A}_L$ by

$$c_t^\tau = c_t 1_{\{t<\tau\}} + c_{t-\tau}^{(i)} 1_{\{x_\tau \in S_i\}} 1_{\{t \geq \tau\}}, \quad for \ \ x_\tau \in S_i.$$

Hence,

$$V_{c^{(i)}}(x_\tau) = V_{c^{(i)}}(x_\tau) - V_{c^{(i)}}(x^{(i)}) + V_{c^{(i)}}(x^{(i)})$$
$$\leq \Xi_\rho(x_\tau) + V_{c^{(i)}}(x^{(i)})$$
$$\leq \Xi_\rho(x_\tau) + \inf_{c \in \mathbf{A}_L} V_c(x^{(i)}) + \rho$$
$$= \Xi_\rho(x_\tau) + v(x^{(i)}) + \rho$$
$$\leq 2\Xi_\rho(x_\tau) + v(x_\tau) + \rho$$

Now, by the definition of $v^r(x)$, we can find $c \in \mathbf{A}_L$ such that

$$v^r(x) + \rho \geq E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] + e^{-\beta\tau} v(x_\tau) \right]$$

Thus, using the formal Markov property [5], we have

$$v^r(x) + \rho \geq \sum_i E\Big[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau}(V_{c^{(i)}}(x_\tau) -$$
$$2\Xi_\rho(x_\tau) - \rho) : \ x_\tau \in S_i \Big]$$
$$= E\Big[\int_0^\tau e^{-\beta t}\{h(x_t^\tau) + |c_t^\tau|^2\}dt + \int_\tau^\infty e^{-\beta t}\{h(x_t^\tau) + |c_t^\tau|^2\}dt \,|\, \mathbf{F}_\tau \Big]$$
$$- 2E[e^{-\beta\tau}\Xi_\rho(x_\tau)] - \rho$$
$$\geq v(x) - 2\Xi_\rho(x) - \rho,$$

where x_t^τ is the response to c_t^τ with $x_0^\tau = x_\tau$. Letting $\rho \to 0$, we deduce $v^r(x) \geq v(x)$, which completes the proof.

Theorem 2

We assume (3), (4), (5) and (9). Then v_L is a viscosity solution of (20). Furthermore, v_L converges locally uniformly to a viscosity solution $u \in C(\mathbf{R})$ of (6) satisfying (3), (4) as $L \to \infty$.

Proof

We note that (11) gives $E[\int_0^g |x_t|^n dt] \leq e^{\beta_0 g} g f_n(x) \ \ for \ \ g > 0$, and

$$E\left[\sup_{0\le s\le g} |x_s - x|^n\right] \le 3^n E[(\int_0^g |Ax_t|dt)^n + (\int_0^g |c_t|dt)^n + (\sup_{0\le s\le g} |\int_0^s \sigma x_t dw_t|)^n]$$

$$\le 3^n \Big(|A|^n g^{n-1} E[\int_0^g |x_t|^n dt] + g^n L^n + Kg^{n/2-1} E[\int_0^g |x_t|^n dt]\Big)$$

for some constant $K > 0$. Hence, taking $n = 2$, we have

$$\limsup_{g\to 0} \sup_{c\in \mathbf{A}_L} E[\sup_{0\le s\le g} |x_s - x|^2] = 0.$$

Thus we can apply a standard result of viscosity solutions [3, Thm.3.1, p.220] to obtain the viscosity solution of

$$-\beta v_L + \frac{1}{2}\sigma^2 x^2 v_L'' + Axv_L' + \min_{|a|\le L}(a^2 + av_L') + h(x) = 0 \text{ in } \mathbf{R},$$

taking into account the uniform continuity of h on each compact interval. Since $v_L(x)$ is non-increasing, $v_L(x)$ converges to $u(x)$ as $L\to\infty$. By the convexity of v_L and Dini's theorem, we can observe the locally uniform convergence and the viscosity property of u [3]. Clearly, v_L fulfills (3) and (4). The proof is complete.

Classical solutions

Here we study the smoothness of the viscosity solution u of (6).

Theorem 3

We assume (3), (4), (5) and (9). Then we have

$$u \in C^2(\mathbf{R}\setminus\{0\}). \tag{20}$$

Proof

Step 1: By the convexity of u we recall a classical result of Alexandrov [5] to see that Lebesgue measure of $\mathbf{R}\setminus \mathbf{D}\cup\{0\} = 0$, where $\mathbf{D} = \lceil x\in\mathbf{R} : u \text{ is twice differentiable at } x\rceil$. By the definition of twice-differentiability, we have $(u'(x), u''(x)) \in \mathbf{J}^{+2}u(x) \cap \mathbf{J}^{-2}u(x)$ for all $x\in\mathbf{D}$, and hence

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + Axu' - \frac{(u')^2}{4} + h(x) = 0, \quad \forall x\in\mathbf{D}.$$

Let $d^+u(x)$ and $d^-u(x)$ denote the right- and left-hand derivatives respectively. Define $r^{\pm}(x)$ by

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 r^{\pm}(x) + Axd^{\pm}u(x) - . \frac{(d^{\pm}u(x))^2}{4} + h(x) = 0 \quad \forall x\in(\mathbf{R}\setminus\{\mathbf{0}\}) \tag{21}$$

Since $d^+u = d^-u = u'$ on $\mathbf{D}$, we have $r^+ = r^- = u''$ a.e. By definition, $d^+u(x)$ is right continuous, and so is $r^+(x)$. Hence it is easy to see that

$$u(y) - u(x) = \int_x^y d^+u(s)ds$$
$$d^+u(s) - d^+u(x) = \int_x^s r^+(t)dt, \quad s > x.$$

Thus we get

$$R(u;y) := u(y) - u(x) - d^+u(x)(y-x) - \frac{1}{2}r^+(x)|y-x|^2 \;/\, |y-x|^2$$
$$= \int_x^y d^+u(s) - d^+u(x) - r^+(x)(s-x) \; ds/|y-x|^2$$
$$= \int_x^y \int_x^s r^+(t) - r^+(x) \; dt \; ds/|y-x|^2 \to 0 \text{ as } y\downarrow x. \tag{22}$$

Step 2: We claim that $u(x)$ is differentiable at $x\in\mathbf{R}\setminus\mathbf{D}\cup\{0\} = 0$. It is well known in [1] that $\delta u(x) = \left| d^+\mathrm{u(x)}, \mathrm{d}^-\mathrm{u}(x)\right|$, for all $x\in(\mathbf{R}\setminus\{0\})$, where $\delta u(x)$ is the generalized gradient of u

at x. Suppose $d^+u(x) > d^-u(x)$. We set

$$\hat{p} = \xi d^+u(x) + (1-\xi)d^-u(x)$$
$$\hat{r} = \xi r^+(x) + (1-\xi)r^-(x), \quad 0<\xi<1.$$

If $\liminf_{y\to x} R(u;y) < 0$, then we can find a sequence $y_m \to x$ such that $\lim_{m\to\infty} R(u;y_m) < 0$. By (22), we may consider that $y_m \le y_{m+1} < x$ for every m, taking a subsequence if necessary. Hence

$$\lim_{m\to\infty} \frac{u(y_m) - u(x) - d^+u(x)(y_m - x)}{|y_m - x|} \le 0,$$

This leads to $d^+u(x) \le d^-u(x)$, which is a contradiction. Thus we have $(d^+u(x), r^+(x)) \in J^{2,-}u(x)$ and similarly, $(d^-u(x), r^-(x)) \in J^{2,-}u(x)$. By the convexity of $J^{2,-}u(x)$, we get $(\hat{p},\hat{r}) \in J^{2,-}u(x)$. Now we note that

$$(\hat{p})^2 < \xi(d^+u(x))^2 + (1-\xi)(d^-u(x))^2,$$

and hence by (21)

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 \hat{r} + Ax\hat{p} - \frac{(\hat{p})^2}{4} + h(x) > 0.$$

On the other hand, by the definition of viscosity solution

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 q + Axp - \frac{p^2}{4} + h(x) \le 0 \ \ \forall (p,q) \in J^{2,-}u(x),$$

which is a contradiction. Therefore we deduce that $\delta u(x)$ is a singleton, and so u is differentiable at x [1].

Step 3: We claim that u' is continuous on $(\mathbf{R}\setminus\{\mathbf{0}\})$. Let $x_m \to x$ and $p_m = u'(x_m) \to p$. Then we have by convexity $u(y) \ge u(x) + p(y-x)$, for all y. Hence we see that $p \in D^-u(x)$, where

$$D^-u(x) = \{p \in \mathbf{R} \ : \ \liminf_{y\to x}\{u(y) - u(x) - p(y-x)\}/|y-x| \ge 0\}.$$

Since $\delta u(x) = D^-u(x)$ and $\delta u(x)$ is a singleton, we deduce $p = u'(x)$ [1, prop.4.7, p.66].

Step 4: We set $w = u'$. Since

$$-\beta w(x_m) + \frac{1}{2}\sigma^2 {x_m}^2 w'(x_m) + Ax_m w(x_m) - \frac{(w(x_m))^2}{4} + h(x_m) = 0 \ \ x_m \in \mathbf{D},$$

the sequence $\{w'(x_m)\}$ converges uniquely as $x_m \to x \in \mathbf{R}\setminus \mathbf{D} \cup \{0\}$, and w is Lipschitz near x by monotonicity. Hence, we have a well-known result in non-smooth analysis that $\delta w(x)$ coincides with the convex hull of the set

$$\mathbf{D}^* w(x) = \left| q \in \mathbf{R} \ : \ q = \lim_{m\to\infty} w'(x_m), \ \ x_m \in \mathbf{D} \to x \right|$$

Then

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 q + Axw(x) - \frac{(w'(x))^2}{4} + h(x) = 0 \ \ \forall q \in \delta w(x).$$

Hence we observe that $\delta w(x)$ is a singleton, and then $w(x)$ is differentiable at x. The continuity of $w'(x)$ follows immediately. Thus we conclude that $w \in C^1(\mathbf{R}\setminus\{0\})$ and $(\mathbf{R}\setminus \mathbf{D} \cup \{0\})$ is empty. The proof is complete.

Theorem 4

We make the assumptions of Theorem 3. Further we assume that

$$h(x)/x^2 \to \hat{h} \in \mathbf{R}_+ \text{ as } x \to 0. \tag{23}$$

Then we have

$$u \in C^1(\mathbf{R}) \cap C^2(\mathbf{R} \setminus \{0\}). \quad (24)$$

In addition, if $\hat{h} = 0$, then

$$u \in C^2(\mathbf{R}). \quad (25)$$

Proof

To prove (24), it suffices to show that u has the following property:

$$u'(x) = o(1) \text{ as } x \to 0. \quad (26)$$

By (23), there exists $\lambda > 0$, for any $\varepsilon > 0$ such that $h(x) \le (\hat{h} + \varepsilon)x^2$ for $|x| < \lambda$, and hence, by (2b)

$$h(x) \le (\hat{h} + \varepsilon)x^2 + C(1/\lambda^m + 1)|x|^m, \quad \forall x \in \mathbf{R}. \quad (27)$$

Note that $u(x) \le E[\int_0^\infty e^{-\beta t} h(x_t^0) dt]$. Then we have by (11)

$$u'(x) = 0(x^2) \text{ as } x \to 0. \quad (28)$$

Now, by convexity

$$u(y) \ge u(x) + u'(x)(y - x), x \ne 0.$$

Substituting $y = 2x$, and $y = 0$ we get $u(2x) \ge u(x) + u'(x)x$ and $u(x) - u'(x)x \le u(0) = 0$ by (28). Hence

$$\frac{u(2x)}{x^2} \ge \frac{u'(x)}{x} \ge \frac{u(x)}{x^2}, \quad (29)$$

which implies (26).

Finally, suppose $\hat{h} = 0$. Then, by virtue of (27), we have $u(x) = o(x^2)$ as $x \to 0$. Moreover, by (29), $u'(x) = o(x)$ as $x \to 0$. Dividing (6) by x^2 and passing to the limit, we get $u''(0) = 0$, which implies (25).

An application to control theory

We shall study the stochastic control problem (1) over the class A_{ad} of admissible controls, subject to (2), where $\mathsf{A}_{ad} = \{c = (c_t) \in \mathsf{A}$: $\lim_{T\to\infty} E[e^{-\beta T} |x_T|^n] = 0$ for the response x_t to $c_t\}$. We consider the stochastic differential equation

$$dx_t^* = [Ax_t^* - u'(x_t^*)/2]dt + \sigma x_t^* dw_t, \quad x_0^* = x. \quad (30)$$

Theorem 5

We assume (3), (4), (5), (9) and (23). Then the optimal control c_t^* is given by

$$c_t^* = -u'(x_t^*)/2. \quad (31)$$

Proof

Since u' is continuous, (30) admits a weak solution x_t^* up to explosion time $\sigma = \inf\{t : |x_t^*| = \infty\}$ [6]. Taking into account $xu'(x) \ge 0$, we can show $(x_t^*)^2 \le (x_t^0)^2$ by the comparison theorem. Hence $\sigma = \infty$. By the monotonicity of $u'(x)$, the uniqueness of (30) holds. Thus we conclude that (30) has a unique strong solution (x_t^*). It follows from (12) that

$$E[e^{-\beta T}(1 + |x_T^*|^n)] \le e^{-(\beta - \beta_0)T}$$
$$E[e^{-\beta_0 T} f_n(x_T^0)] \to 0 \text{ as } T \to \infty,$$

where x_t^0 is a unique solution given by (16). So $(c_t^*) \in \mathsf{A}_{ad}$. Since u satisfies (4), we see by (29) and (11) that

$$E[\int_0^T e^{-2\beta t}(x_t^* u'(x_t^*))^2 dt] \le E[\int_0^T e^{-2\beta t} u(2x_t^*)^2 dt]$$
$$\le CE[\int_0^T e^{-2\beta t}(1 + |x_t^*|^{2n}) dt]$$
$$\le CE[\int_0^T e^{-2\beta t} f_{2n}(x_t^0) dt] < \infty,$$

and hence $\int_0^t e^{-\beta s}\sigma x_s^* u'(x_s^*)dw_s$ is a martingale. Then we apply Ito's formula for convex functions [7, p.219] to obtain

$$E[e^{-\beta T}u(x_T^*)] = u(x) +$$

$$E\left[\int_0^T e^{-\beta t}\left(-\beta u + Axu' + c_t^* u' + \frac{1}{2}\sigma^2 x^2 u''\right)|_{x=x_t^*}\, dt\right]$$

$$= u(x) - E[\int_0^T e^{-\beta t}\{h(x_t^*) + |c_t^*|^2\}dt].$$

Passing to the limit, we have $J(c^*) = u(x)$. By the same calculation as above, we can see that

$$E[e^{-\beta T\wedge\tau_n}u(x_{T\wedge\tau_n})] \geq u(x) - E[\int_0^{T\wedge\tau_n} e^{-\beta t}\{h(x_t) + |c_t|^2\}dt],$$

where $\{\tau_n\}$ is a sequence of localizing stopping times for the local martingale. Letting $\tau_n \to \infty$ and then $T \to \infty$, we obtain $u(x) \leq J(c)$ for all $c \in \mathbf{A}_{ad}$. The proof is complete.

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DETERMINATION OF CLOBAZAM BY VISIBLE SPECTROPHOTOMETRY IN PURE AND PHARMACEUTICAL PREPARATIONS

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Abstract: A sensitive and simple spectrophotometric method for the estimation of clobazam in both pure form and its pharmaceutical formulation is described. The method is based on the interaction of clobazam with dichloronitrobenzene in alkaline medium. Absorbance of the resulting orange complex is measured at 450 nm and is stable for more than 24 hours. The reaction obeys Beer's law from 0.05 – 1.5 mg/10 mL of clobazam. Molar absorptivity is 0.5510 x 10^4 mol^{-1} cm^{-1} and the relative standard deviation 0.96 %. The quantitative assessment of tolerable amounts of other drugs not interfering have also been studied.

Keywords: Clobazam, dichloronitrobenzene and spectrophotometry.

Introduction

Clobazam is a benzodiazepine (Fig. 1) with acticonvulsant and acute-anxiety properties. It is effective against a wide variety of epileptic seizures. The most frequent side effects include drowsiness, hangover effects, dizziness and lightheadedness. Less frequent adverse reactions include weight gain, orthostatic hypotension, syncope, headache, dry mouth and incoordination [1].

Different analytical techniques have been employed for the estimation of clobazam. In HPLC [2,3], HPLC-photodiode array [4,5], solid-phase column extraction is performed to cleanup blood samples before running the analytical HPLC system [2]. After washing with potassium phosphate buffer, the retained substances were back flashed into

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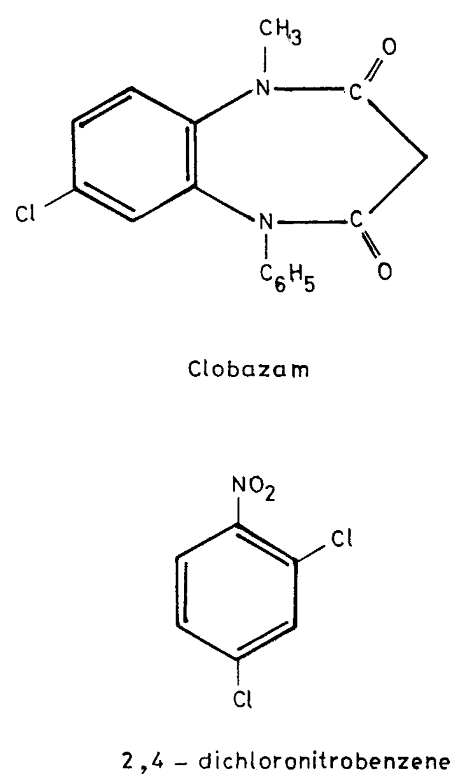

Figure 1. Structural formulas of clobazam and 2, 4 – dichloronitrobenzene.

reversed- phase column with a mobile phase of acetonitrile-phosphate-buffer-diethylamine [3]. Also liquid-liquid extraction with n-hexane:ethylacetate [4], solid-phase extraction and a photodiode array detection is employed [5].

Similarly, in gas chromatography samples need extraction [6,7] and have given negative result in 284 cases [8]. In micellar electrokinetic chromatography a number of solvents [9] and in capillary electrophoresis large injection volume of sample are involved [10]. While derivative spectrophotometric method can not be directly used toxicologically and its application to biological fluids requires prior chromatographic separation and use of diode array detection [11]. Long and tedious procedures are involved in LC/MS [12] and HPLC/GC/MS [13].

During the present study it was found that clobazam reacts with dichloronitrobenzene to give orange color having maximum absorbance at 450 nm. The reaction is selective for clobazam with 0.1 mg/ml as visual limit of identification. The method is simple, accurate, precise and sensitive. Also studied was percentage of non-interfering drugs.

Materials and Methods

Apparatus and reagents

Hitachi u-1100 spectrophotometer with 1 cm silica cells was used to measure the absorbance. Analytical grade chemicals and doubly distilled water were used. Standard solution of clobazam (1 mg/ml, w/v) was prepared by dissolving 100 mg of the substance in 30 ml alcohol (BDH) and volume was made up to 100 ml with distilled water to get a stock solution which was diluted further as required. Dichloronitrobenzene (BDH) solution (2.0 %, w/v) was prepared by dissolving 2 g of dichloronitrobenzene in 100 ml of ethyl alcohol.

General Procedures

To an aliquot of clobazam containing 0.05 to 1.5 mg/10 ml was added 0.1 ml of 2 N sodium hydroxide, 1 ml of 2 % dichloronitrobenzene and the contents were heated for 60 seconds in a water bath at 90°C. The contents were cooled at room temperature and the volume was made up to 10 ml with ethyl alcohol. The resulting color was measured at 450 nm, employing all reagents except clobazam as blank. The experiment was repeated with different volumes of standard clobazam solution and a calibration curve was prepared (Fig. 2). The color reaction obeys Beer's Law from 0.05 to 1.5 mg/10 ml of clobazam.

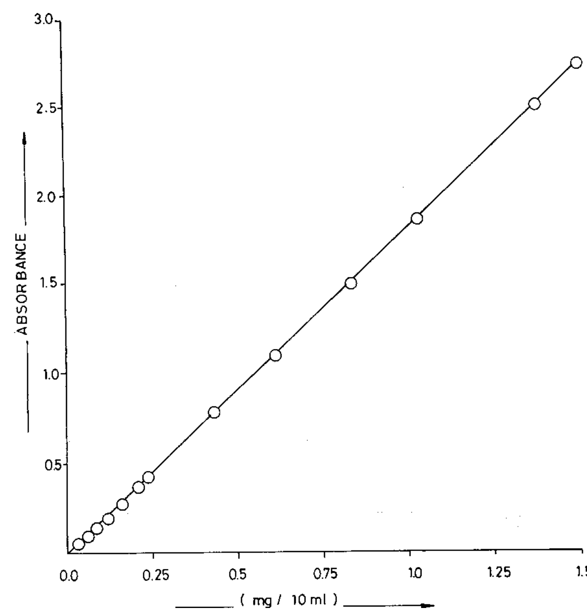

Figure 2. Calibration curve of clobazam with dichloronitrobenzene.

Procedure for studying the interfering compounds

To an aliquot containing 1 mg/ml of clobazam different amounts of various organic compounds (1 mg/ml) (w/v) were added individually as long as the solution showed the same (+0.01) absorbance as that of pure clobazam solution without the addition of the interfering organic compound, under experimental conditions, as described in the general procedure. The value was calculated as the percentage of organic compound with respect to the amount of clobazam.

Procedure for the determination of clobazam in pharmaceutical preparations

Tablets containing 10 mg of clobazam were powdered, weighed, dissolved in 30 ml ethyl alcohol and filtered. The filtrate was diluted with distilled water to get a 1 mg/ml solution of clobazam. An aliquot containing 0.05 to 1.5 mg/10 ml was taken, the procedure was followed as described above and the absorbance was measured at 450 nm. The quantity per tablet was calculated from the standard calibration curve.

Results and Discussion

Clobazam reacts with dichloronitrobenzene when heated for 60s at 90°C to give an orange coloured complex, the absorption spectra of which under the optimum condition lies at 450 nm (Fig. 3). Hence all measurements for further studies were carried out at this wavelength.

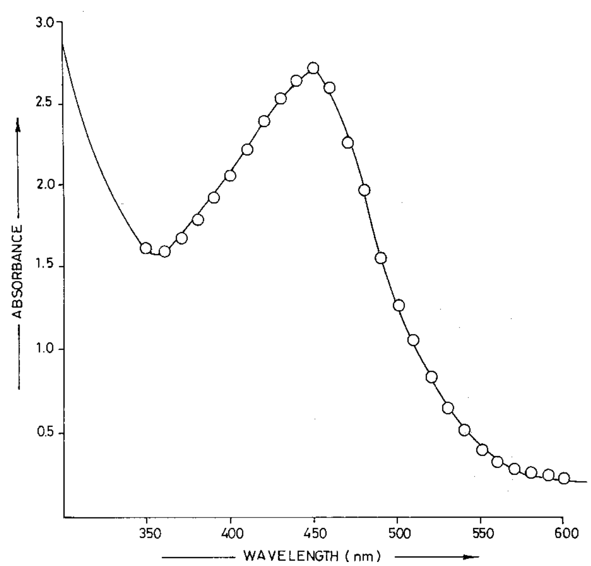

Figure 3. Absorption spectra of clobazam with dichloronitrobenzene.

Dichloronitrobenzene was used as color producing reagent. It was found that 20 mg/10 ml of dichloronitrobenzene gave maximum color (Fig. 4). Above and below this concentration the color intensity diminished and became unstable. Effect of pH is shown in Fig. 5. Maximum color intensity was obtained at pH 11.9. The pH was maintained by addition of 0.1 ml of 2 N sodium hydroxide.

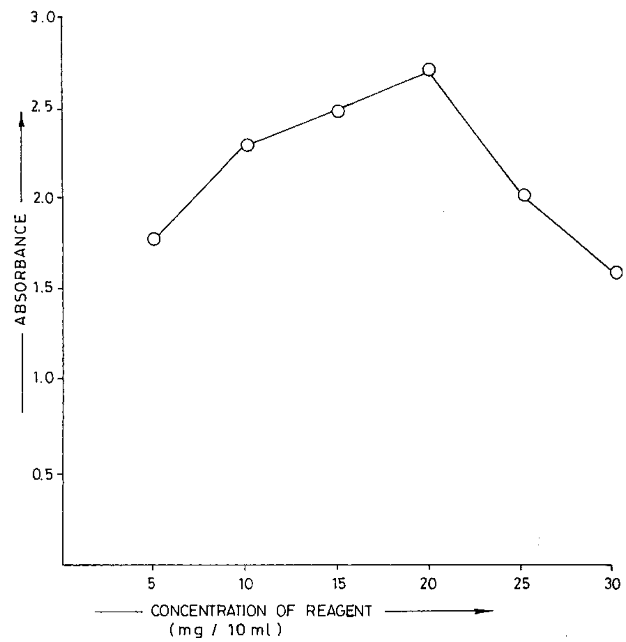

Figure 4. Effect of dichloronitrobenzene.

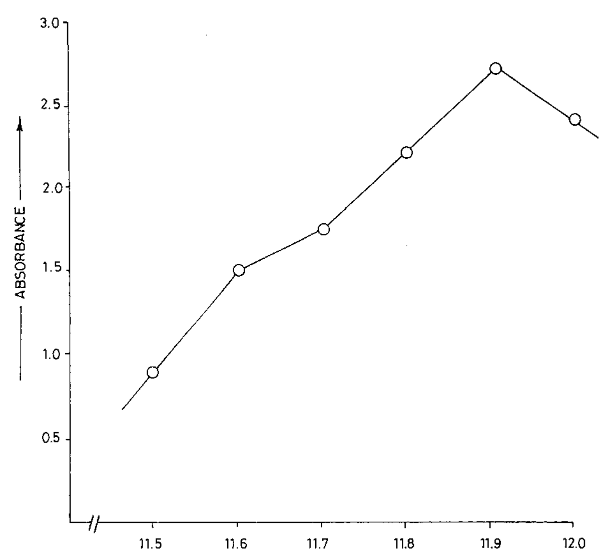

Figure 5. Effect of pH.

The effect of temperature is shown in Fig. 6. It was found that with the rise of temperature the color intensity increased and was maximum and stable at 90°C. The color did not develop at room temperature. The absorbance of the developed color remained stable for more than 24h. A water bath was used to carry out the temperature studies. After production of the color, the contents of the test tube were cooled at room temperature prior to dilution with ethyl alcohol and measurement of the absorbance. The effect of heating time on color intensity is shown in Fig. 7. It was found that heating for 60 seconds at 90°C gave maximum color. Above and below these timings the color intensity was less and also unstable.

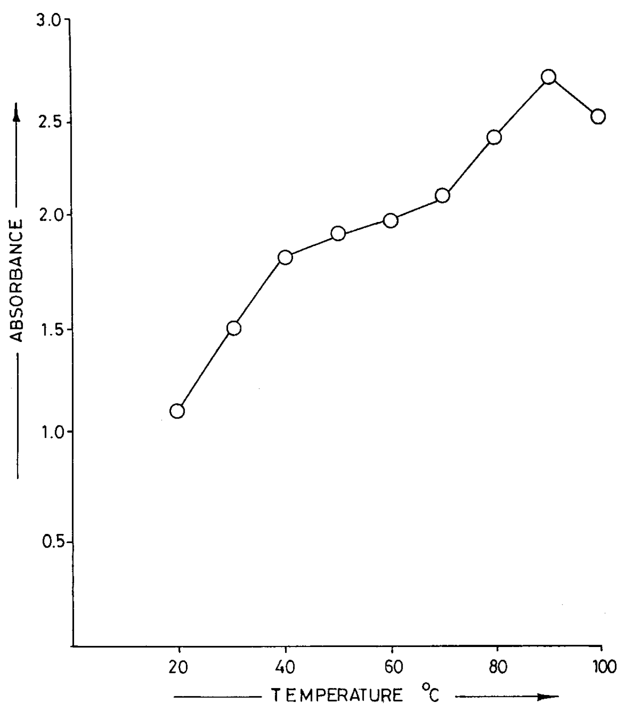

Figure 6. Effect of temperature.

Different organic solvents such as dichloromethane, benzene, hexane, chloroform, methyl ethyl ketone, tetrahydrofuran, carbontetrachloride and trichlorobenzene, were tested for color extraction and stability. Since none were effective, no solvent was employed except for ethyl alcohol which was used for dilution and stability of the colored complex.

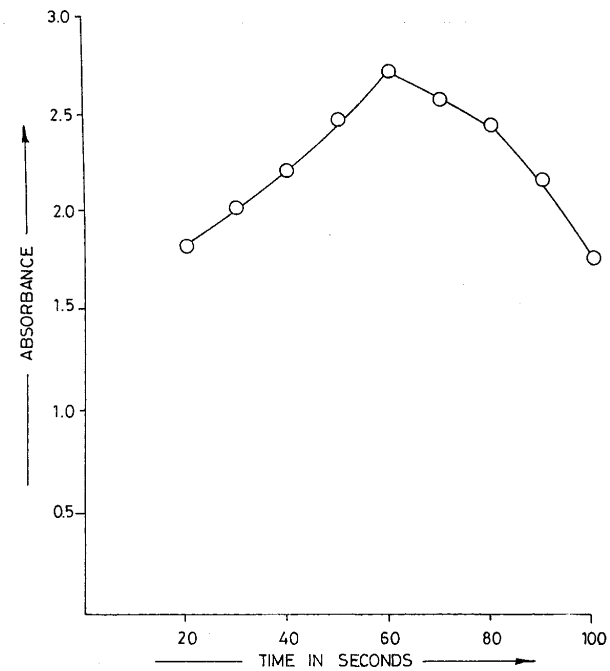

Figure 7. Effect of heating time.

The sudden fading of the color of the product formed by the interaction of 2,4- dichloronitrobenzene and chlobazam in dilute acidic medium indicates that true bonded structure is not formed. It also indicates the existence of an ionic (charge transfer) complex which is probably formed by the interaction of carbonyl oxygen of clobazam and nitrogen of nitro group of 2,4-dichloronitrobenzene. The probable structure of the color product is shown in Fig. 8.

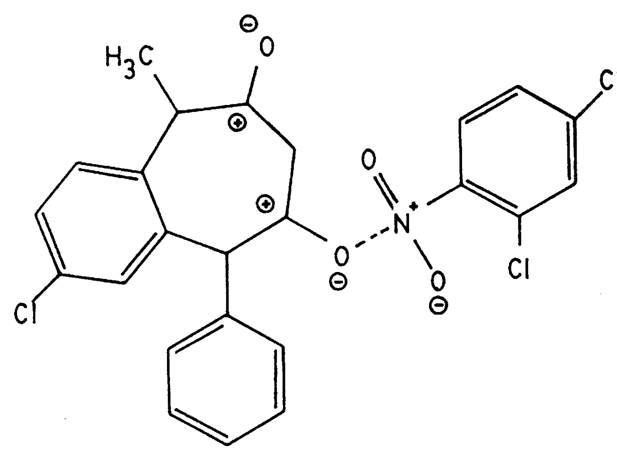

Figure 8. Probable structure of the coloured compound formed by the interaction of colbazam and 2, 4-dichloronitrobenzene.

The results of the determination of clobazam are shown in Tables 1 and 2, which reveal the sensitivity, validity and repeatability of the method. It is shown that the method is reasonably precise and accurate, as the amount taken for identical sample is known and the amount found by the above procedure does not exceed the relative standard deviation of 0.96 % (replicate of five independent-measurements, Table 1). The calibration graph is linear in the range of 0.05 to 1.5 mg/10 ml. The apparent molar absorptivity calculated was 0.5510 x 10^4 mol^{-1} cm^{-1}. The regression equation [15] was calculated by the method of least squares from nineteen points, each of which was the average of five determinations. The correlation between absorbance and concentration was 0.8259 in terms of correlation coefficient (r).

The quantitative assessment of different organic compounds (w/v) under the experimental conditions is given in Table 3. Various amounts of diverse interfering compounds having similar actions were added to a fixed amount of clobazam (1 mg/ml) and the recommended procedure for the spectrophotometric determination was followed.

Table 1. Determination of clobazam from pure solution.

Clobazam taken mg/10ml	Clobazam found * mg/10ml	Relative Standard Deviation %
0.100	0.102	0.98
0.150	0.151	0.66
0.200	0.203	0.50
0.300	0.290	0.39
0.500	0.504	0.31
1.000	1.042	0.10
1.200	1.210	0.08
1.500	1.515	0.06

*Every reading is an average of five independent measurements.

Table 2. Optical characteristics precision and accuracy of the proposed method.

Parameters	Values
λ_{max} (nm)	450
Molar absorptivity (mol^{-1} cm^{-1})	0.5510 x 10^4
Regression Equation (Y)* Slope (b)	0.897
Intercept (a)	0.078
Correlation coefficient (r)	0.826
Relative Standard Deviation (RSD%)**	0.96 %
%Range of Error (Confidence Limit) at 95% Confidence Level	9.02 + 0.024 %

*Y = a + bC where C is the concentration of analyte (mg/10 ml) and Y is the absorbance unit.
**Calculated from five determinations.

Table 3. Quantitative assessment of tolerable amount of other drugs.

Drugs	Maximum Amount Not Interfering* (%)
Aspirin	200
Chloroquine phosphate	100
Diclofenac sodium	100
Diazepam	200
Metamizol sodium	100
Mefenamic acid	200
Lorazepam	30
Pheniramine maleate	400
Dipotassium chlorazepate	300
Paracetamol	250
Ibuprofen	300
Promazine	100
Aldomet	100
Chloral Hydrate	350

*The value is the percentage of the drug with respect to 1mg/10ml of clobazam that causes +0.01 change in absorbance.

Table 4. Determination of clobazam from pharmaceutical preparations.

Drug	Trade Name	Pharmaceutical Preparation	Amount Present (Manufacturer's Specifications) (mg)	Amount Found*	Percentage Recovery(%)
Clobazam	Frisium (Aventis Pharma, Karachi, Pakistan)	Tablet	10	10.2	102

*Every reading is an average of five determinations.

The proposed method is successfully applied for the quality control of pure clobazam and in the pharmaceutical dosage form as shown in Table 4. Thus, the spectrophotometric method for determination of clobazam is reliable, simple, sensitive and reproducible. It is selective for clobazam. The method can be successfully applied for microdetermination of clobazam either in pure form or in pharmaceutical preparations. The colour reaction has 0.1 mg/10 ml as visual limit of quantization. The advantage of the present procedure is that it neither requires many reagents nor many solvents and has low RSD (0.96 %), whereas HPLC [3,4] procedures are long, tedious and expensive involving many reagents and solvents. The literature indicates that this color reaction has not been reported previously. A significant advantage of the spectrophotometric determination is its application to the determination of individual compounds. This aspect of spectrophotometric analysis is of major interest in analytical pharmacy, since it offers a distinct possibility of quality control in the assay of pharmaceutical dosage formulation.

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WAVELET TRANSFORM FOR WATERMARKS IN DIGITAL IMAGES

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Abstract: In this paper we study wavelet filters applied to watermarking in order to protect copyrights. Information is inserted in the transformed domain of the image. This transform is based on the analysis of the main components and wavelet transform. The watermarked image is reconstructed applying the inverse transform. We perform mathematical proofs over the image to demonstrate that the original image is slightly altered after the watermarking process. Finally, we simulate different attacks such as JPEG compression and adding noise to the watermarked image. We argue that this method is efficient based on robustness and security.

Keywords: Wavelets, watermarks, copyrights

Introduction

In these days, due to the increment of illegal copies and espionage in different communications media, digital watermarking is essential to protect copyrights in digital images [7]. This lead us to interchange information in a secure manner over insecure communication channels. Watermarking techniques slightly modify the original data, hence these are almost invisible [8].

In steganography, the main objective is to insert a message as a watermark inside a carrier image [1,9]. Watermarking technique requires the following properties in order to use it: legibility, security, invisibility and robustness. Legibility refers to the ability to detect the embedded information whenever it is required to extract it, security involves camouflaging the watermark in such a way as to make it unnoticeable for the rest of the people; for purposes of invisibility it is very important to select the carrier images and finally, robustness refers to the ability of the watermark to resist a number of attacks [3,4,10]. These attacks include digital image processing operations such as compression, geometric distortion and different kinds of noise.

Discrete Wavelet Transform (DWT) and the Discrete Cosine Transform (DCT) are the most popular domains for watermarks [11]. In general, the DWT produces images with more visible watermarks and with more storage capacity [12].

The first part of the paper focuses on the description of the DWT, the inverse DWT (which is used to reconstruct the image with an embedded watermark) and finally we explain the procedure to recover the embedded information. Then, we describe the different attacks that can alter the watermark, showing a table comparing the results from the tests by submitting the watermarked images to the different attacks.

Materials and Methods

Wavelet discret transform (DWT)

Signal x(n) is passed trough a bank of mirror filters in cuadrature [13]. The resulting signal of each filter is decimated by a factor of 2.

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Signal resolution, which accounts for detailed amount of information inside the signal, is modified by the filters and is scaled by the decimation operation. Decimation of a signal corresponds to a reduction of the sampling frequency or to the discarding of some of the samples of the signal. This process of filtering and decimation is known as sub-band coding as shown in Fig 1.

This procedure can be expressed as:

$$y_{high}[k] = \sum_n x[n] \cdot g[2k-n] \tag{1}$$

$$y_{low}[k] = \sum_n x[n] \cdot h[2k-n] \tag{2}$$

where $y_{high}[k]$ and $y_{low}[k]$ are the output of the high pass filters and low pass filters respectively, after the decimation by 2.

This is the operation mode of the DWT. This procedure analyzes the signal in different frequency bands with different resolutions through decomposition of the different components of the signal from high energy to low energy. This decomposition of the signal in different frequency bands is accomplished through successive filtering of the signal in the time domain as show in Fig 1. The original sequence x[n] is passed through a high pass filter g[n] and a low pass filter h[n].

Sub-band coding can be repeated to achieve more decomposition. Each level of filtering and decimation will result in half the number of samples (and hence, half the number of time resolution.) Depending of the chosen wavelet, coefficients of g and h will change.

There are a number of different wavelet transforms. However, we have only worked with the Haar wavelet transform, which after a number of tests was the more appropriate for this particular application and indicates application of the transform only at the first level. [2,5,6].

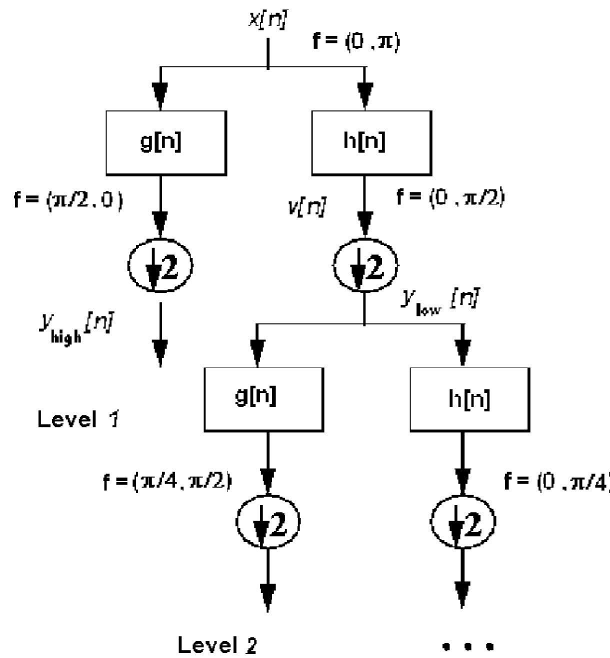

Figure 1. Sub-band coding algorithm .

DWT in two dimensions

A discrete image X is a matrix with M rows and N columns of real numbers, where M and N have to be even integers:

$$x = \begin{pmatrix} x_{1,M} & x_{2,M} & \cdots & x_{N,M} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,2} & x_{2,2} & \cdots & x_{N,2} \\ x_{1,1} & x_{2,1} & \cdots & x_{N,1} \end{pmatrix} \tag{3}$$

The wavelet transform in two dimensions is obtained with the same equations as for one dimension; performing the next steps:

A. Applying the wavelet transform to each row X, which will produce a new matrix.

B. Applying to this new matrix obtained in step A the wavelet transform again, but this time to each column.

This results in four sub-images of M/2 rows and N/2 columns:

$$f \rightarrow \begin{pmatrix} h^1 & | & d^1 \\ - & & - \\ a^1 & | & v^1 \end{pmatrix} \qquad (4)$$

a^1 is calculated with the average of the rows followed by the average of the columns, resulting in this sub-image; a compression of the original. It contains low frequency components.

h^1 is calculated with the average of the rows and the difference of the columns. Here the horizontal details of the image are preserved and contain middle-low frequency components.

v^1 is similar to h, except that the vertical and horizontal components are interchanged. This sub-image contains the vertical details, conserving components of middle-low frequency components.

Finally, d^1 contains the diagonal details and it is calculated as the difference of the columns and rows and contains the high frequency components. [2,5,6].

DWT applied to watermarking

Once we have sub-matrices a^1, h^1, v^1 and d^1, matrix a^1 is kept intact since it contains the low frequency components. If it is altered it could have a big impact in the image which is not recommendable. Matrices v^1 and d^1 are not used either to embed a watermark since they contain middle high and high frequencies, respectively, being the most vulnerable in case of an attack. This only leaves matrix h^1, which contains the middle low frequencies; this makes it the must robust part to attacks after low frequencies, but the difference is that changes in this matrix will not be perceptible.

The watermark is inserted following the next procedure. The first pairs of the components of the matrix are compared. If the first is higher than the second one it is considered as a "1", otherwise it will be considered as "0". Then the next pair of values are compared and the same insertion criteria are used until the whole matrix is compared. This is also the general procedure for a steganographic application.

For watermarking we have make several tests with different images. We have confirmed that by distributing the information among the luminance and crominance matrices it is not robust against compression. We have also proved that the higher percentage of information is found in the luminance matriz. This means that the crominance matrices will be the more affected in case of compression.

By storing information only in the middle low frequencies of sub-matrix h^1 of luminance we obtained satisfactory results.

Attacks

There are various types of attacks to try to eliminate a watermark. It is important to notice that for an attack to be efficient it should eliminate the watermark without modifying the image visibly.

Types of attacks

A. *Compression*. When a commercial method of compression is applied to the image to eliminate the watermark, and then it is returned to the original format.

B. *Geometric Distortion*. Consists in summing to each pixel of image a small value to modify it completely without noticing visually.

C. *Noise*. There are different types of noise that can be summed to the image to alter it, such as:

Multiplicative Noise: it uses the next equation $g = f + n * f$ to sum the noise to the image, where f is the image and n is a random variable uniformly distributed. The range in which n takes values depends on the variance. This noise softens the image in a uniform manner, hence for a small variance the effect on the image is also small.

Impulsive Noise: it adds random values to some of the pixels of the image, the amount of pixels that are affected is related to the variance. The elements altered by this noise are very noticeable in the image.

Gaussian Noise: it adds noise normally distributed (as shown in Table 1). This noise is more aggressive than the rest, since it distorts the whole image, making it very susceptible.

Table 1. Noise Models.

Type	PDF	Mean and Variance
Impulsive	$p_z(z)=\begin{cases}P_a & z=a\\ P_b & z=b\\ 0 & \text{otherwise}\end{cases}$ $b>a$	$m=aP_a+bP_b$ $\sigma^2=(a-m)^2P_a+(b-m)^2P_b$
Gaussian	$p_z(z)=\frac{1}{\sqrt{2\pi}\,b}e^{-(z-a)^2/2b^2}$ $-\infty<z<\infty$	$m=a$ $\sigma^2=b^2$

Results and Discussion

Implementation and tests

First we show, graphically and mathematically, how the original image changes with respect to the image with the embedded watermark, then it can be seen how the image is affected with the watermark with each attack. The Figures only show the test in one image due to the extension it occupies, but a Table is annexed with the mathematical prove of five different images. All tests were done by introducing information at its maximum capacity [12].

Figs. 2, 3 and 4 show 3500 samples taken to each matrix R, G and B. The one in red corresponds to the original image and the one in blue shows the image with the watermark.

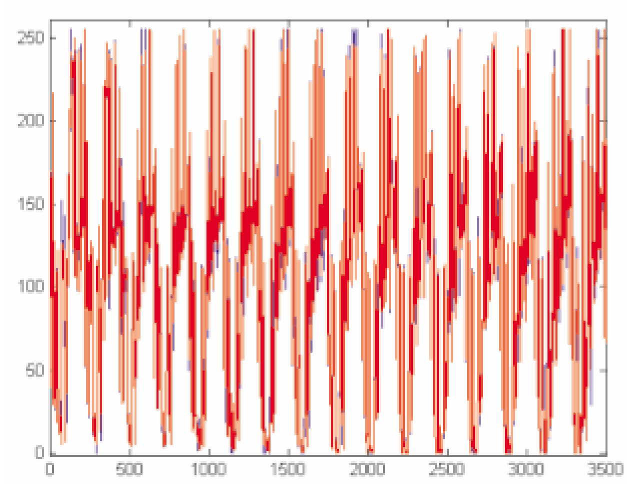

Figure 2. Graphic of 3500 samples of matrix R, in red the original image and in blue the watermarked image.

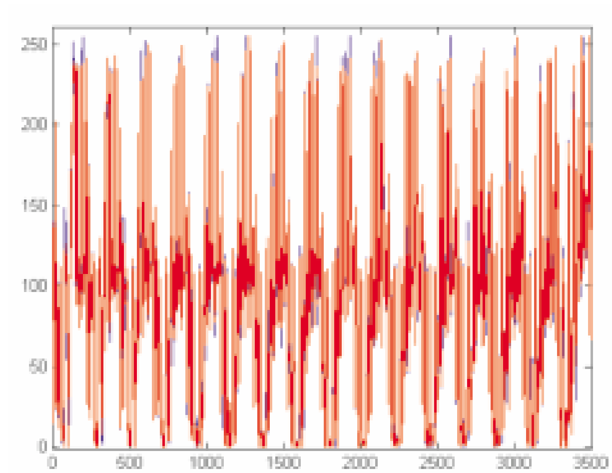

Figure 3. Graphic of 3500 samples of matrix G, in red the original image and in blue the watermarked image.

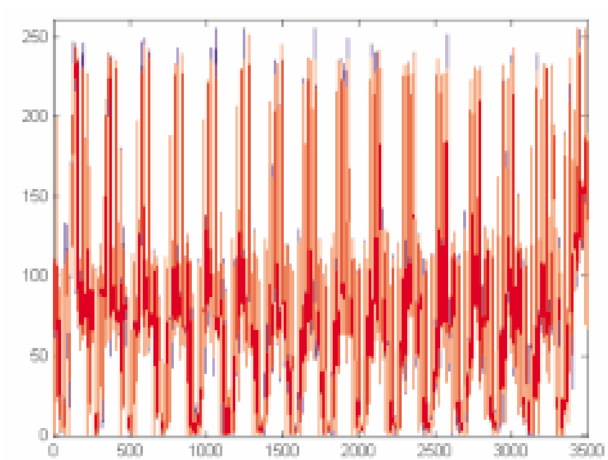

Figure 4. Graphic of 3500 samples of matrix B, in red the original image and in blue the watermarked image.

In the above three Figures, we can see the changes suffered by the watermark in each plane, although when recovering the image, these changes are not noticeable.

Figs. 5, 6 and 7 show the same 3500 samples taken in each matrix R, G and B, but now in red we have the image with the watermark and in blue the same image with multiplicative noise with variance of 0.001.

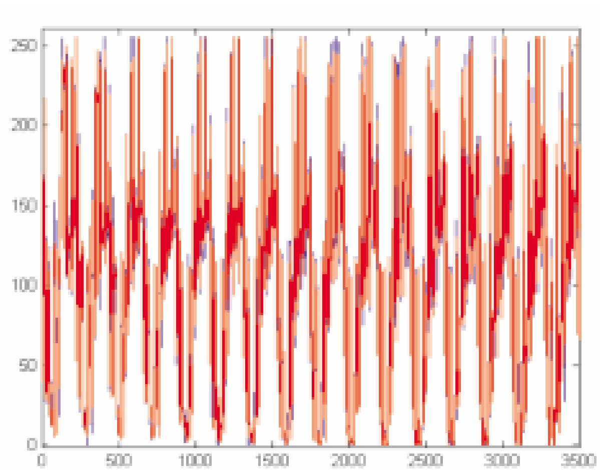

Figure 5. Graphic of 3500 samples of matrix R, in red the watermarked image and in blue the watermarked image with multiplicative noise.

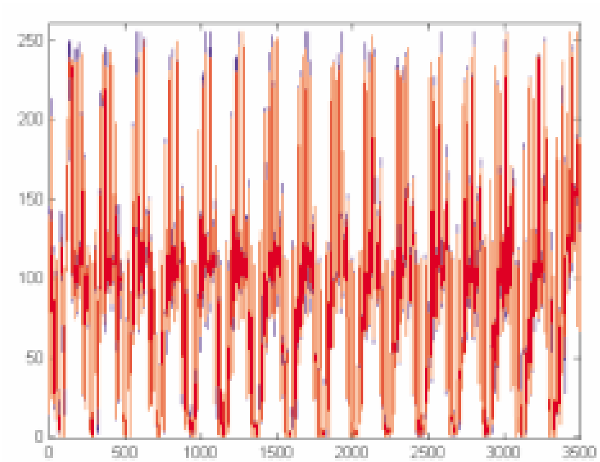

Figure 6. Graphic of 3500 samples of matrix G, in red the watermarked image and in blue the watermarked image with multiplicative noise.

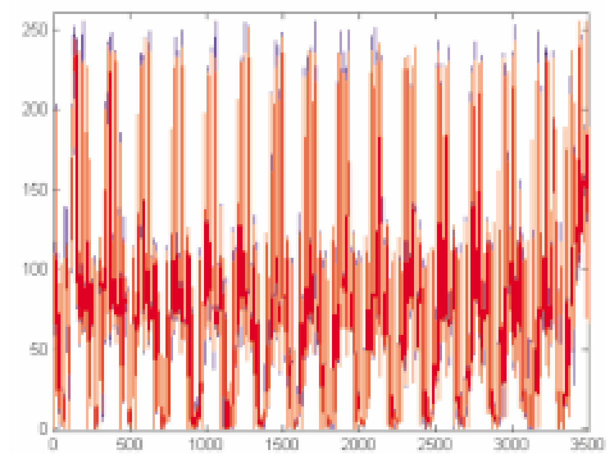

Figure 7. Graphic of 3500 samples of matrix B, in red the watermarked image and in blue the watermarked image with multiplicative noise.

Multiplicative noise softens the image and with small variance, as the one used here of 0.001, the alteration is almost not perceptible, and the watermark resisted 100% in different images.

Figs. 8, 9 and 10 show the same samples selected in the previous graphs. In red we have the watermarked image and in blue the same image after adding the impulsive noise with variance of 0.001.

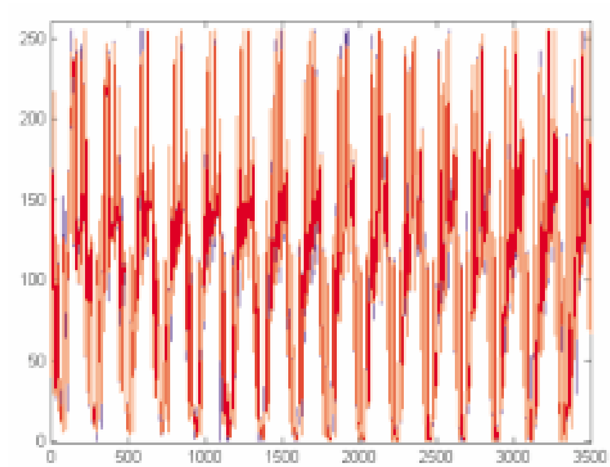

Figure 8. Graphic of 3500 samples of matrix R, in red the watermarked image and in blue the watermarked image with impulsive noise.

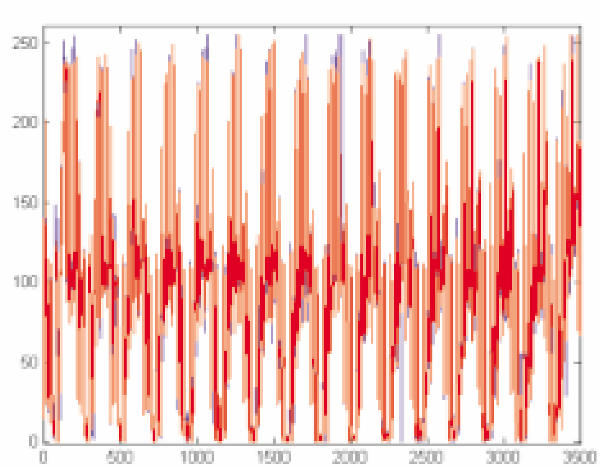

Figure 9. Graphic of 3500 samples of matrix G, in red the watermarked image and in blue the watermarked image with impulsive noise.

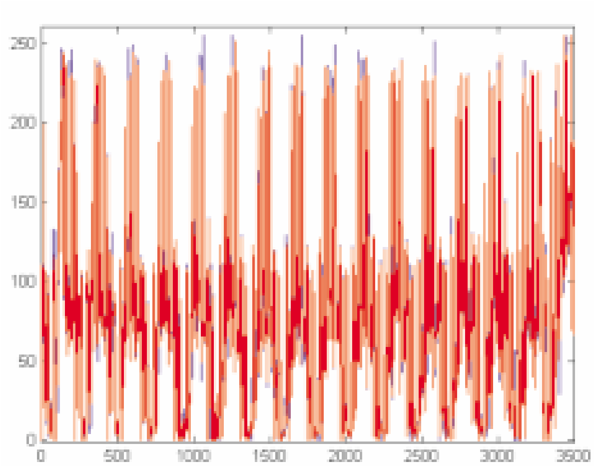

Fig ure 10. Graphic of 3500 samples of matrix B, in red the watermarked image and in blue the watermarked image with impulsive noise.

Adding impulsive noise to the image visually is very notorious, hence it is a very perceptible attack. After several tests with different images, 12% of the embedded information was lost.

In Figs. 11, 12 and 13 we have the same samples taken in each matrix R, G and B, where the graph in red shows the watermarked image and in blue the same image with noise but now Gaussian with variance of 0.001.

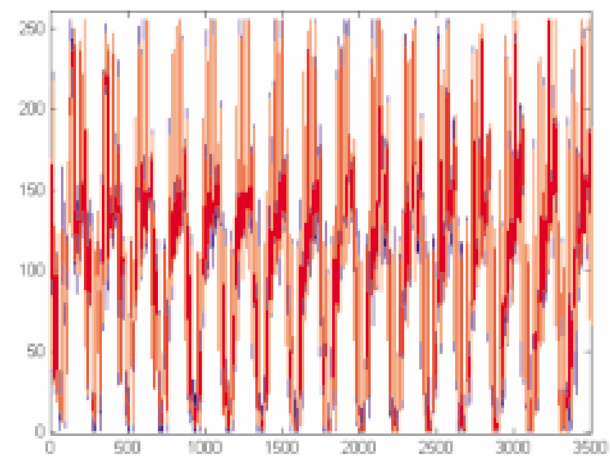

Figure 11. Graphic of 3500 samples of matrix R, in red the watermarked image and in blue the watermarked image with Gaussian noise.

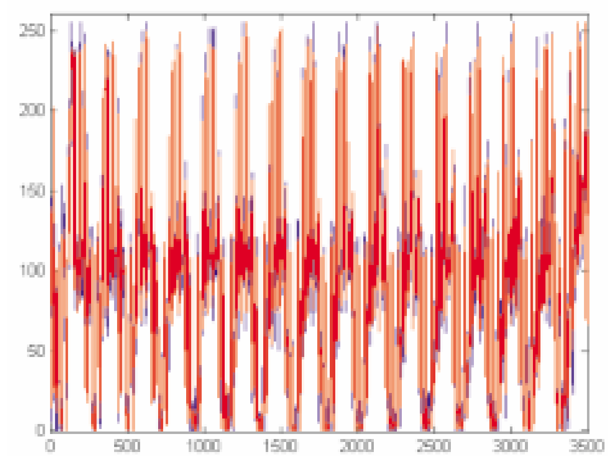

Figure 12. Graphic of 3500 samples of matrix G, in red the watermarked image and in blue the watermarked image with Gaussian noise.

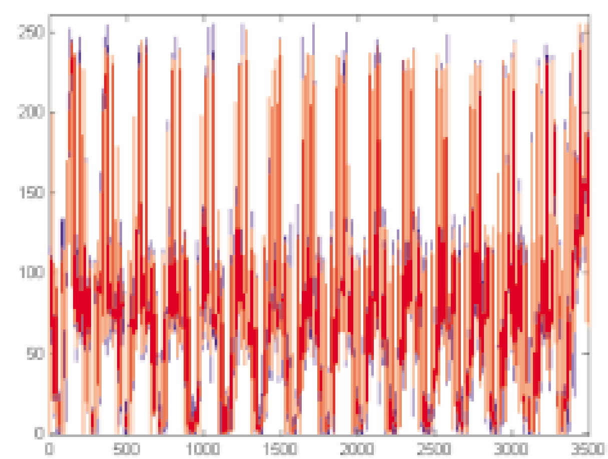

Figure 13. Graphic of 3500 samples of matrix B, in red the watermarked image and in blue the watermarked image with Gaussian noise.

As in the previous examples, the tests correspond to a variance of 0.001, losing an average of 16% of the watermark.

Gaussian noise and impulsive noise affect the image in different manner but they are much more notorious than the multiplicative noise, which is reflected in these Figures.

For the geometric distortion, we have done several tests using shifting in the range of 1 to 80. It is important to mention that using a shifting higher than 15 will produce a visible modification on the image. The purpose of using higher values of shifting in this work is to prove that the watermark always resists the attack regardless of the value added, since the inserted information is embedded comparing pairs of values, and the added value does not alter this relation. The histogram shown in Figure 14 shows the impact on an image caused by applying a shifting of 50.

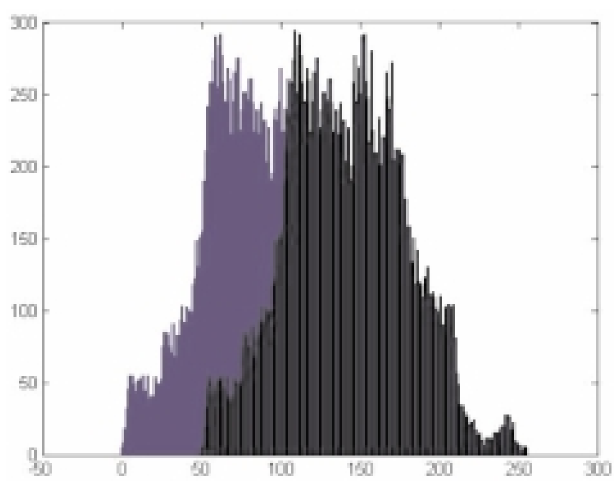

Figure 14. In blue the histogram of the watermarked image and in black the watermarked image with shifting of 50.

Compression was done with commercial software with formats BMP, JPEG and Zip. This software offers a compression rate ranging from 50% to 70%. However, if the watermark is embedded in the zone that we propose in this paper, satisfactory result is achieved when the watermark is recovered.

The mathematical analysis was done with the correlation index, energy percentage recovered and PSNR (Peak Signal to Noise Ratio). The results are shown in Tables 2, 3, 4, 5 and 6.

We calculated the correlation index between the original image and the modified image as follows:

$$\rho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}} \qquad l = 0, \pm 1, \pm 2 \ldots \tag{5}$$

where:

$\rho_{xy}[l]$ Correlation index
$r_{xy}[l]$ Cross correlation between the original image and the modified image.
$r_{xx}[0]$ Autocorrelation of the original image.
$r_{yy}[0]$ Autocorrelation of the modified image.

We calculated the PSNR as follows:

$$PSNR(dB) = 10\log_{10}\left(\frac{XY \max P_{x,y}^2}{\sum_{x,y}\left(P_{x,y} - \tilde{p}_{x,y}\right)^2}\right) \tag{6}$$

where:

X and Y are the number of rows and columns, respectively.
$P_{x,y}$ represents a pixel, whose coordinates are (x,y) in the original image.
$\tilde{P}_{x,y}$ represents a pixel, whose coordinates are (x,y) in the watermarked image.

Table 2. Matrix R correlation.

	Original Image	Multipli-cative Noise	Impulsive Noise	Gaussian Noise	Compre-ssion Noise
Image 1 water-marked	0.9999802	0.9995113	0.9990186	0.9971078	0.9998389
Image 2 water-marked	0.9999452	0.9995288	0.9974101	0.9936245	0.9997492
Image 3 water-marked	0.9999680	0.9994625	0.9993945	0.9985986	0.9998637
Image 4 water-marked	0.9999527	0.9995152	0.9995084	0.9983921	0.9999640
Image 5 water-marked	0.9998189	0.9994995	0.9994154	0.9981030	0.9999698

Table 3. Matrix G correlation.

	Original Image	Multipli-cative Noise	Impulsive Noise	Gaussian Noise	Compre-ssion Noise
Image 1 water-marked	0.9999890	0.9995032	0.9992693	0.9975235	0.9999630
Image 2 water-marked	0.9999852	0.9994990	0.9967019	0.9912684	0.9998664
Image 3 water-marked	0.9999759	0.9995145	0.9988478	0.9967613	0.9998922
Image 4 water-marked	0.9999742	0.9995039	0.9995321	0.9983535	0.9999862
Image 5 water-marked	0.9998982	0.9994947	0.9989822	0.9968874	0.9999764

Table 4. Matrix B correlation.

	Original Image	Multipli-cative Noise	Impulsive Noise	Gaussian Noise	Compre-ssion Noise
Image 1 water-marked	0.9999960	0.9994985	0.9985537	0.9956487	0.9995689
Image 2 water-marked	0.9999853	0.9994938	0.9936931	0.9912684	0.9990533
Image 3 water-marked	0.9999510	0.9996421	0.9985542	0.9956853	0.9991845
Image 4 water-marked	0.9999733	0.9995036	0.9993014	0.9979530	0.9992126
Image 5 water-Marked	0.9998845	0.9994977	0.9988778	0.9960856	0.9991357

Table 5. Energy percentage of the recovered watermarked image.

	Original Image	Multipli-cative Noise	Impulsive Noise	Gaussian Noise	Compre-ssion Noise
Image 1 water-marked	99.980944	100.05124	100.19359	100.54094	99.880423
Image 2 water-marked	99.950691	99.928101	100.73372	101.78531	99.751472
Image 3 water-marked	99.915630	100.02666	100.15554	100.46218	100.15554
Image 4 water-marked	99.969529	100.04989	100.07206	100.33814	99.936251
Image 5 water-marked	100.40056	100.08783	100.17762	100.56995	99.929731

Table 6. Metric distortion by PSNR (dB).

	Image 1 water-marked	Image 2 water-marked	Image 3 water-marked	Image 4 water-marked	Image 5 water-marked
Original image	67.553861	67.301299	66.811934	67.201236	67.132489

In conclusion, the objective of this research is to demonstrate the legibility, security and robustness of this technique of embedding digital watermark, which in our opinion is accomplished for the following reasons. It is legible since the embedded information can be recovered using the inverse transform procedure. It is secure since for any non-authorized person the existence of the watermark is not evident. And more importantly, it resists compression such as JPEG, geometric distortion and multiplicative noise, attacks that alter the image in a uniform manner. These types of attacks will not affect the embedded information, hence it will resist a watermark. On the other hand, impulsive noise and Gaussian noise affect the watermark, but we should remember that if an attack is effective it should not alter visibly the image, and these two attacks clearly affect the image.

It should be noted that watermarked images have a high quality. This means that by embedding the watermark the image is not altered significantly. This can be seen in the first value of both the correlation and the energy Tables and in Table 6 of metric distortion.

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Short Communication

TWO NEW RECORDS OF SNAKE SPECIES FROM MACHIARA NATIONAL PARK, AZAD JAMMU & KASHMIR

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Abstract: The present study was conducted in and around Machiara National Park, Azad Jammu & Kashmir (AJK) in 2004 to document the faunistic diversity of the Park. Two species of snakes *Coluber ladacensis* and *Amphiesma platyceps* (Reptilia) are being reported for the first time from AJK in general and Machiara National Park in particular.

Keywords**:** Herpetology, biodiversity, *Coluber ladacensis*, *Amphiesma platyceps*

Introduction

Machiara National Park in Azad Jammu & Kashmir (AJK) lies on the western side of Punjal Range and is zoogeographically part of the Himalayas. The park possesses unique physical as well as biogeographical characteristics and presents a variety of microhabitats. It falls in the monsoon belt and receives sufficient precipitation in the form of snow and rainfall. Regarding the herpetology of AJK, very limited studies are so far available [1-5]. The present study reports two species of snakes from this area for the first time. Previously, Smith [6] is known to have made the first and the most comprehensive studies on snakes of the Indian sub-continent, including Pakistan. Later on Minton [7] and Mertens [8] also extensively studied the amphibians and reptiles of Pakistan. However, they mainly concentrated on the southern provinces of Pakistan with some records from northern Pakistan as well.

Study Area

Machiara National Park located at 34^0 40′ N latitude and 730 10′ E longitude covers an area of about 33136 acres. The park area represents temperate forest and alpine meadows. Top of the Makra Mountain here represents an area of permanent snow. Although the park is spread over a relatively smaller area, it has a steep vertical cline. The area starts at an altitude of about 800m and attains a height of about 3000m at Makra Mountain top. Several visits were made during the year 2004 to the park area to study its faunistic diversity.

Results and Discussion

The species *ladacensis* was first described as *Zaminis ladacensis* Anderson, 1871. Smith synonymised the species with *Coluber rhodorachis* [6], which was followed by Minton [7]. Mertens [8] once again considered it as an independent taxon and identified it at sub-specific level as *Coluber rhodorachis ladacensis*. He reported its presence from different areas of Pakistan, including some parts of NWFP. Khan [9] followed the arrangements of Mertens [3]. Whitaker and Captain [10] raised it to species level and stretched the limit of its subcaudal scales to 93.

The species *platyceps* was first described as *Tropidonotus platyceps* Blyth, 1845. Smith [6] identified it as *Natrix platyceps*. Minton [7] did not show the presence of this species in Pakistan. However, Mertens [8] reported its presence as *Natrix* (*Amphiesma*) *platyceps* from Abbotabad. Khan [9] and Whitaker and Captain [10] placed it in the genus *Amphiesma*.

The presence of *Coluber ladacensis* in AJK is based on specimen # PMNH 1650 collected from Chogali (34º 25´ 27" N, 74º 44´ 52" E) at an elevation of 2385 m. Chogali is a temperate forest with thick conifer trees and shrubs, representing extremely mesic habitat. The specimen is characterized by19 mid-dorsals; 208 ventrals; 98 subcaudals, divided; 8 supralabials, of which 4th and 5th touch the eye; 2 preoculars; 2 postoculars and presence of loreal.

The presence of *Amphiesma platyceps* is based on specimen # PMNH 1654, collected from Pathra (34º 32´ 58" N, 73º 31´ 53" E) at an elevation of 2490 m. The specimen is characterized by 19 mid-dorsals, middle ones keeled and lateral smooth; 194 ventrals; 63 subcaudals, divided; 8 supralabials, of which 3rd, 4th and 5th touch the eye; 1 preocular; 2 postoculars; loreal and anal, entire.

The existence of these two species in the Park is not surprising since both of these have been previously reported from the geographically adjacent North West Frontier Province (N.W.F.P.) of Pakistan. However, it is worth mentioning that the historic report of these species in NWFP was based on old taxonomic arrangements that do not agree with the recent characterization of these species made by Whitaker & Captain [10]. The latter authors [10] have clearly delineated the species from the other closely related taxa and have set new ranges of the phollidosis of the species. The present study not only stretches the occurrence of these species from Pakistan well into Azad Jammu & Kashmir reaffirming their existence in the northern temperate forests of Pakistan but also shows that a consistent and long term effort is needed to document the total natural wealth of the Park.

Acknowledgements

We are deeply indebted to the management of Machiara National Park (Protected area Management Project-PAMP), Azad Jammu & Kashmir for providing funds and opportunity to visit the area. We also gratefully acknowledge the assistance of our colleagues, Dr. S. Azhar Hassan and Mr. Riaz Ahmad. Our thanks are also due to Mr. Akram of PAMP, who assisted me in many ways during my stay in Machiara.

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Letter

A NOTE ON THE LANCZOS POTENTIAL FOR THE GÖDEL SOLUTION

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We show that the Gauss equation employed in the embedding of R_4 into E_5, yields a symmetric tensor b_{ij} which generates a Lanczos potential for the Gödel cosmological model.

A spacetime can be embedded into E_5 if and only if there exists the second fundamental form $b_{ac} = b_{ca}$ fulfilling the Gauss – Codazzi equations [1]:

$$R_{acij} = \in \left(b_{ai}b_{cj} - b_{aj}b_{ci}\right) \quad , \tag{1}$$

$$B_{cij} \equiv b_{ji;c} - b_{jc;i} = 0, \tag{2}$$

where $\in = \pm 1$, R_{acij} is the curvature tensor and *;j* denotes the covariant derivative. Then we say that such-space is said to be of class one.

From the Gauss relation (1) it is possible to show the identity [2-6]:

$$pb_{ij} = \frac{K_2}{48} g_{ij} - \frac{1}{2} R_{iacj} G^{ac} \quad , \tag{3}$$

where $G_{ac} = R_{ac} - \frac{R}{2} g_{ac}$ and $R_{ac} = R^r{}_{acr}$ are the Einstein and Ricci tensors, respectively, and $K_2 = {}^*R^{*\,ijac} R_{ijac}$ is a Lanczos invariant [7,8] in terms of the double dual [9] of the Riemann tensor ${}^*R^{*ij}{}_{ac} = \frac{1}{4} \eta^{ijrm} R_{rm}{}^{nr} \eta_{nrac}$, where η_{ijac} is the Levi-Civita tensor, and

$$p^2 = -\frac{\in}{6}\left(\frac{R}{24} K_2 + R_{imnj} G^{ij} G^{mn}\right) \geq 0. \tag{4}$$

If $p \neq 0$ then (3) can be used to obtain explicitly a b_{ij} verifying (1).

Now we apply (3) to the Gödel metric [10] (using signature +2):

$$ds^2 = -\left(dx^1\right)^2 - 2\, e^{x^4} dx^1 dx^2 - \frac{1}{2} e^{2x^4} \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2, \tag{5}$$

therefore $\in = 1$, $p = \frac{\sqrt{2}}{4}$ and:[1]

$$\left(b_{ij}\right) = -\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & e^{x^4} & 0 & 0 \\ e^{x^4} & \frac{3}{2} e^{x^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

The tensor (6) does not satisfy (2) because we know [11-15] that (5) is not of class one; in fact, $b_{12;4} \neq b_{14;2}$. Thus, for this Gödel cosmological model we have a B_{cij} whose only non-zero independent components are:

$$B_{124} = B_{421} = -\frac{\sqrt{2}}{2} e^{x^4} \quad , \quad B_{242} = \frac{3}{2}\sqrt{2}\, e^{2x^4} \quad , \tag{7}$$

with the same symmetries as the Lanczos potential K_{cij} [9,16,17]:

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$$B_{ijr} = -B_{jir} \quad , \quad B_{ijr} + B_{jri} + B_{rij} = 0 \, ,$$

$$B_{ir}^{\ \ r} = 0 \quad , \quad B_{ijr;}^{\ \ \ \ r} = 0 \quad .$$

Then "ansatz"

$$K_{ijr} = Q\, B_{ijr}, \quad Q = constant, \tag{8}$$

must generate the conformal tensor via the expression [18,19]:

$$C_{aijr} = K_{aij\,;\,r} - K_{air\,;\,j} + K_{jra\,;\,i} - K_{jri\,;\,a} +$$

$$g_{ar}K_{ji} - g_{aj}K_{ri} + g_{ij}K_{ra} - g_{ir}K_{ja} \, , \tag{9}$$

where $K_{ij} \equiv K_{i\ j\,;\,r}^{\ r} = K_{ji}$. With (7) and (8) we find that (9) implies correctly all components of the Weyl tensor if $Q = \sqrt{2}/18$, which means that (6) produces a Lanczos potential for Gödel geometry:

$$K_{ijr} = \frac{\sqrt{2}}{18}\left(b_{rj\,;\,i} - b_{ri\,;\,j}\right) \tag{10}$$

We know that (5) does not accept embedding into E_5. However, the study of the Gauss-Codazzi equations is important because it permits construction of the Lanczos generator (10) for the Gödel spacetime. Then, if a metric is not of class one, perhaps a b_{ij} verifying (1) may have a relationship similar to (10) with a Lanczos potential for this metric.

Our work calls attention towards an interesting connection between the embedding of Riemannian 4-spaces and the Lanczos generator, which needs be investigated further. For example, it is still not known if (5) admits embedding into E_6 [13-15,20]. The Lanczos potential gives us a new approach to this open problem.

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Obituary

Mr. A. H. Chotani
1923-2004

Mr. Abdul Hamid Chotani breathed his last on 17-10-2004, at the age of 81 years, after a prolonged illness. May Allah rest his soul in peace (ameen).

One of us (M. Aslam) met Mr. Chotani when I (S.M. Jaffar) joined PCSIR in October 1954. However, I recalled having seen his photograph in 1940 in a newspaper, when he topped in the matriculation Examination of Bombay University (India) where I too was a student. I believe, he was the first Muslim in the province to have achieved this distinction. He went on to do his B.Sc. (Hon's) in 1945 from Bombay University, and then B.S. and MS in Chemical Engineering from University of Wisconsin, Madison, U.S.A., in 1947&1948.

While working for a master's degree in Chemical Engineering, he was also teaching in the Mathematics Department, which supplemented his meager financial support from philanthropic organizations in Bombay. His strong background in Physics, Chemistry and Mathematics helped him to achieve high grades in the courses required in the Chemical Engineering Department. In Wisconsin, he provided voluntary assistance to his Indian colleagues in the evenings, in working on class-assignments, which included both Muslims and Hindus.

Mr. Chotani participated actively in the Islamic Cultural Association of the University of Wisconsin, which included students from Pakistan, Egypt, Turkey, India and Iraq. Independence Day of Pakistan was celebrated on 14th August 1947 in the student union hall. I (S.M.Jaffar) recall that a National anthem of Pakistan had not been decided upon by that time. Pakistani students presented a substitute anthem by signing a poem of Allama Iqbal, which was translated into English poetry by Mr. Chotani and distributed to the audience. Once, after Friday prayers, a recitation of Holy Qur'an was held by some Muslim students who possessed Qirat certificates from religious schools. Mr. Chotani surprised them when he recited the Qur'an with excellent "Tajwid". He told us that he was able to do this because his teacher taught him, in his early years, to read Qur'an with correct pronunciation (Tajwid).

He was an active member of International Club of the University of Wisconsin, which had a membership of seventy countries. He represented Pakistan effectively and spoke convincingly on the Two Nation Theory, ideological basis of Pakistan and issues concerning partition of the sub-continent. After completion of studies, he took an internship with a paper-manufacturing plant in Wisconsin State. Using his creative genius he succeeded in solving a lingering problem affecting the quality of the paper. This was highly appreciated by the management and they offered him a regular job in their plant but Mr. Chotani preferred to return and serve Pakistan.

Initially he joined the Association Cement Industry at Wah, as a chemist, but routine nature of his job did not satisfy his passion for undertaking challenging assignments. He resigned after sometime and moved to Karachi in search of a suitable job where he could make use of his talent in Chemical Engineering. Fortunately, Dr. Salimuzzaman Siddiqui, newly appointed Chairman of P.C.S.I.R., was looking for some dedicated scientists and engineers. He was impressed by the qualifications and experience of Mr. Chotani and he offered him a research position in P.C.S.I.R., which turned out to be a lifetime assignment till his retirement in 1983.

Mr. A. R. Chotani was working on a mini-plant for Dr. Salimuzzaman's project relating to "Use of super-heated steam for desulphurization of Pakistani coal", when I (M. Aslam) joined the organization. A project for waterproofing roofing–felt had been assigned to me, and I realized that laboratory experimentation had to be done on pilot-plant scale in order to ensure proper commercialisation of this product. I discussed this problem with Mr. Chotani. He was very excited and we worked out a plan, got a rather reluctant approval from Dr. Salimuzaman, and went to work. Early results were very discouraging and, to top it all, there was a fire in the pilot-plant, which also partly damaged the barrack in which it was housed. Dr. Salimuzaman called me and said, "I don't have the heart to tell this to Chotani, but you should convey to him that a failure in research should not become a prestige-issue", and suggested winding up the project. I conveyed this to Chotani. He was absolutely dumbfounded; he just sat there for almost half an hour saying nothing and I was looking at his mental state. I went along with him in deciding to undertake three further trials, before giving up the project. A main problem had been to cool and get the right structure in the felt after it came out of the steeping tank. It was our last experiment and we resorted to direct spray with water, which gave us success!

We produced 5,000 sq. ft. of the felt and then showed the results to Dr. Salimuzaman, who was no less excited. This shows one aspect of Chotani's character. Whenever he undertook any project/ assignment, he would not accept failure— this was his great strength and, in some cases, this trait was also his weak point.

Subsequently, looking after the Chemical Engineering & Pilot Plant Division of P.C.S.I.R., Chotani was involved in upscaling the coal-project into a major pilot-plant, in which German experts also provided inputs. Sometime thereafter, Salimuzaman decided to take him on to the Administration side, as Secretary of P.C.S.I.R., a responsibility handled well by him for almost a decade. Later, he returned to the R&D side in 1967/68, to become the head of the Fuel Research Center at Karachi. He was awarded Tamgha-e-Quaid-e-Azam by the Government of Pakistan in recognition of his meritorious services to PCSIR. Then, in 1977 he went to head the Hydrocarbon Research Institute, where he did important spade work on utilization of LPG and CNG for automobiles, leading to development of appropriate conversion-kits. Finally he returned to P.C.S.I.R. as Member (Technology) of the Governing Body in 1980, and he retired in 1983.

Chotani was a hardworking person, took all his responsibilities and assignments very seriously. He played a great role in establishing Pakistan Association of Scientists and Scientific Professions in 1955/56 and remained associated with this organisation in one capacity or the other till 1988. He was also involved in various social projects of his Memon community.

On of the sources of his energy and strength was undoubtedly the fact that he was a good Muslim, steadfast in his prayers. Chotani was a person with many personal gifts. Not only was he a good friend and easy to get along with, but he had a multitude of interests—professional as well as social. At one stage, he was one of a group of four persons, in the early days of P.C.S.I.R, who were constantly engaged in thinking and working together on a variety of S&T problems of industrial interest.

Mr. Chotani is survived by a family of four; a widowed wife, two sons and a daughter. His memory will always remain in our hearts as a scientist as well as a human being of highest moral values. May Allah almighty give his family the strength to bear this irreparable loss.

Muhammad Aslam
Fellow, Pakistan Academy of Sciences
and
Syed Muhammad Jaffar (Ph.D. Wisconsin)

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2. **Williams, R.L., Hilton, D.J., Pease, S., Wilson, T.A., Stewart, C.L., Grearing, D.P., Wagner, .F., Metcalf, D., Nicola, N.A. and Gough, N.M.** 1988. Myeloid leukaemia inhibitory factor (LIF) maintains the developmental potential of embryonic stem cells. Nature 336: 684-687

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5. **Leeper, G.W.** 1978. *Managing Heavy Metals on the Land*. Marcel Dekker, New York.
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