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γ-compactness in Fuzzifying Topology

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Abstract:
In this paper the concepts of fuzzifying γ-irresolute functions and fuzzifying γ-compact spaces were characterized in terms of fuzzifying γ-open sets and some of their properties are discussed.

Keywords and Phrases:
Lukasiewicz logic; fuzzifying topology; fuzzifying γ-open set; fuzzifying compactness.
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1. Introduction

Fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [5-7, 10-11, 16]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [6], the kind of topologies defined by Chang [1] and Goguen [2] is called the topologies of fuzzy subsets, and further is naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L-topological space) is a family τ of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and τ satisfies the basic conditions of classical topologies [9].

On the other hand, the authors of [8, 13] proposed the terminologies I-fuzzy topologies (if the set of membership values is chosen to be the unit interval [0,1]) and L-fuzzy topologies (if the corresponding set of membership values is chosen to be lattice L). More specifically, an I-fuzzy topology (resp. an L-fuzzy topology) is a (resp. an L-) fuzzy family τ over $P(X)$, where $P(X)$ denotes the class of all crisp subsets of nonempty set X. They were defined and extensively studied by Höhle, Šostak, Rodabaugh, Kubiak, and others [4, 8, 10-11, 13].

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In general, L-fuzzy topologies are investigated and described with algebraic and analytic methods.

In 1991, Ying [17-20] used the semantic method of continuous valued logic to propose the so-called fuzzifying topology as a preliminary of the research on bifuzzy topology and elementally develop topology in the theory of fuzzy sets from completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. In a way, fuzzifying topologies are analogous to I-fuzzy topologies, but the former appeal to some semantical expressions of Lukasiewicz logic as a basic tool, and thus can be viewed as an alternative approach to fuzzy topology. Particularly, as the author [17-20] indicated, by investigating fuzzifying topology we may partially answer an important question proposed by Rosser and Turquette [14] in 1952, which asked wether there are many valued theories beyond the level of predicates calculus.

Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implicaton formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzifying topologies [12, 15, 20]. For example, Ying [20] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [12] the concepts of fuzzifying γ-open set and fuzzifying γ-continuity were introduced and studied. Also, sayed [15] introduced and studied the concept of fuzzifying γ-Hausdorff separation axiom.

In [3] the concepts of γ-irresolute function and γ-compactness for fuzzy topological spaces were introduced.

In this paper we introduce and study the concept of γ-irresolute function between fuzzifying topological spaces. Furthermore, we introduce and study the concept of γ-compactness in the framework of fuzzifying topology. We use the finite intersection property to give a characterization of the fuzzifying γ-compact spaces.

2. Preliminaries

In this section, we offer some concepts and results in fuzzifying topology, which will used in the sequel. For the details, we refer to [12, 17-20]. First, we display the Lukasiewicz logic and corresponding set theoretical notations used in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0,1]$. We write $\models \varphi$ if $[\varphi]=1$ for any interpretation. By $\models^{w} \varphi$ (φ is feebly valid) we mean that for any valuation it always holds that $[\varphi]>0$, and $\varphi \models^{w} \psi$ we mean that $[\varphi]>0$ implies $[\psi]=1$. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) (a) $[\alpha]=\alpha(\alpha \in [0,1])$;

(b) $[\varphi \wedge \psi]=\min([\varphi],[\psi])$;

(c) $[\varphi \rightarrow \psi]=\min(1,1-[\varphi]+[\psi])$.

(2) If $\tilde{A} \in \Im(X)$, $[x \in \tilde{A}] := \tilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x\in X}[\varphi(x)]$.

In addition the following derived formulae are given,

(1) $[\neg\varphi] := [\varphi \to 0] = 1-[\varphi]$;

(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi],[\psi])$;

(3) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi)\wedge(\psi \to \varphi)]$;

(4) $[\varphi \dot{\wedge} \psi] := [\neg(\varphi \to \neg\psi) = \max(0,[\varphi]+[\psi]-1)$;

(5) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \sup_{x\in X}[\varphi(x)]$;

(6) If $\tilde{A}, \tilde{B} \in \Im(X)$, then

$$[\tilde{A} \subseteq \tilde{B}] := [\forall x(x \in \tilde{A} \to x \in \tilde{B})] = \inf_{x\in X}\min(1, 1-\tilde{A}(x)+\tilde{B}(x)),$$

where $\Im(X)$ is the family of all fuzzy sets in X.

Often we do not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying [17-20] did.

Secondly, we give some definitions and results in fuzzifying topology.

Definition 2.1 [17]. Let X be a universe of discourse, $\tau \in \Im(P(X))$ satisfy the following conditions:

(1) $\tau(X)=1$, $\tau(\phi)=1$;

(2) for any $A, B, \tau(A\cap B) \geq \tau(A)\wedge\tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}$, $\tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda}\tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X,τ) is a fuzzifying topological space.

Definition 2.2 [17]. The family of all fuzzifying closed sets, denoted by $\mathcal{F} \in \Im(P(X))$, is defined as follows: $A \in \mathcal{F} := X - A \in \tau$, where $X-A$ is the complement of A.

Definition 2.3 [17]. The fuzzifying neighborhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows: $N_x(A) = \sup_{x\in B\subseteq A}\tau(B)$.

Definition 2.4 [17,Lemma 5.2]. The closure $\overline{A}$ of A is defined as $\overline{A}(x) = 1 - N_x(X - A)$. In Theorem 5.3 [17], Ying proved that the closure $^- : P(X) \to \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [17]) because its extension $^- : \Im(X) \to \Im(X)$,

$\overline{\tilde{A}} = \bigcup_{\alpha\in[0,1]} \alpha\overline{\tilde{A}_\alpha}$, $\tilde{A} \in \Im(X)$, where $\tilde{A}_\alpha = \{x : \tilde{A}(x) \geq \alpha\}$ is the α-cut of $\tilde{A}$ and $\alpha\tilde{A}(x) = \alpha \wedge \tilde{A}(x)$ satisfies the following Kuratowski closure axioms:

(1) $\models \overline{\phi} = \phi$;

(2) for any $\tilde{A} \in \Im(X)$, $\models \tilde{A} \subseteq \overline{\tilde{A}}$;

(3) for any $\tilde{A}, \tilde{B} \in \Im(X)$, $\models \overline{\tilde{A} \cup \tilde{B}} \equiv \overline{\tilde{A}} \cup \overline{\tilde{B}}$

(4) for any $\tilde{A}, \tilde{B} \in \Im(X)$, $\models \overline{\left(\overline{\overline{\tilde{A}}}\right)} \subseteq \tilde{A}$.

Definition 2.5 [18]. For any $A \subseteq X$, the fuzzy set of interior points of A is called the interior of A, and given as follows: $A^{\circ}(x) \coloneqq N_x(A)$. From Lemma 3.1 [17] and the definitions of $N_x(A)$ and A° we have $\tau(A) = \inf_{x\in A} A^{\circ}(x)$.

Definition 2.6 [12]. For any $\tilde{A} \in \Im(X)$, $\models (\tilde{A})^{\circ} \equiv X - \overline{(X - \tilde{A})}$.

Lemma 2.1 [12]. *If* $\left[\tilde{A} \subseteq \tilde{B}\right] = 1$, *then (1)* $\models \overline{\tilde{A}} \subseteq \overline{\tilde{B}}$ *(2)* $\models (\tilde{A})^{\circ} \subseteq (\tilde{B})^{\circ}$.

Definition 2.7 [12]. Let (X, τ) be a fuzzifying topological space.

(1) The family of all fuzzifying γ-open sets, denoted by $\tau_\gamma \in \Im(P(X))$, is defined as follows: $A \in \tau_\gamma \coloneqq \forall x\left(x \in A \rightarrow x \in A^{-\circ} \cap A^{\circ -}\right)$, i.e., $\tau_\gamma(A) = \inf_{x\in A} \min(A^{-\circ}(x), A^{\circ -}(x))$.

(2) The family of all fuzzifying γ-closed sets, denoted by $\mathcal{F}_\gamma \in \Im(P(X))$, is defined as follows: $A \in \mathcal{F}_\gamma \coloneqq X - A \in \tau_\gamma$.

(3) The fuzzifying γ-neighborhood system of a point $x \in X$ is denoted by $N_x^\gamma \in \Im(P(X))$ and defined as follows: $N_x^\gamma(A) = \sup_{x\in A} \tau_\gamma(A)$.

(4) The fuzzifying γ-closure of a set $A \in P(X)$, denoted by $\mathrm{Cl}_\gamma \in \Im(X)$, is defined as follows: $\mathrm{Cl}_\gamma(A)(x) = 1 - N_x^\gamma(X - A)$.

(5) Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $C_\gamma \in \Im(Y^X)$, called fuzzifying γ-continuity, is given as follows: $C_\gamma(f) \coloneqq \forall B\left(B \in \sigma \rightarrow f^{-1}(B) \in \tau_\gamma\right)$.

Definition 2.8 [15]. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicate T_2^γ (fuzzifying γ-Hausdorff) $\in \Im(\Omega)$ is defined as follows:

$$T_2^\gamma(X,\tau) := \forall x \forall y \left((x \in X \wedge y \in X \wedge x \neq y) \to \exists B \exists C\left(B \in N_x^\gamma \wedge C \in N_y^\gamma \wedge B \cap C = \phi\right)\right).$$

Definition 2.9 [20]. Let X be a set. If $\tilde{A} \in \Im(X)$ such that the support $\operatorname{supp}\tilde{A} = \{x \in X : \tilde{A}(x) > 0\}$ of A is finite, then $\tilde{A}$ is said to be finite and we write $F(\tilde{A})$. A unary fuzzy predicate $FF \in \Im(\Im(X))$, called fuzzy finiteness, is given as $FF(\tilde{A}) := (\exists \tilde{B})\left(F(\tilde{B}) \wedge (\tilde{A} \equiv \tilde{B})\right) = 1 - \inf\{\alpha \in [0,1] : F(\tilde{A}_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\tilde{A}_{[\alpha]})\}$, where $\tilde{A}_\alpha = \{x \in X : \tilde{A}(x) \geq \alpha\}$ and $\tilde{A}_{[\alpha]} = \{x \in X : \tilde{A}(x) > \alpha\}$.

Definition 2.10 [20]. Let X be a set.

(1) A binary fuzzy predicate $K \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying covering, is given as follows: $K(\Re, A) := \forall x\left(x \in A \to \exists B(B \in \Re \wedge x \in B)\right)$.

(2) Let (X,τ) be a fuzzifying topological space. A binary fuzzy predicate $K_\circ \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying open covering, is given as follows: $K_\circ(\Re, A) := K(\Re, A) \dot{\wedge} (\Re \subseteq \tau)$.

Definition 2.11 [20]. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness, is given as follows:

$$(X,\tau) \in \Gamma := (\forall \Re)\left(K_\circ(\Re, X) \to (\exists \wp)\left((\wp \leq \Re) \wedge K(\wp, A) \dot{\wedge} FF(\wp)\right)\right),$$

where $\wp \leq \Re$ means that for any $M \in P(X)$, $\wp(M) \leq \Re(M)$.

Definition 2.12 [20]. Let X be a set. A unary fuzzy predicate $fI \in \Im(\Im(P(X)))$, called fuzzifying finite intersection property, is given as follows:

$$fI(\Re) := (\forall \mathcal{B})\left((\mathcal{B} \leq \Re) \wedge FF(\mathcal{B}) \to (\exists x)(\forall B)\left((B \in \mathcal{B}) \to (x \in B)\right)\right).$$

Lemma 2.2 [1.2]. *Let (X,τ) be a fuzzifying topological space. Then*
(1) $\models \tau \subseteq \tau_\gamma$; (2) $\models \mathcal{F} \subseteq \mathcal{F}_\gamma$, (3) $\models \mathcal{F}_\gamma\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \geq \bigwedge_{\lambda \in \Lambda} \mathcal{F}_\gamma(A_\lambda)$.

Corollary 2.1 [12]. $\tau_\gamma(A) = \inf_{x \in A} N_x^\gamma(A)$.

Theorem 2.1 [12]. *For any x, A, B,* $\models A \subseteq B \to \left(A \in N_x^\gamma \to B \in N_x^\gamma\right)$.

3. Fuzzifying γ-irresolute functions

Definition 3.1. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I_\gamma \in \Im(Y^X)$, called fuzzifying γ-irresolute, is given as follows: $I_\gamma(f) := \forall B(B \in \sigma_\gamma \to f^{-1}(B) \in \tau_\gamma)$. Intuitively, the degree to which f is fuzzifying γ-irresolute function is

$$[I_\gamma(f)] = \inf_{B \subseteq Y} \min(1, 1 - \sigma_\gamma(B) + \tau_\gamma(f^{-1}(B))).$$

Theorem 3.1. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. Then*

$$\models f \in I_\gamma \to f \in C_\gamma.$$

Proof. From Lemma 2.2 we have $\sigma(B) \le \sigma_\gamma(B)$ and the result holds.

Definition 3.2. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predicates $w_k \in \Im(Y^X)$, where $k = 1, \cdots, 5$, as follows:

(1) $f \in w_1 = \forall B(B \in \mathcal{F}_\gamma^Y \to f^{-1}(B) \in \mathcal{F}_\gamma^X)$, where $\mathcal{F}_\gamma^X$ and $\mathcal{F}_\gamma^Y$ are the fuzzifying γ-closed subsets of X and Y, respectively;

(2) $f \in w_2 = \forall x \forall u(u \in N_{f(x)}^{\gamma^Y} \to f^{-1}(u) \in N_x^{\gamma^X})$, where N^{γ^X} and N^{γ^Y} are the family of fuzzifying γ-neighborhood systems of X and Y, respectively;

(3) $f \in w_3 = \forall x \forall u\left(u \in N_{f(x)}^{\gamma^Y} \to \exists v\left(f(v) \subseteq u \to v \in N_x^{\gamma^X}\right)\right)$;

(4) $f \in w_4 = \forall A(f(cl_\gamma^X(A)) \subseteq cl_\gamma^Y(f(A)))$;

(5) $f \in w_5 = \forall B(cl_\gamma^X(f^{-1}(B)) \subseteq f^{-1}(cl_\gamma^Y(B)))$.

Theorem 3.2. $\models f \in I_\gamma \leftrightarrow f \in w_k$, $k = 1, \cdots, 5$.

Proof. (a) We will prove that $\models f \in I_\gamma \leftrightarrow f \in w_1$.

$$[f \in w_1] = \inf_{B \in P(Y)} \min(1, 1 - \mathcal{F}_\gamma^Y(B) + \mathcal{F}_\gamma^X(f^{-1}(B)))$$

$$= \inf_{B \in P(Y)} \min(1, 1 - \sigma_\gamma(Y - B) + \tau_\gamma(X - f^{-1}(B)))$$

$$= \inf_{B \in P(Y)} \min(1, 1 - \sigma_\gamma(Y - B) + \tau_\gamma(f^{-1}(Y - B)))$$

$$= \inf_{u\in P(Y)} \min\left(1, 1-\sigma_\gamma(u)+\tau_\gamma\left(f^{-1}(u)\right)\right) = \left[f \in I_\gamma\right].$$

(b) We will prove that $\models f \in I_\gamma \leftrightarrow f \in w_2$. First, we prove that $[f \in w_2] \ge [f \in I_\gamma]$. If $N^{\gamma^Y}_{f(x)}(u) \le N^{\gamma^X}_x\left(f^{-1}(u)\right)$, then $\min\left(1, 1-N^{\gamma^Y}_{f(x)}(u)+N^{\gamma^X}_x\left(f^{-1}(u)\right)\right) = 1 \ge [f \in I_\gamma]$. Suppose $N^{\gamma^Y}_{f(x)}(u) > N^{\gamma^X}_x\left(f^{-1}(u)\right)$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then,

$$\begin{aligned} N^{\gamma^Y}_{f(x)}(u) - N^{\gamma^X}_x\left(f^{-1}(u)\right) &= \sup_{f(x)\in A\subseteq u} \sigma_\gamma(A) - \sup_{x\in B\subseteq f^{-1}(u)} \tau_\gamma(B) \\ &\le \sup_{f(x)\in A\subseteq u} \sigma_\gamma(A) - \sup_{f(x)\in A\subseteq u} \tau_\gamma\left(f^{-1}(A)\right) \\ &\le \sup_{f(x)\in A\subseteq u} \left(\sigma_\gamma(A) - \tau_\gamma\left(f^{-1}(A)\right)\right) \end{aligned}$$

So

$$1 - N^{\gamma^Y}_{f(x)}(u) + N^{\gamma^X}_x\left(f^{-1}(u)\right) \ge \inf_{f(x)\in A\subseteq u} \left(1 - \sigma_\gamma(A) + \tau_\gamma\left(f^{-1}(A)\right)\right).$$

Therefore

$$\begin{aligned} \min\left(1, 1 - N^{\gamma^Y}_{f(x)}(u) + N^{\gamma^X}_x\left(f^{-1}(u)\right)\right) &\ge \inf_{f(x)\in A\subseteq u} \min\left(1, 1 - \sigma_\gamma(A) + \tau_\gamma\left(f^{-1}(A)\right)\right) \\ &\ge \inf_{v\in P(Y)} \min\left(1, 1 - \sigma_\gamma(v) + \tau_\gamma\left(f^{-1}(v)\right)\right) = \left[f \in I_\gamma\right]. \end{aligned}$$

Hence

$$\inf_{x\in X} \inf_{u\in P(Y)} \min\left(1, 1 - N^{\gamma^Y}_{f(x)}(u) + N^{\gamma^X}_x\left(f^{-1}(u)\right)\right) \ge \left[f \in I_\gamma\right].$$

Second, we prove that $\left[f \in I_\gamma\right] \ge [f \in w_2]$. From Corollary 2.1, we have

$$\begin{aligned} \left[f \in I_\gamma\right] &= \inf_{u\in P(Y)} \min\left(1, 1 - \sigma_\gamma(u) + \tau_\gamma\left(f^{-1}(u)\right)\right) \\ &\ge \inf_{u\in P(Y)} \min\left(1, 1 - \inf_{f(x)\in u} N^{\gamma^Y}_{f(x)}(u) + \inf_{x\in f^{-1}(u)} N^{\gamma^X}_x\left(f^{-1}(u)\right)\right) \\ &\ge \inf_{u\in P(Y)} \min\left(1, 1 - \inf_{x\in f^{-1}(u)} N^{\gamma^Y}_{f(x)}(u) + \inf_{x\in f^{-1}(u)} N^{\gamma^X}_x\left(f^{-1}(u)\right)\right) \\ &\ge \inf_{x\in X} \inf_{u\in P(Y)} \min\left(1, 1 - N^{\gamma^Y}_{f(x)}(u) + N^{\gamma^X}_x\left(f^{-1}(u)\right)\right) = [f \in w_2]. \end{aligned}$$

(c) We prove that $[f \in w_2] = [f \in w_3]$. From Theorem 2.1 we have

$$[f \in w_3] = \inf_{x\in X} \inf_{u\in P(Y)} \min\left(1, 1 - N^{\tau_Y}_{f(x)}(u) + \sup_{v\in P(X), f(v)\subseteq u} N^{\tau_X}_{x}(v)\right)$$
$$\geq \inf_{x\in X} \inf_{u\in P(X)} \min\left(1, 1 - N^{\tau_Y}_{f(x)}(u) + N^{\tau_X}_{x}\left(f^{-1}(u)\right)\right) = [f \in w_2].$$

(d) We prove that $[f \in w_4] = [f \in w_5]$. First, since for any fuzzy set $\tilde{A}$ we have $\left[f^{-1}\left(f(\tilde{A})\right) \supseteq \tilde{A}\right] = 1$, then for any $B \in P(Y)$ we have $\left[f^{-1}\left(f\left(cl^X_\tau\left(f^{-1}(B)\right)\right)\right) \supseteq cl^X_\tau\left(f^{-1}(B)\right)\right] = 1$. Also, since $\left[f\left(f^{-1}(B)\right) \subseteq B\right] = 1$, then we have that

$$\left[cl^Y_\tau\left(f\left(f^{-1}(B)\right)\right) \subseteq cl^Y_\tau(B)\right] = 1.$$

Then from Lemma 1.2 (2) [22] we have

$$\left[cl^X_\tau\left(f^{-1}(B)\right) \subseteq f^{-1}\left(cl^Y_\tau(B)\right)\right] \geq \left[f^{-1}\left(f\left(cl^X_\tau\left(f^{-1}(B)\right)\right)\right) \subseteq f^{-1}\left(cl^Y_\tau(B)\right)\right]$$
$$\geq \left[f^{-1}\left(f\left(cl^X_\tau\left(f^{-1}(B)\right)\right)\right) \subseteq f^{-1}\left(cl^Y_\tau\left(f\left(f^{-1}(B)\right)\right)\right)\right]$$
$$\geq \left[f\left(cl^X_\tau\left(f^{-1}(B)\right)\right) \subseteq cl^Y_\tau\left(f\left(f^{-1}(B)\right)\right)\right].$$

Therefore

$$[f \in w_5] = \inf_{B\in P(Y)} \left[cl^X_\tau\left(f^{-1}(B)\right) \subseteq f^{-1}\left(cl^Y_\tau(B)\right)\right]$$
$$\geq \inf_{B\in P(Y)} \left[f\left(cl^X_\tau\left(f^{-1}(B)\right)\right) \subseteq cl^Y_\tau\left(f\left(f^{-1}(B)\right)\right)\right]$$
$$\geq \inf_{A\in P(X)} \left[f\left(cl^X_\tau(A)\right) \subseteq cl^Y_\tau\left(f(A)\right)\right] = [f \in w_4].$$

Second, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence from Lemma 1.2 (1) [22] we have

$$[f \in w_4] = \inf_{A\in P(X)} \left[f\left(cl^X_\tau(A)\right) \subseteq cl^Y_\tau\left(f(A)\right)\right]$$
$$\geq \inf_{A\in P(X)} \left[f\left(cl^X_\tau(A)\right) \subseteq f\left(f^{-1}\left(cl^Y_\tau\left(f(A)\right)\right)\right)\right]$$
$$\geq \inf_{A\in P(X)} \left[cl^X_\tau(A) \subseteq f^{-1}\left(cl^Y_\tau\left(f(A)\right)\right)\right]$$
$$\geq \inf_{B\in P(Y), B=f(A)} \left[cl^X_\tau\left(f^{-1}(B)\right) \subseteq f^{-1}\left(cl^Y_\tau(B)\right)\right]$$

$$\geq \inf_{B\in P(Y)}\left[cl_\gamma^X\left(f^{-1}(B)\right)\subseteq f^{-1}\left(cl_\gamma^Y(B)\right)\right]=\left[f\in w_5\right].$$

(6) We want to prove that $\models f\in w_2 \leftrightarrow f\in w_5$.

$$\left[f\in w_5\right]=\inf_{B\in P(Y)}\left[cl_\gamma^X\left(f^{-1}(B)\right)\subseteq f^{-1}\left(cl_\gamma^Y(B)\right)\right]$$

$$=\inf_{B\in P(Y)}\inf_{x\in X}\min\left(1,1-\left(1-N_x^{\tau_\gamma^X}\left(X-f^{-1}(B)\right)\right)+1-N_{f(x)}^{\tau_\gamma^Y}(Y-B)\right)$$

$$=\inf_{B\in P(Y)}\inf_{x\in X}\min\left(1,1-N_{f(x)}^{\tau_\gamma^Y}(Y-B)+N_x^{\tau_\gamma^X}\left(f^{-1}(Y-B)\right)\right)$$

$$=\inf_{B\in P(Y)}\inf_{x\in X}\min\left(1,1-N_{f(x)}^{\tau_\gamma^Y}(u)+N_x^{\tau_\gamma^X}\left(f^{-1}(u)\right)\right)=\left[f\in w_2\right].$$

4. Fuzzifying γ -compact space

Definition 4.1. A fuzzifying topological space (X,τ) is said to be γ -fuzzifying topological space if $\tau_\gamma(A\cap B)\geq\tau_\gamma(A)\wedge\tau_\gamma(B)$.

Definition 4.2. A binary fuzzy predicate $K_\gamma\in\Im\left(\Im(P(X))\times P(X)\right)$, called fuzzifying γ -open covering, is givenas $K_\gamma(\Re,A):=K(\Re,A)\wedge\left(\Re\subseteq\tau_\gamma\right)$.

Definition 4.3. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma_\gamma\in\Im(\Omega)$, called fuzzifying γ -compactness, is given as follows:

(1) $(X,\tau)\in\Gamma_\gamma:=(\forall\Re)\left(K_\gamma(\Re,X)\to(\exists\wp)\left((\wp\leq\Re)\wedge K(\wp,X)\wedge FF(\wp)\right)\right)$;

(2) If $A\subseteq X$, then $\Gamma_\gamma(A):=\Gamma_\gamma(A,\tau/A)$.

Lemma 4.1. $\models K(\Re,A)\to K_\gamma(\Re,A)$.

Proof. Since from Lemma 2.2 $\models\tau\subseteq\tau_\gamma$, then we have $[\Re\subseteq\tau]\leq\left[\Re\subseteq\tau_\gamma\right]$. So, $\left[K(\Re,A)\right]\leq\left[K_\gamma(\Re,A)\right]$.

Theorem 4.1. $\models(X,\tau)\in\Gamma_\gamma\to(X,\tau)\in\Gamma$.

Proof. From Lemma 4.1 the proof is immediate.

Theorem 4.2. *For any fuzzifying topological space* (X,τ) *and* $A\subseteq X$,

$$\Gamma_\gamma(A)\leftrightarrow(\forall\Re)\left(K_\gamma(\Re,A)\to(\exists\wp)\left((\wp\leq\Re)\wedge K(\wp,A)\wedge FF(\wp)\right)\right).$$

where K_γ is related to τ.

Proof. For any $\Re \in \Im(\Im(X))$, we get $\bar{\Re} \in \Im(\Im(A))$ defined as $\bar{\Re}(C) = \Re(B)$ with $C = A \cap B$, $B \subseteq X$. Then

$$K(\bar{\Re}, A) = \inf_{x \in A} \sup_{x \in C} \bar{\Re}(C) = \inf_{x \in A} \sup_{x \in C = A \cap B} \Re(B) = \inf_{x \in A} \sup_{x \in B} \Re(B) = K(\Re, A),$$

because $x \in A$ and $x \in B$ if and only if $x \in A \cap B$. Therefore

$$\begin{aligned}
[\bar{\Re} \subseteq \tau_\gamma /_A] &= \inf_{C \subseteq A} \min(1, 1 - \bar{\Re}(C) + \tau_\gamma /_A(C)) \\
&= \inf_{C \subseteq A} \min\left(1, 1 - \sup_{C = A \cap B, B \subseteq X} \Re(B) + \sup_{C = A \cap B, B \subseteq X} \tau_\gamma(B)\right) \\
&\geq \inf_{C \subseteq A, C = A \cap B, B \subseteq X} \min(1, 1 - \Re(B) + \tau_\gamma(B)) \\
&\geq \inf_{B \subseteq X} \min(1, 1 - \Re(B) + \tau_\gamma(B)) = [\Re \subseteq \tau_\gamma].
\end{aligned}$$

For any $\wp \leq \bar{\Re}$, we define $\wp' \in \Im(P(X))$ as follows:

$$\wp'(B) = \begin{cases} \wp(B) & \text{if } B \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\wp' \leq \Re$, $FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$.

Furthermore, we have

$$\begin{aligned}
[\Gamma_\gamma(A) \mathbin{\dot\wedge} K_\gamma(\Re, A)] &\leq [\Gamma_\gamma(A) \mathbin{\dot\wedge} K'_\gamma(\bar{\Re}, A)] \\
&\leq [(\exists \wp)((\wp \leq \bar{\Re}) \wedge K(\wp, A) \mathbin{\dot\wedge} FF(\wp))] \\
&\leq [(\exists \wp')((\wp' \leq \Re) \wedge K(\wp', A) \mathbin{\dot\wedge} FF(\wp'))] \\
&\leq [(\exists \mathcal{B})((\mathcal{B} \leq \Re) \wedge K(\mathcal{B}, A) \mathbin{\dot\wedge} FF(\mathcal{B}))].
\end{aligned}$$

Then $\Gamma_\gamma(A) \leq [K_\gamma(\Re, A)] \to [(\exists \mathcal{B})((\mathcal{B} \leq \Re) \wedge K(\mathcal{B}, A) \mathbin{\dot\wedge} FF(\mathcal{B}))]$, where $K'_\gamma(\bar{\Re}, A) = [K(\bar{\Re}, A) \mathbin{\dot\wedge} (\bar{\Re} \subseteq \tau_\gamma /_A)]$. Therefore

$$\begin{aligned}
\Gamma_\gamma(A) &\leq \inf_{\Re \in \Im(P(X))} [K_\gamma(\Re, A) \to (\exists \mathcal{B})((\mathcal{B} \leq \Re) \wedge K(\mathcal{B}, A) \mathbin{\dot\wedge} FF(\mathcal{B}))] \\
&= [(\forall \Re)(K_\gamma(\Re, A) \to (\exists \mathcal{B})((\mathcal{B} \leq \Re) \wedge K(\mathcal{B}, A) \mathbin{\dot\wedge} FF(\mathcal{B})))].
\end{aligned}$$

Conversely, for any $\Re \in \Im(P(A))$, if $[\Re \subseteq \tau_\gamma/_A] = \inf_{B\subseteq A} \min(1, 1-\Re(B)+\tau_\gamma/_A(B)) = \lambda$, then for any $n \in N$ and $B \subseteq A$, $\sup_{B=A\cap C, C\subseteq X} \tau_\gamma(C) = \tau_\gamma/_A(B) > \lambda + \Re(B)-1-1/n$, and there exists $C_B \subseteq X$ such that $C_B \cap A = B$ and $\tau_\gamma(C_B) > \lambda + \Re(B)-1-1/n$. Now, we define $\bar{\Re} \in \Im(P(X))$ as $\bar{\Re}(C) = \max_{B\subseteq A} (0, \lambda+\Re(B)-1-1/n)$. Then $[\bar{\Re} \subseteq \tau_\gamma] = 1$ and

$$K(\bar{\Re}, A) = \inf_{x\in A} \sup_{x\in C\subseteq X} \bar{\Re}(C) = \inf_{x\in A} \sup_{x\in B} \bar{\Re}(C_B) \geq \inf_{x\in A} \sup_{x\in B} \left(\lambda + \Re(B) - 1 - \frac{1}{n}\right)$$

$$= \inf_{x\in A} \sup_{x\in B} \Re(B) + \lambda - 1 - \frac{1}{n} = K(\Re, A) + \lambda - 1 - \frac{1}{n},$$

$$K_\gamma(\bar{\Re}, A) = [K(\bar{\Re}, A) \mathbin{\dot\wedge} (\bar{\Re} \subseteq \tau_\gamma)] = [K(\bar{\Re}, A)] \geq \max\left(0, K(\Re, A) + \lambda - 1 - \frac{1}{n}\right)$$

$$\geq \max(0, K(\Re, A) + \lambda - 1) - \frac{1}{n} = K'_\gamma(\Re, A) - \frac{1}{n}.$$

For any $\wp \leq \bar{\Re}$, we set $\wp' \in \Im(P(A))$ as $\wp'(B) = \wp(C_B)$, $B \subseteq A$. Then $\wp' \leq \Re$, $FF(\wp') = FF(\wp)$ and $K(\wp', A) = K(\wp, A)$. Therefore

$$[(\forall \Re)(K_\gamma(\Re, A) \to (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \mathbin{\dot\wedge} FF(\wp)))] \mathbin{\dot\wedge} [K'_\gamma(\Re, A)] - \frac{1}{n}$$

$$\leq [(\forall \Re)(K_\gamma(\Re, A) \to (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \mathbin{\dot\wedge} FF(\wp)))] \mathbin{\dot\wedge} \left([K'_\gamma(\Re, A)] - \frac{1}{n}\right)$$

$$\leq [K_\gamma(\bar{\Re}, A) \to (\exists \wp)((\wp \leq \bar{\Re}) \wedge K(\wp, A) \mathbin{\dot\wedge} FF(\wp))] \mathbin{\dot\wedge} [K_\gamma(\bar{\Re}, A)]$$

$$\leq [(\exists \wp)((\wp \leq \bar{\Re}) \wedge K(\wp, A) \mathbin{\dot\wedge} FF(\wp))]$$

$$\leq [(\exists \wp')((\wp' \leq \Re) \wedge K(\wp', A) \mathbin{\dot\wedge} FF(\wp'))]$$

$$\leq [(\exists B)((B \leq \Re) \wedge K(B, A) \mathbin{\dot\wedge} FF(B))].$$

Let $n \to \infty$. We obtain

$$[(\forall \Re)(K_\gamma(\Re, A) \to (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \mathbin{\dot\wedge} FF(\wp)))] \mathbin{\dot\wedge}$$

$$[K'_\gamma(\Re, A)] \leq [(\exists B)((B \leq \Re) \wedge K(B, A) \mathbin{\dot\wedge} FF(B))].$$

Then

$$\left[(\forall \Re)\left(K_\gamma(\Re,A)\to(\exists\wp)\left((\wp\le\Re)\wedge K(\wp,A)\dot\wedge FF(\wp)\right)\right]$$

$$\le\left[K'_\gamma(\Re,A)\to(\exists\mathcal{B})\left((\mathcal{B}\le\Re)\wedge K(\mathcal{B},A)\dot\wedge FF(\mathcal{B})\right)\right]$$

$$\le\inf_{\Re\in\Im(P(X))}\left[K'_\gamma(\Re,A)\to(\exists\mathcal{B})\left((\mathcal{B}\le\Re)\wedge K(\mathcal{B},A)\dot\wedge FF(\mathcal{B})\right)\right]$$

$$=\Gamma_\gamma(A).$$

Theorem 4.3. *Let (X,τ) be a fuzzifying topological space.*

$$\pi_1:=(\forall\Re)\left((\Re\in\Im(P(X)))\wedge(\Re\subseteq\mathcal{F}_\gamma)\dot\wedge \text{fI}(\Re)\to(\exists x)(\forall A)(A\in\Re\to x\in A)\right);$$

$$\pi_2:=(\forall\Re)(\exists B)\Big(\left((\Re\subseteq\mathcal{F}_\gamma)\wedge(B\in\tau_\gamma)\right)\dot\wedge$$
$$(\forall\wp)\left((\wp\le\Re)\dot\wedge FF(\wp)\to\neg(\cap\wp\subseteq B)\right)\to\neg(\cap\Re\subseteq B)\Big).$$

Then $\models\Gamma_\gamma(X,\tau)\leftrightarrow\pi_i,\ i=1,2$.

Proof. (a) We prove $\Gamma_\gamma(X,\tau)=[\pi_1]$. For any $\Re\in\Im(P(X))$, we set $\Re^c(X-A)=\Re(A)$. Then

$$[\Re\subseteq\tau_\gamma]=\inf_{A\in P(X)}\min\left(1,1-\Re(A)+\tau_\gamma(A)\right)$$
$$=\inf_{X-A\in P(X)}\min\left(1,1-\Re^c(X-A)+\mathcal{F}_\gamma(X-A)\right)=[\Re^c\subseteq\mathcal{F}_\gamma],$$

$$FF(\Re)=1-\inf\{\alpha\in[0,1]:F(\Re_\alpha)\}=1-\inf\{\alpha\in[0,1]:F(\Re^c_\alpha)\}=FF(\Re^c)$$

and

$$\mathcal{B}\le\Re^c\Leftrightarrow\mathcal{B}(M)\Leftrightarrow\mathcal{B}^c(X-M)\le\Re(X-M)\Leftrightarrow\mathcal{B}^c\le\Re.$$

Therefore

$$\Gamma_\gamma(X,\tau)=\left[(\forall\Re)\left(K_\gamma(\Re,X)\to(\exists\wp)\left((\wp\le\Re)\wedge K(\wp,X)\dot\wedge FF(\wp)\right)\right)\right]$$

$$=\left[(\forall\Re)\left((\Re\subseteq\tau_\gamma)\dot\wedge K(\Re,X)\to(\exists\wp)\left((\wp\le\Re)\wedge K(\wp,X)\dot\wedge FF(\wp)\right)\right)\right]$$

$$=\left[(\forall\Re)\left((\Re\subseteq\tau_\gamma)\to\left(K(\Re,X)\to(\exists\wp)\left((\wp\le\Re)\wedge K(\wp,X)\dot\wedge FF(\wp)\right)\right)\right)\right]$$

$$=\Big[(\forall\Re)\Big((\Re^c\subseteq\mathcal{F}_\gamma)\to\big((\forall x)(\exists A)(A\in\Re\wedge x\in A)\to$$
$$(\exists\wp)\left((\wp\le\Re)\wedge K(\wp,X)\dot\wedge FF(\wp)\right)\big)\Big)\Big]$$

$$=\left[(\forall\Re)\left((\Re^c\subseteq\mathcal{F}_\gamma)\rightarrow((\forall x)(\exists A)(A\in\Re\wedge x\in A)\rightarrow (\exists\mathcal{B}^c)\left((\mathcal{B}^c\leq\Re)\wedge K(\mathcal{B}^c,X)\wedge FF(\mathcal{B}^c)\right)\right)\right)\right]$$

$$=\left[(\forall\Re)\left((\Re^c\subseteq\mathcal{F}_\gamma)\rightarrow((\forall x)(\exists A)(A\in\Re\wedge x\in A)\rightarrow (\exists\mathcal{B})\left((\mathcal{B}\leq\Re^c)\wedge FF(\mathcal{B})\wedge K(\mathcal{B}^c,X)\right)\right)\right)\right]$$

$$=\left[(\forall\Re)\left((\Re^c\subseteq\mathcal{F}_\gamma)\rightarrow((\forall x)(\exists A)(A\in\Re\wedge x\in A)\rightarrow (\exists\mathcal{B})\left((\mathcal{B}\leq\Re^c)\wedge FF(\mathcal{B})\wedge(\forall x)(\exists B)(B\in\mathcal{B}^c\wedge x\in B)\right)\right]$$

$$=\left[(\forall\Re)\left(\Re^c\subseteq\mathcal{F}_\gamma\rightarrow\left(\neg\left((\exists\mathcal{B})(\mathcal{B}\leq\Re^c\wedge FF(\mathcal{B})\wedge (\forall x)(\exists B)(B\in\mathcal{B}^c\wedge x\in B)\right)\rightarrow\neg((\forall x)(\exists A)(A\in\Re\wedge x\in A))\right)\right)\right]$$

$$=\left[(\forall\Re)\left((\Re^c\subseteq\mathcal{F}_\gamma)\rightarrow\left(\text{fI}(\Re^c)\rightarrow\neg((\forall x)(\exists A)(A\in\Re\wedge x\in A))\right)\right)\right]$$

$$=\left[(\forall\Re)\left((\Re^c\subseteq\mathcal{F}_\gamma)\wedge\text{fI}(\Re^c)\rightarrow(\exists x)(\forall A)(A\in\Re^c\rightarrow x\in A)\right)\right]=[\pi_1].$$

(b) We prove $[\pi_1]=[\pi_2]$. Let $X-B\in P(X)$. For any $\Re\in\Im(P(X))$.

$$\left[(\Re\subseteq\mathcal{F}_\gamma)\wedge(B\in\tau_\gamma)\right]=\left[(\Re\subseteq\mathcal{F}_\gamma)\wedge(X-B\in\mathcal{F}_\gamma)\right]$$

$$=\inf_{A\in P(X)}\min(1,1-\Re(A)+\mathcal{F}_\gamma(A))\wedge\mathcal{F}_\gamma(X-B)$$

$$=\inf_{A\in P(X)}\min(1,1-\Re(A)+\mathcal{F}_\gamma(A))\wedge\inf_{A\in P(X)}\min(1,1-[A\in\{X-B\}]+\mathcal{F}_\gamma(A))$$

$$=\inf_{A\in P(X)}\min\left(1,1-[(\Re\cup\{X-B\})(A)]+\mathcal{F}_\gamma(A)\right)$$

$$=\left[(\Re\cup\{X-B\})\subseteq\mathcal{F}_\gamma\right].$$

Therefore, for any $\mathcal{B}\in\Im(P(X))$, let $\wp=\mathcal{B}\setminus\{X-B\}\in\Im(P(X))$.

$$\wp(A)=\begin{cases}\mathcal{B}(A), & A\neq X-B\\ 0, & A=X-B\end{cases}.$$

Then $\wp\leq\mathcal{B}$, $\wp\cup\{X-B\}\geq\mathcal{B}$, $[FF(\wp)]=[FF(\mathcal{B})]$,

$$[\wp\leq\Re]=\left[\mathcal{B}\leq(\Re\cup\{X-B\})\right]$$

and

$$\left[(\forall\wp)\left((\wp\leq\Re)\mathbin{\dot\wedge}FF(\wp)\to(\exists x)(\forall A)\left(A\in\left(\wp\cup\{X-B\}\right)\to(x\in A)\right)\right)\right]$$

$$=\inf_{\wp\leq\Re}\min\left(1,1-\left[FF(\wp)\right]+\sup_{x\in X}\inf_{A\in P(X)}\left(\left(\wp\cup\{X-B\}\right)(A)\to A(x)\right)\right)$$

$$\leq\inf_{\mathcal{B}\leq(\Re\cup\{X-B\})}\min\left(1,1-\left[FF(\mathcal{B})\right]+\sup_{x\in X}\inf_{A\in P(X)}\left(\mathcal{B}(A)\to A(x)\right)\right)$$

$$=\text{fI}\left(\Re\cup\{X-B\}\right).$$

Furthermore, we have

$$\pi_1\mathbin{\dot\wedge}\left[\left((\Re\subseteq\mathcal{F}_\gamma)\wedge(B\in\tau_\gamma)\right)\wedge(\forall\wp)\left((\wp\leq\Re)\mathbin{\dot\wedge}FF(\wp)\to\neg\left(\bigcap\wp\subseteq B\right)\right)\right]$$

$$=\pi_1\mathbin{\dot\wedge}\left[\left(\Re\cup\{X-B\}\subseteq\mathcal{F}_\gamma\right)\mathbin{\dot\wedge}(\forall\wp)\right.$$

$$\left.\left((\wp\leq\Re)\mathbin{\dot\wedge}FF(\wp)\to(\exists x)(\forall A)\left(A\in\left(\wp\cup\{X-B\}\right)\to x\in A\right)\right)\right]$$

$$=\pi_1\mathbin{\dot\wedge}\left[\left(\Re\cup\{X-B\}\subseteq\mathcal{F}_\gamma\right)\mathbin{\dot\wedge}\text{fI}\left(\Re\cup\{X-B\}\right)\right]$$

$$\leq\left[(\exists x)(\forall A)\left(A\in\left(\Re\cup\{X-B\}\right)\to x\in A\right)\right]=\left[\neg\left(\bigcap\Re\subseteq B\right)\right].$$

Therefore

$$\pi_1\leq\inf_{\Re\in\Im(P(X))}\sup_{B\subseteq X}\left(\left(\Re\subseteq\mathcal{F}_\gamma\wedge B\in\tau_\gamma\right)\wedge(\forall\wp)\left((\wp\leq\Re)\mathbin{\dot\wedge}FF(\wp)\to\right.\right.$$
$$\left.\left.\neg\left(\bigcap\wp\subseteq B\right)\right)\to\neg\left(\bigcap\Re\subseteq B\right)\right)=\pi_2.$$

Conversely,

$$\pi_2\mathbin{\dot\wedge}\left[\left(\Re\subseteq\mathcal{F}_\gamma\right)\mathbin{\dot\wedge}\text{fI}(\Re)\right]=\pi_2\mathbin{\dot\wedge}\left[\left((\Re\setminus\{B\})\cup\{B\}\right)\subseteq\mathcal{F}_\gamma\right]\mathbin{\dot\wedge}\left[\text{fI}\left((\Re\setminus\{B\})\cup\{B\}\right)\right]$$

$$=\pi_2\mathbin{\dot\wedge}\left[\left(\Re'\subseteq\mathcal{F}_\gamma\right)\wedge\left(X-B\in\tau_\gamma\right)\mathbin{\dot\wedge}(\forall\wp)\left((\wp\leq\Re')\mathbin{\dot\wedge}FF(\wp)\to\right.\right.$$
$$\left.\left.(\exists x)(\forall A)\left(A\in\left(\wp\cup\{B\}\right)\to x\in A\right)\right)\right]$$

$$=\pi_2\mathbin{\dot\wedge}\left[\left(\Re'\subseteq\mathcal{F}_\gamma\right)\wedge\left(X-B\in\tau_\gamma\right)\mathbin{\dot\wedge}(\forall\wp)\left((\wp\leq\Re')\mathbin{\dot\wedge}FF(\wp)\to\right.\right.$$
$$\left.\left.\neg\left(\bigcap\wp\subseteq X-B\right)\right)\right]$$

$$\leq\left[\neg\left(\bigcap\Re'\subseteq X-B\right)\right]=\left[(\exists x)(\forall A)\left(\left(A\in\left(\Re'\cup\{B\}\right)\to(x\in A)\right)\right)\right]$$

$$=\left[(\exists x)(\forall A)\left(A\in\Re\to(x\in A)\right)\right].$$

Therefore

$$\pi_2 \le \inf_{\Re\in\Im(P(X))}\left[\left(\Re\subseteq\mathcal{F}_\gamma\right)\dot\wedge \text{fI}(\Re)\to(\exists x)(\forall A)\left(A\in\Re\to(x\in A)\right)\right]=\pi_1 .$$

5. Some properties of fuzzifying γ-compactness

Theorem 5.1. *For any fuzzifying topological space* (X,τ) *and* $A\subseteq X$,

$$\vDash \Gamma_\gamma(X,\tau)\dot\wedge A\in\mathcal{F}_\gamma\to\Gamma_\gamma(A).$$

Proof. For any $\Re\in\Im(P(A))$, we define $\bar{\Re}\in\Im(P(X))$ as follows:

$$\bar{\Re}(B)=\begin{cases}\Re(B) & \text{if } B\subseteq A,\\ 0 & \text{otherwise.}\end{cases}$$

Then $FF(\bar{\Re})=1-\inf\{\alpha\in[0,1]:F(\bar{\Re}_\alpha)\}=1-\inf\{\alpha\in[0,1]:F(\Re_\alpha)\}=FF(\Re)$ and

$$\begin{aligned}\sup_{x\in X}\inf_{x\notin B\subseteq X}\left(1-\bar{\Re}(B)\right)&=\sup_{x\in X}\left(\left(\inf_{x\notin B\subseteq A}\left(1-\bar{\Re}(B)\right)\right)\wedge\left(\inf_{x\notin B\nsubseteq A}\left(1-\bar{\Re}(B)\right)\right)\right)\\&=\sup_{x\in X}\left(\inf_{x\notin B\subseteq A}\left(1-\bar{\Re}(B)\right)\right)\wedge\sup_{x\in X}\left(\inf_{x\notin B\nsubseteq A}\left(1-\bar{\Re}(B)\right)\right)\\&=\sup_{x\in X}\left(\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)\\&=\sup_{x\in A}\left(\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)\vee\sup_{x\notin A}\left(\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)\end{aligned}$$

If $x\notin A$, then for any $x'\in A$ we have

$$\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)=\inf_{B\subseteq A}\left(1-\Re(B)\right)\le\inf_{x'\notin B\subseteq A}\left(1-\Re(B)\right).$$

Therefore, $\sup_{x\in X}\inf_{x\notin B\subseteq A}\left(1-\bar{\Re}(B)\right)=\sup_{x\in A}\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)$,

$$\begin{aligned}\left[\text{fI}(\bar{\Re})\right]&=\left[(\forall\bar{B})\left((\bar{B}\le\bar{\Re})\wedge FF(\bar{B})\to(\exists x)(\forall B)\left((B\in\bar{\Re})\to(x\in B)\right)\right)\right]\\&=\inf_{B\le\bar{\Re}}\min\left(1,1-FF(\bar{B})+\sup_{x\in X}\inf_{x\notin B\subseteq X}\left(1-\bar{\Re}(B)\right)\right)\\&=\inf_{B\le\Re}\min\left(1,1-FF(B)+\sup_{x\in A}\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)=\left[\text{fI}(\Re)\right].\end{aligned}$$

We want to prove that $\mathcal{F}_\gamma(A)\dot\wedge\left[\Re\subseteq\mathcal{F}_\gamma/_A\right]\le\left[\bar{\Re}\subseteq\mathcal{F}_\gamma\right]$. In fact, from Lemma 2.2 (3) we have

$$\begin{aligned}
\mathcal{F}_\gamma(A)\dot{\wedge}\left[\Re\subseteq\mathcal{F}_\gamma/_A\right]&=\max\left(0,\mathcal{F}_\gamma(A)+\inf_{B\subseteq A}\min\left(1,1-\Re(B)+\mathcal{F}_\gamma/_A(B)\right)-1\right)\\
&\le\inf_{B\subseteq A}\left(1-\Re(B)\right)+\left(\mathcal{F}_\gamma(A)+\mathcal{F}_\gamma/_A(B)-1\right)\\
&\le\inf_{B\subseteq A}\left(1-\Re(B)\right)+\left(\mathcal{F}_\gamma(A)\wedge\mathcal{F}_\gamma/_A(B)\right)\\
&=\inf_{B\subseteq A}\left(1-\Re(B)\right)+\left(\mathcal{F}_\gamma(A)\wedge\sup_{B'\cap A=B,B'\subseteq X}\mathcal{F}_\gamma(B')\right)\\
&=\inf_{B\subseteq A}\left(1-\Re(B)\right)+\sup_{B'\cap A=B,B'\subseteq X}\left(\mathcal{F}_\gamma(A)\wedge\mathcal{F}_\gamma(B')\right)\\
&=\inf_{B\subseteq A}\left(1-\Re(B)\right)+\sup_{B'\cap A=B,B'\subseteq X}\left(\mathcal{F}_\gamma(A\cap B')\right)\\
&=\inf_{B\subseteq A}\left(1-\Re(B)\right)+\mathcal{F}_\gamma(B)\\
&=\inf_{B\subseteq A}\min\left(1,1-\Re(B)+\mathcal{F}_\gamma(B)\right)\\
&=\inf_{B\subseteq A}\min\left(1,1-\bar{\Re}(B)+\mathcal{F}_\gamma(B)\right)=\left[\bar{\Re}\subseteq\mathcal{F}_\gamma\right].
\end{aligned}$$

Furthermore, from Theorem 4.3 we have

$$\begin{aligned}
\Gamma_\gamma(X,\tau)\dot{\wedge}\mathcal{F}_\gamma(A)\dot{\wedge}\left[\Re\subseteq\mathcal{F}_\gamma/_A\right]\dot{\wedge}\mathrm{fI}(\Re)&\le\Gamma_\gamma(X,\tau)\dot{\wedge}\left[\bar{\Re}\subseteq\mathcal{F}_\gamma\right]\dot{\wedge}\mathrm{fI}(\bar{\Re})\\
&\le\sup_{x\in X}\inf_{x\in B\subseteq A}\left(1-\bar{\Re}(B)\right)=\sup_{x\in A}\inf_{x\in B\subseteq A}\left(1-\Re(B)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma_\gamma(X,\tau)\dot{\wedge}\mathcal{F}_\gamma(A)&\le\left[\Re\subseteq\mathcal{F}_\gamma/_A\right]\dot{\wedge}\mathrm{fI}(\Re)\to\sup_{x\in A}\inf_{x\in B\subseteq A}\left(1-\Re(B)\right)\\
&\le\inf_{\Re\in\Im(P(A))}\left(\left[\Re\subseteq\mathcal{F}_\gamma/_A\right]\dot{\wedge}\mathrm{fI}(\Re)\to\sup_{x\in A}\inf_{x\in B\subseteq A}\left(1-\Re(B)\right)\right)=\Gamma_\gamma(A).
\end{aligned}$$

Theorem 5.2. *Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f\in Y^X$ is surjection. Then* $\models\Gamma_\gamma(X,\tau)\dot{\wedge}C_\gamma(f)\to\Gamma(f(X))$.

Proof. For any $\mathcal{B}\in\Im(P(Y))$, we define $\Re\in\Im(P(X))$ as follows:

$$\Re(A)=f^{-1}(\mathcal{B})(A)=\mathcal{B}(f(A)).$$

Then

$$\begin{aligned}
K(\Re,X)&=\inf_{x\in X}\sup_{x\in A}\Re(A)=\inf_{x\in X}\sup_{x\in A}\mathcal{B}(f(A))\\
&=\inf_{x\in X}\sup_{f(x)\in B}\mathcal{B}(B)=\inf_{y\in f(X)}\sup_{y\in B}\mathcal{B}(B)=K(\mathcal{B},f(X)),
\end{aligned}$$

$$
\begin{aligned}
[\mathcal{B}\subseteq\sigma]\wedge[C_\gamma(f)] &= \inf_{B\subseteq Y}\min(1,1-\mathcal{B}(B)+\sigma(B))\wedge\inf_{B\subseteq Y}\min\left(1,1-\sigma(B)+\tau_\gamma\left(f^{-1}(B)\right)\right)\\
&=\max\left(0,\inf_{B\subseteq Y}\min(1,1-\mathcal{B}(B)+\sigma(B))+\inf_{B\subseteq Y}\min\left(1,1-\sigma(B)+\tau_\gamma\left(f^{-1}(B)\right)\right)-1\right)\\
&\le\inf_{B\subseteq Y}\max\left(0,\min(1,1-\mathcal{B}(B)+\sigma(B))+\min\left(1,1-\sigma(B)+\tau_\gamma\left(f^{-1}(B)\right)\right)-1\right)\\
&\le\inf_{B\subseteq Y}\min\left(1,1-\mathcal{B}(B)+\tau_\gamma\left(f^{-1}(B)\right)\right)=\inf_{A\subseteq X}\inf_{f^{-1}(B)=A}\min\left(1,1-\mathcal{B}(B)+\tau_\gamma\left(f^{-1}(B)\right)\right)\\
&=\inf_{A\subseteq X}\inf_{f^{-1}(B)=A}\min\left(1,1-\mathcal{B}(B)+\tau_\gamma(A)\right)=\inf_{A\subseteq X}\min\left(1,1-\sup_{f^{-1}(B)=A}\mathcal{B}(B)+\tau_\gamma(A)\right)\\
&=\inf_{A\subseteq X}\min\left(1,1-\Re(A)+\tau_\gamma(A)\right)=\left[\Re\subseteq\tau_\gamma\right].
\end{aligned}
$$

For any $\wp\le\Re$, we set $\bar{\wp}\in\Im(P(Y))$ defined as follows:

$$\bar{\wp}(f(A))=f(\wp)(f(A))=\wp(A),\ A\subseteq X.$$

Then $\bar{\wp}(f(A))=f(\wp)(f(A))\le f(\Re)(f(A))=f\left(f^{-1}(\mathcal{B})(f(A))\le\mathcal{B}(f(A)),\right.$

$$
\begin{aligned}
FF(\wp)&=1-\inf\left\{\alpha\in[0,1]:F\left(\wp_{[\alpha]}\right)\right\}=1-\inf\left\{\alpha\in[0,1]:F\left(f(\wp)_{[\alpha]}\right)\right\}\\
&=FF(f(\wp)\le FF(\bar{\wp})
\end{aligned}
$$

and

$$
\begin{aligned}
K(\bar{\wp},f(X))&=\inf_{y\in f(X)}\sup_{y\in B}\bar{\wp}(B)=\inf_{y\in f(X)}\sup_{y\in B=f(A)}\wp(A)\ge\inf_{y\in f(X)}\sup_{f^{-1}(y)\in A}\wp(A)\\
&=\inf_{x\in X}\sup_{x\in A}\wp(A)=K(\wp,X).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\left[\Gamma_\gamma(X,\tau)\right]\dot{\wedge}\left[C_\gamma(f)\right]\dot{\wedge}\left[K'(\mathcal{B},f(X))\right]\\
&\quad=\left[\Gamma_\gamma(X,\tau)\right]\dot{\wedge}\left[C_\gamma(f)\right]\dot{\wedge}\left[K(\mathcal{B},f(X))\right]\dot{\wedge}\left[\mathcal{B}\subseteq\sigma\right]\\
&\quad\le\left[\Gamma_\gamma(X,\tau)\right]\dot{\wedge}\left[\Re\subseteq\tau_\gamma\right]\dot{\wedge}\left[K(\Re,X)\right]=\left[\Gamma_\gamma(X,\tau)\right]\dot{\wedge}\left[K_\gamma(\Re,X)\right]\\
&\quad\le\left[(\exists\wp)((\wp\le\Re)\wedge K(\wp,X)\dot{\wedge}FF(\wp))\right]\\
&\quad\le\left[(\exists\wp')((\wp'\le\Re)\wedge K(\wp',f(X))\dot{\wedge}FF(\wp'))\right],
\end{aligned}
$$

where K' is related to σ. Therefore, from Theorem 4.2 we obtain

$$
\begin{aligned}
&[\Gamma_\gamma(X,\tau)] \mathbin{\dot\wedge} [C_\gamma(f)] \\
&\quad \le K'(\mathcal{B}, f(X)) \to (\exists \wp')((\wp' \le \Re) \wedge K(\wp', f(X)) \mathbin{\dot\wedge} FF(\wp')) \\
&\quad \le \inf_{\mathcal{B}\in\Im(P(X))} \left(K'(\mathcal{B}, f(X)) \to (\exists \wp')((\wp' \le \Re) \wedge K(\wp', f(X)) \mathbin{\dot\wedge} FF(\wp'))\right) \\
&\quad = [\Gamma(f(X))].
\end{aligned}
$$

Theorem 5.3. *Let (X,τ) and (Y,σ) be any two fuzzifying topological space and $f \in Y^X$ is surjection.*

$$\models \Gamma_\gamma(X,\tau) \mathbin{\dot\wedge} I_\gamma(f) \to \Gamma_\gamma(f(X)).$$

Proof. From the proof of Theorem 5.2 we have for any $\mathcal{B} \in \Im(P(Y))$ we define $\Re \in \Im(P(X))$ as

$$\Re(A) = f^{-1}(\mathcal{B})(A) = \mathcal{B}(f(A)).$$

Then $K(\Re, X) = K(\mathcal{B}, f(X))$ and $[\mathcal{B} \subseteq \sigma_\gamma] \mathbin{\dot\wedge} [I_\gamma(f)] \le [\Re \subseteq \tau_\gamma]$. For any $\wp \le \Re$, we get $\bar\wp \in \Im(P(Y))$ defined as $\bar\wp(f(A)) = f(\wp)(f(A)) = \wp(A)$, $A \subseteq X$ and we have $FF(\wp) \le FF(\bar\wp)$, $K(\bar\wp, f(X)) \ge K(\wp, X)$. Therefore

$$
\begin{aligned}
&[\Gamma_\gamma(X,\tau)] \mathbin{\dot\wedge} [I_\gamma(f)] \mathbin{\dot\wedge} [K'_\gamma(\mathcal{B}, f(X))] \\
&\quad = [\Gamma_\gamma(X,\tau)] \mathbin{\dot\wedge} [I_\gamma(f)] \mathbin{\dot\wedge} [K(\mathcal{B}, f(X))] \mathbin{\dot\wedge} [\mathcal{B} \subseteq \sigma_\gamma] \\
&\quad \le [\Gamma_\gamma(X,\tau)] \mathbin{\dot\wedge} [\Re \subseteq \tau_\gamma] \mathbin{\dot\wedge} [K(\Re, X)] = [\Gamma_\gamma(X,\tau)] \mathbin{\dot\wedge} [K(\Re, X)] \\
&\quad \le [(\exists \wp)((\wp \le \Re) \wedge K(\wp, X) \mathbin{\dot\wedge} FF(\wp))] \\
&\quad \le [(\exists \wp)((\wp \le \Re) \wedge K(\bar\wp, f(X)) \mathbin{\dot\wedge} FF(\bar\wp))] \\
&\quad \le [(\exists \wp')((\wp' \le \mathcal{B}) \wedge K(\wp', f(X)) \mathbin{\dot\wedge} FF(\wp'))],
\end{aligned}
$$

where K'_γ is related to σ. Therefore, from Theorem 4.2 we obtain

$$
\begin{aligned}
&[\Gamma_\gamma(X,\tau)] \mathbin{\dot\wedge} [I_\gamma(f)] \\
&\quad \le K'_\gamma(\mathcal{B}, f(X)) \to (\exists \wp')((\wp' \le \mathcal{B}) \wedge K(\wp', f(X)) \mathbin{\dot\wedge} FF(\wp')) \\
&\quad \le \inf_{\mathcal{B}\in\Im(P(X))} \left(K'_\gamma(\mathcal{B}, f(X)) \to (\exists \wp')((\wp' \le \mathcal{B}) \wedge K(\wp', f(X)) \mathbin{\dot\wedge} FF(\wp'))\right)
\end{aligned}
$$

$=[\Gamma_\gamma(f(X))]$.

Theorem 5.4. *Let (X,τ) be any fuzzifying γ -topological space and $A,B\subseteq X$. Then*

(1) $T_2^\gamma(X,\tau)\wedge(\Gamma_\gamma(A)\wedge\Gamma_\gamma(B))\wedge A\cap B=\phi\models^{ws} T_2^\gamma(X,\tau)\to$

$$(\exists U)(\exists V)((U\in\tau_\gamma)\wedge(V\in\tau_\gamma)\wedge(A\subseteq U)\wedge(B\subseteq V)\wedge(U\cap V=\phi));$$

(2) $T_2^\gamma(X,\tau)\wedge\Gamma_\gamma(A)\models^{ws} T_2^\gamma(X,\tau)\to A\in\mathcal{F}_\gamma$.

Proof. (1) Assume $A\cap B=\phi$ and $T_2^\gamma(X,\tau)=t$. Let $x\in A$. Then for any $y\in B$ and $\lambda<t$, we have from Corollary 2.1 that

$$\begin{aligned}\sup\{\tau_\gamma(P)\wedge\tau_\gamma(Q): x\in P, y\in Q, P\cap Q=\phi\}\\ =\sup\{\tau_\gamma(P)\wedge\tau_\gamma(Q): x\in P\subseteq U, y\in Q\subseteq V, U\cap V=\phi\}\\ =\sup_{U\cap V=\phi}\left\{\sup_{x\in P\subseteq U}\tau_\gamma(P)\wedge\sup_{y\in Q\subseteq V}\tau_\gamma(Q)\right\}=\sup_{U\cap V=\phi}\{N_x^\gamma(U)\wedge N_y^\gamma(V)\}\\ \geq\inf_{x\neq y}\sup_{U\cap V=\phi}\{N_x^\gamma(U)\wedge N_y^\gamma(V)\}=T_2^\gamma(X,\tau)=t>\lambda,\text{ i.e.,}\end{aligned}$$

there exist P_y, Q_y such that $x\in P_y$, $y\in Q_y$, $P_y\cap Q_y=\phi$ and $\tau_P(P_y)>\lambda$, $\tau_P(Q_y)>\lambda$. Set $B(Q_y)=\tau_P(Q_y)$ for $y\in B$. Since $[B\subseteq\tau_\gamma]=1$, we have

$$[K_\gamma(B,B)]=[K(B,B)]=\inf_{y\in B}\sup_{y\in C}B(C)\geq\inf_{y\in B}B(Q_y)=\inf_{y\in B}\tau_P(Q_y)\geq\lambda.$$

On the other hand, since $T_2^\gamma(X,\tau)\wedge(\Gamma_\gamma(A)\wedge\Gamma_\gamma(B))>0$, then $1-t<\Gamma_\gamma(A)\wedge\Gamma_\gamma(B)\leq\Gamma_\gamma(A)$.

Therefore, for any $\lambda\in(1-\Gamma_\gamma(A),t)$, it holds that

$$\begin{aligned}1-\lambda<\Gamma_\gamma(A)\leq 1-[K_\gamma(B,B)]+\sup_{\wp\leq B}\{K(\wp,B)]\wedge FF(\wp)\}\\ \leq 1-\lambda+\sup_{\wp\leq B}\{K(\wp,B)]\wedge FF(\wp)\},\end{aligned}$$

i.e., $\sup_{\wp\leq B}\{K(\wp,B)]\wedge FF(\wp)\}>0$ and there exists $\wp\leq B$ such that $K(\wp,B)+FF(\wp)-1>0$, i.e., $1-FF(\wp)<K(\wp,B)$. Then, $\inf\{\theta:F(\wp_\theta)\}<K\ (\wp,B)$. Now, there exists θ_1 such that $\theta_1<K(\wp,B)$ and $F(\wp_{\theta_1})$. Since $\wp\leq B$, we may write $\wp_{\theta_1}=\{Q_{y_1},\cdots,Q_{y_n}\}$. We put $U_x=\{P_{y_1}\cap\cdots\cap P_{y_n}\}$, $V_x=\{Q_{y_1}\cap\cdots\cap Q_{y_n}\}$ and have

$V_x \supseteq B$, $U_x \cap V_x = \phi$, $\tau_\gamma(U_x) \geq \tau_\gamma(P_{y_1}) \wedge \cdots \wedge \tau_\gamma(P_{y_n}) > \lambda$ because (X, τ) is fuzziyfing γ-topological space. Also, $\tau_\gamma(V_x) \geq \tau_\gamma(Q_{y_1}) \wedge \cdots \wedge \tau_\gamma(Q_{y_n}) > \lambda$. In fact, $\inf_{y \in B} \sup_{y \in D} \wp(D) = K(\wp, B) > \theta_1$, and for any $y \in B$, there exists D such that $y \in D$ and $\wp(D) > \theta_1$, $D \in \wp_{\theta_1}$. Similarly, if $\lambda \in (1 - [\Gamma_\gamma(A) \wedge \Gamma_\gamma(B)], t)$, then we can find $x_1, \cdots x_m \in A$ with $U_\cdot = U_{x_1} \cup \cdots \cup U_{x_m} \supseteq A$. By putting $V_\cdot = V_{x_1} \cap \cdots \cap V_{x_m}$ we obtain $V_\cdot \supseteq B$, $U_\cdot \cap V_\cdot = \phi$ and

$$(\exists U)(\exists V)((U \in \tau_\gamma) \wedge (V \in \tau_\gamma) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V = \phi))$$

$$\geq \tau_\gamma(U_\cdot) \wedge \tau_\gamma(V_\cdot) \geq \min_{i=1,\cdots,n} \tau_\gamma(U_{x_i}) \wedge \min_{i=1,\cdots,n} \tau_\gamma(V_{x_i}) > \lambda.$$

Finally, we let $\lambda \to t$ and complete the proof.

(2) Assume $\models^{ws} T_2^\gamma(X, \tau) \mathbin{\dot\wedge} \Gamma_\gamma(A)$. For any $x \in X - A$ we have from (1)

$$\sup_{x \in U \subseteq X-A} \tau_\gamma(U) \geq \sup\{\tau_\gamma(U) \wedge \tau_\gamma(V) : x \in U, A \subseteq V, U \cap V = \phi\} \geq [T_2^\gamma(X, \tau)].$$

From Corollary 2.1, we obtain,

$$\mathcal{F}_\gamma(A) = \inf_{x \in X-A} N_x^\gamma(X - A) = \inf_{x \in X-A} \sup_{x \in U \subseteq X-A} \tau_\gamma(U) \geq [T_2^\gamma(X, \tau)].$$

Definition 5.1. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces. A unary fuzzy predicate $Q_\gamma \in \Im(Y^X)$, called fuzzifying γ-closedness, if given as follows:

$$Q_\gamma(f) := \forall B(B \in \mathcal{F}_\gamma^X \to f^{-1}(B) \in \mathcal{F}_\gamma^Y),$$

where $\mathcal{F}_\gamma^X$ and $\mathcal{F}_\gamma^Y$ are the fuzzy families of τ, σ - γ-closed in X and Y respectively.

Theorem 5.5. *Let (X, τ) a fuzzifying topological space, (Y, σ) be an γ-fuzzifying topological space and $f \in Y^X$. Then $\models \Gamma_\gamma(X, \tau) \mathbin{\dot\wedge} T_2^\gamma(Y, \sigma) \mathbin{\dot\wedge} I_\gamma(f) \to Q_\gamma(f)$.*

Proof. For any $A \subseteq X$, we have the following:

(i) From Theorem 5.1 we have $[\Gamma_\gamma(X, \tau) \mathbin{\dot\wedge} \mathcal{F}_\gamma^X(A)] \leq \Gamma_\gamma(A)$;

(ii) $I_\gamma(f_{/A}) = \inf_{U \in P(Y)} \min(1, 1 - \sigma_\gamma(U) + \tau_\gamma/_A((f/_A)^{-1}(U)))$

$= \inf_{U \in P(Y)} \min(1, 1 - \sigma_\gamma(U) + \tau_\gamma/_A(A \cap f^{-1}(U)))$

$$= \inf_{U \in P(Y)} \min\left(1, 1 - \sigma_\gamma(U) + \sup_{A \cap f^{-1}(U) = f(A)} \tau_\gamma(B)\right)$$

$$\geq \inf_{U \in P(Y)} \min\left(1, 1 - \sigma_\gamma(U) + \tau_\gamma\left(f^{-1}(U)\right)\right) = I_\gamma(f).$$

(iii) From Theorem 5.3, we have $\left[\Gamma_\gamma(A) \wedge I_\gamma(f_A)\right] \leq \Gamma_\gamma(f(A))$.

(iv) From Theorem 5.4 (2) we have $T_2^\gamma(Y,\sigma) \dot{\wedge} \Gamma_\gamma(f(A)) \models T_2^\gamma(Y,\sigma) \to f(A) \in \mathcal{F}_\gamma^Y$, which implies $\models T_2^\gamma(Y,\sigma) \dot{\wedge} \Gamma_\gamma(f(A)) \to f(A) \in \mathcal{F}_\gamma^Y$. By combining (i)-(iv) we have

$$\begin{aligned}\left[\Gamma_\gamma(X,\tau) \dot{\wedge} T_2^\gamma(Y,\sigma) \dot{\wedge} I_\gamma(f)\right] &\leq \left[\left(\mathcal{F}_\gamma^X(A) \to \Gamma_\gamma(A)\right) \dot{\wedge} I_\gamma(f_A) \dot{\wedge} T_2^\gamma(Y,\sigma)\right] \\ &\leq \left[\left(\mathcal{F}_\gamma^X(A) \to \left(\Gamma_\gamma(A)\right) \dot{\wedge} I_\gamma(f_A)\right)\right) \dot{\wedge} T_2^\gamma(Y,\sigma)\right] \\ &\leq \left[\mathcal{F}_\gamma^X(A) \to \Gamma_\gamma(f(A)) \dot{\wedge} T_2^\gamma(Y,\sigma)\right] \\ &\leq \left[\mathcal{F}_\gamma^X(A) \to \mathcal{F}_\gamma^Y(f(A))\right].\end{aligned}$$

Therefore

$$\begin{aligned}\left[\Gamma_\gamma(X,\tau) \dot{\wedge} T_2^\gamma(X,\tau) \dot{\wedge} I_\gamma(f)\right] &\leq \left[\mathcal{F}_\gamma^X(A) \to \mathcal{F}_\gamma^Y(f(A))\right] \\ &\leq \inf_{A \subseteq X}\left(\left[\mathcal{F}_\gamma^X(A) \to \mathcal{F}_\gamma^Y(f(A))\right]\right) = Q_\gamma(f).\end{aligned}$$

References

[1] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24 (1968), 182-190.

[2] J. A. Goguen, The fuzzy tychonoff theorem, *J. Math. Anal. Applc.*, 43 (1973), 182-190.

[3] I. M. Hanafy and H. S. Al-Saadi, Strong formas of continuity in fuzzy topological spaces, *Kyungpook Math. J.*, 41 (2001), 137-147.

[4] U. Höhle, Uppersemicontinuous fuzzy sets and applications, *J. Math. Anal. Appl.*, 78 (1980), 659-673.

[5] U. Höhle, Many valued topology and its application, Kluwer Academic Publishers, Dordrecht, (2001).

[6] U. Höhle and S. E. Rodabaugh, Mathematics of fuzzy sets, Logic, Topology, and Measure Theory, in : Handboolk of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, (1999).

[7] U. Höhle, S. E. Rodabaugh and A. Šostak, (Eds.), Special Issue on Fuzzy Topology, *Fuzzy Sets and Systems*, 73 (1995), 1-183.

[8] U. Höhle and A. Šostak, Axiomatic foundations of fixedbasis fuzzy topology, in : U. Höhle, S. E. Rodabaugh, (Eds.), Mathematics of fuzzy sets: Logic, Topology, and Measure Theory, in: Handbook of fuzzy sets series, vol. 3, *Kluwer academic Publishers, Dordrecht*, (1999), 123-272.

[9] J. L. Kelley, General Topology, Van Nostrand, New York, (1995).

[10] T. Kubiak, On fuzzy topologies, Ph. D. Thesis, Adam Mickiewicz University, Poznan, Poland, (1985).

[11] Y. M. Liu and M. K. Luo, Fuzzy topology, World Scientific, Singapore, (1998).

[12] T. Noiri and O. R. Sayed, Fuzzy γ -open sets and fuzzy γ -continuity in fuzzifying topology, *Sci. Math. Jpn.*, 55 (2002) (2), 255-263.

[13] S. E. Rodabaugh, Categorical foundations of variable-basis fuzzy topology, in: U. Höhle, S. E. Rodabaugh, (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, in: Handbook of Fuzzy Sets Series, vol. 3, *Kluwer Academic Publishers, Dordrecht*, (1999), 273-388.

[14] J. B. Rosser and A. R. Turquette, Many-Valued Logics, North-Holland, Amsterdam, *J. Fuzzy Math.* (to appear).

[15] O. R. Sayed, On fuzzy γ -separation Axioms in fuzzifying topology, (Submitted).

[16] G. J. Wang, Theory of L -fuzzy topological spaces, Shanxi Normal University Press, Xi an, (1988) (in Chinese).

[17] M. S. Ying, A new approach for fuzzy topology (I), *Fuzzy Sets and Systems*, 39 (1991), 303-321.

[18] M. S. Ying, A new approach for fuzzy topology (II), *Fuzzy Sets and Systems*, 47 (1992), 221-23.

[19] M. S. Ying, A new approach for fuzzy topology (III), *Fuzzy Sets and Systems*, 55 (1993), 193-207.

[20] M. S. Ying, Compactness in fuzzifying topology, *Fuzzy Ses and Systems*, 55 (1993), 79-92.

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