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Completely continuous functions and R-map in fuzzifying topological space

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Abstract

This paper considers fuzzifying topologies, a special case of I-fuzzy topologies introduced by Ying. The concepts of fuzzifying regular derived set, fuzzifying regular interior and fuzzifying regular convergence are studied and some results on above concepts are obtained. Also, the concepts of fuzzifying completely continuous functions and fuzzifying R-map are introduced and some important characterizations are obtained. Furthermore, some compositions of fuzzifying continuity with fuzzifying completely continuous functions and fuzzifying R-map are presented.

Keywords: Lukasiewicz logic; Fuzzifying topology; Fuzzifying regular open; Fuzzifying regular convergence; Completely continuous functions and R-map

1. Introduction

Fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [7,17,8,9,13,14,19,20]. In contrast to classical topology, fuzzy topology is endowed with richer structure, which, to a certain extent, is manifested by different ways to generalize certain classical concepts. So far, according to [8], the kind of topologies defined by Chang [3] and Goguen [5] comprises topologies of fuzzy subsets, which with the underlying carrier sets, are naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, an L-topological space is a carrier set along an L-topology, a family of its L-subsets (or fuzzy subsets) which satisfy the basic conditions analogous to those for classical topologies [11].

On the other hand, Hoehle in [6] proposed the terminology L-fuzzy topology for an L-valued mapping on the traditional powerset $P(X)$ of X satisfying certain conditions. The authors of [10,13–15,19] defined an L-fuzzy topology to be an L-valued mapping on the L-powerset L^X of X satisfying conditions analogous to those of [6].

Recently, with the semantical method of continuous-valued logic, Ying [21–23] defined so-called fuzzifying topologies. In fact fuzzifying topologies are a special case of the L-fuzzy topologies in [10,15] since all the t-norms on I are included as a special class of tensor products in these papers.

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Ying uses *one* particular tensor product, namely Lukasiewicz conjunction. Thus, his fuzzifying topologies are a special class of all the I-fuzzy topologies considered in the categorical frameworks of [10,15]. Particularly, as the author [21–24] indicated, by investigating fuzzifying topology we may partially answer an important question proposed by Rosser and Turquette [16] in 1952, which asked whether there are many valued theories beyond the level of predicate calculus.

Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies [1,12,17,18,24]. For example, the concept of regular open set, regular neighborhood system, regular closure and almost continuity in fuzzifying topological space were introduced and studied [18,24].

In classical mathematics, completely continuous functions and R-map [2,4] have been defined and their properties have been obtained.

The rest of the paper is organized as follows: in Section 2, we briefly present some concepts and results in fuzzifying topology, which are used in the sequel. Afterwards, in Section 3, in the framework of fuzzifying topology, the concepts of regular derived set, regular interior and regular boundary are established, some of their properties are discussed and some fundamental results in classical topology are generalized. In Section 4, we define the fuzzifying regular convergence of a net and describe the fuzzifying regular accumulation point, the fuzzifying regular closure of a set, and in fact the fuzzifying topology of a space in terms of it (Theorem 4.1 (1), (2) and (3)). Also, we prove that if a subnet of a net S is fuzzifying regular converges to a point x in fuzzifying topological space, then x is a fuzzifying regular accumulation point of S (Theorem 4.1 (4)). In Section 5, a theory of fuzzifying regular convergence has been built on the concept of filter. In Section 6, we define the fuzzifying completely continuous function and fuzzifying R-map between fuzzifying topological spaces. Also, we give a list of conditions (Definitions 6.2 and 6.4), each equivalent to fuzzifying completely continuous function and fuzzifying R-map (Theorems 6.1 and 6.2). In addition, some compositions of fuzzifying completely continuous functions and fuzzifying R-map with fuzzifying continuity are presented. In the last section, the main results obtained are briefly summarized, and a number of related directions are addressed for further study.

2. Preliminaries

In this section, we offer some concepts and results in fuzzifying topology, which will be used in the sequel. For the details, we refer to [21–24].

First, we display the Lukasiewicz logic and corresponding set theoretical notations used in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0, 1]$. We write $\vDash \varphi$ (a formula φ is valid) if and only if $[\varphi] = 1$ for every interpretation. The truth valuation rules for primary fuzzy logical formal and corresponding set theoretical rotations are:

(1) $[\alpha] := \alpha(\alpha \in [0, 1])$, $[\varphi \wedge \psi] := \min([\varphi], [\psi])$, $[\varphi \longrightarrow \psi] := \min(1, 1 - [\varphi] + [\psi])$.
(2) If $\tilde{A} \in \Im(X)$, where $\Im(X)$ is the family of all fuzzy subsets of X, then
$[x \in \tilde{A}] := \tilde{A}(x)$.
(3) If X is the universe of discourse, then
$[\forall x \varphi(x)] := \inf_{x\in X} [\varphi(x)]$.

In addition, the truth valuation rules for some derived formulae are

(1) $[\neg\varphi] := [\varphi \longrightarrow 0] = 1 - [\varphi]$,
(2) $[\varphi \longleftrightarrow \psi] := [(\varphi \longrightarrow \psi) \wedge (\psi \longrightarrow \varphi)]$,
(3) $[\varphi \dot{\wedge} \psi] := [\neg(\varphi \longrightarrow \neg\psi)] = \max(0, [\varphi] + [\psi] - 1)$,
(4) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] = \sup_{x\in X} [\varphi(x)]$,
(5) If $\tilde{A}, \tilde{B} \in \Im(X)$, then
 (a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x(x \in \tilde{A} \longrightarrow x \in \tilde{B})] = \inf_{x\in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x))$,
 (b) $[\tilde{A} \equiv \tilde{B}] := [(\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A})]$,
 (c) $[\tilde{A} \doteq \tilde{B}] := [(\tilde{A} \subseteq \tilde{B}) \dot{\wedge} (\tilde{B} \subseteq \tilde{A})]$.

Second, we give the following definitions and results in fuzzifying topology which are useful in the rest of the present paper.

Definition 2.1 (*Ying [21]*). Let X be a universe of discourse, $\tau \in \Im(P(X))$ satisfy the following conditions:

(1) $\tau(X) = \tau(\emptyset) = 1$,
(2) for any $A, B \in P(X)$, $\tau(A \cap B) \geqslant \tau(A) \wedge \tau(B)$,
(3) for any $\{A_\lambda \in P(X) : \lambda \in \Lambda\}$, $\tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geqslant \bigwedge_{\lambda\in\Lambda} \tau(A_\lambda)$.

Then τ is a fuzzifying topology and (X, τ) is a fuzzifying topological space.

Definition 2.2 (*Ying [21]*). The family of all fuzzifying closed sets, denoted by $F \in \Im(P(X))$, is defined as $A \in F := (X - A) \in \tau$, where $X - A$ is the complement of A.

Definition 2.3 (*Ying [21]*). Let $x \in X$. The neighborhood system of x, denoted by $N_x \in \Im(P(X))$, is defined as $N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$.

Definition 2.4 (*Lemma 5.2. [21]*). The closure $Cl(A)$ or $\bar{A}$ of A defined as
$Cl(A)(x) = 1 - N_x(X - A)$.

In Theorem 5.3 [21] Ying proved that the closure $Cl : P(X) \longrightarrow \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [21]) since its extension $Cl : \Im(X) \longrightarrow \Im(X)$, $Cl(\tilde{A}) = \bigcup_{\alpha\in[0,1]} \alpha Cl(\tilde{A}_\alpha)$, $\tilde{A} \in \Im(X)$, where $\tilde{A}_\alpha = \{x : \tilde{A}(x) \geqslant \alpha\}$ is the α-cut of A and $\alpha\tilde{A} = \alpha \wedge \tilde{A}(x)$ satisfies the following Kuratowski closure axioms:

(1) $\models Cl(\emptyset) = \emptyset$,
(2) for any $\tilde{A} \in \Im(X)$, $\models \tilde{A} \subseteq Cl(\tilde{A})$,
(3) for any $\tilde{A}, \tilde{B} \in \Im(X)$, $\models Cl(\tilde{A} \cup \tilde{B}) = Cl(\tilde{A}) \cup Cl(\tilde{B})$,
(4) for any $\tilde{A} \in \Im(X)$, $\models Cl(Cl(\tilde{A})) \subseteq Cl(\tilde{A})$.

Definition 2.5 (*Ying [22]*). For any $A \in P(X)$, the interior of A, denoted by $Int(A) \in \Im(P(X))$, is defined as $Int(A)(x) = N_x(A)$.

Definition 2.6 (*Khedr et al. [12]*). For any $\tilde{A} \in \Im(X)$, $\models Int(\tilde{A}) \equiv X - Cl(X - \tilde{A})$.

Lemma 2.1 (*Khedr et al. [12]*). *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then* (1) $\models Cl(\tilde{A}) \subseteq Cl(\tilde{B})$, *and* (2) $\models Int(\tilde{A}) \subseteq Int(\tilde{B})$.

Lemma 2.2 (*Khedr et al. [12]*). *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A} \in \Im(X)$,

(1) $\models X - Cl(Int(\tilde{A})) \equiv Int(Cl(X - \tilde{A}))$, *and*
(2) $\models X - Int(Cl(\tilde{A})) \equiv Cl(Int(X - \tilde{A}))$.

Lemma 2.3 (*Khedr et al. [12]*). *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then* $\models Cl(Int(\tilde{A})) \subseteq Cl(Int(\tilde{B}))$.

Definition 2.7 (*Ying [23]*). Let (X, τ) and (Y, σ) be two fuzzifying topological spaces. A unary fuzzy predicate $C \in \Im(Y^X)$ called fuzzifying continuity, is given as

$$C(f) := \forall U(U \in \sigma \longrightarrow f^{-1}(U) \in \tau).$$

Definition 2.8 (*Zahran [24]*). The family of all fuzzifying regular open sets, denoted by $\tau_R \in \Im(P(X))$, is defined as $A \in \tau_R := A \equiv Int(Cl(A))$, i.e.,

$$[A \in \tau_R] = \min\left(\inf_{x\in A} (Int(Cl(A)(x))),\ \inf_{x\in X-A} (1 - Int(Cl(A)(x)))\right).$$

Definition 2.9 (*Zahran [24]*). The family of all fuzzifying regular closed sets, denoted by $F_R \in \Im(P(X))$, is defined as $A \in F_R := (X - A) \in \tau_R$.

Definition 2.10 (*Zahran [24]*). Let $x \in X$. The fuzzifying regular neighborhood system of x is denoted by $N_x^R \in \Im(P(X))$ and defined as $A \in N_x^R := \exists B((B \in \tau_R) \wedge (x \in B \subseteq A))$, i.e., $N_x^R(A) = \sup_{x \in B \subseteq A} \tau_R(B)$.

Theorem 2.1 (*Zahran [24]*). *The mapping* $N^R : X \longrightarrow \Im^N(P(X)), x \longmapsto N_x^R$, *where* $\Im^N(P(X))$ *is the set of all normal fuzzy subsets of* $P(X)$ *has the following properties*:

(1) *for any* $x, A, \vDash A \in N_x^R \longrightarrow x \in A$,
(2) *for any* $x, A, B, \vDash A \subseteq B \longrightarrow (A \in N_x^R \longrightarrow B \in N_x^R)$,
(3) *for any* $x, A, \vDash A \in N_x^R \longrightarrow \exists C((C \in N_x^R) \wedge (C \subseteq A) \wedge \forall y(y \in C \longrightarrow C \in N_y^R))$.

Definition 2.11 (*Zahran [24]*). The fuzzifying regular closure, denoted by $Cl_R(A) \in \Im(X)$, is defined as $x \in Cl_R(A) := \forall B((B \supseteq A) \wedge (B \in F_R) \longrightarrow x \in B)$, i.e., $Cl_R(A)(x) = \inf_{x \notin B \supseteq A} (1 - F_R(B))$.

Lemma 2.4 (*Zahran [24]*). $Cl_R(A)(x) = 1 - N_x^R(X - A)$.

Definition 2.12 (*Ying [22]*). For any $A \subseteq X$, the boundary of A, denoted by $b(A)$, is defined as $x \in b(A) := (x \notin Int(A) \wedge x \notin Int(X - A))$, i.e., $b(A)(x) = \min(1 - Int(A)(x), 1 - Int(X - A)(x))$.

Lemma 2.5 (*Ying [22]*). $Cl(A) = A \cup b(A)$.

Definition 2.13. The family of all fuzzifying semi-open (resp., pre-open) sets is denoted by τ_S [12] (resp., τ_P [1]) $\in \Im(P(X))$, and defined as $A \in \tau_S = \inf_{x \in A} Cl(Int(A)(x))$ (resp., $A \in \tau_P = \inf_{x \in A} Int(Cl(A)(x))$).

Definition 2.14. The family of all fuzzifying semi-closed (resp., pre-closed) sets is denoted by F_S [12] (resp., F_P [1]) $\in \Im(P(X))$, and defined as $A \in F_S = (X - A) \in \tau_S$ (resp., $A \in F_P = (X - A) \in \tau_P$).

Lemma 2.6 (*Ying [23]*). *Let* $\tilde{C}, \tilde{D} \in \Im(Y)$ *and* $f \in Y^X$, *then* $\vDash (\tilde{C} \subseteq \tilde{D}) \longrightarrow (f^{-1}(\tilde{C}) \subseteq f^{-1}(\tilde{D}))$.

3. Regular open set

The following lemma gives the relationship between the regular open set, the pre-open set and the semi-open set in fuzzifying topology.

Lemma 3.1. *Let* (X, τ) *be a fuzzifying topological space. Then for any* A, *we have*

(1) $\vDash A \in \tau_R \longleftrightarrow (A \in \tau_P \wedge A \in F_S)$,
(2) $\vDash A \in F_R \longleftrightarrow A \equiv Cl(Int(A))$,
(3) $\vDash A \in F_R \longleftrightarrow (A \in \tau_S \wedge A \in F_P)$.

Proof. It is clear. □

Definition 3.1. Let (X, τ) be a fuzzifying topological space. The fuzzifying regular derived set of A, denoted by $d_R(A) \in \Im(X)$, is defined as $x \in d_R(A) := \forall B(B \in N_x^R \longrightarrow (B \cap (A - \{x\}) \neq \emptyset))$, i.e., $d_R(A)(x) = \inf_{B \cap (A - \{x\}) = \emptyset} (1 - N_x^R(B))$.

Lemma 3.2. (1) $\vDash d_R(A)(x) = 1 - N_x^R((X - A) \cup \{x\})$, *and*
(2) $\vDash d_R(\emptyset) \equiv \emptyset$.

Proof. (1) It is similar to the proof of Lemma 5.1 [21].
(2) From (1) above and since N_x^R is normal we have

$$d_R(\emptyset)(x) = 1 - N_x^R((X - \emptyset) \cup \{x\}) = 1 - N_x^R(X) = 1 - 1 = 0. \quad \square$$

Theorem 3.1. *For any* $A \in P(X), \vDash A \in F_R \longrightarrow d_R(A) \subseteq A$.

Proof.

$$\begin{aligned}[d_R(A) \subseteq A] &= \inf_{x\in X-A} (1 - d_R(A)(x)) \\ &= \inf_{x\in X-A} N_x^R((X-A)\cup\{x\}) \\ &= \inf_{x\in X-A} N_x^R((X-A)) \\ &= \inf_{x\in X-A} \sup_{x\in B\subseteq X-A} \tau_R(B) \geqslant \tau_R(X-A) = [A \in F_R]. \qquad \square\end{aligned}$$

Theorem 3.2. *For any x, A,*

(1) $\models Cl_R(\emptyset) \equiv \emptyset$,
(2) $\models A \subseteq Cl_R(A)$,
(3) $\models x \in Cl_R(A) \longleftrightarrow \forall B(B \in N_x^R \longrightarrow A \cap B \neq \emptyset)$,
(4) $\models Cl_R(A) \equiv A \cup d_R(A)$,
(5) $\models A \in F_R \longrightarrow A \equiv Cl_R(A)$,
(6) $\models B \doteq Cl_R(A) \longrightarrow B \in F_R$.

Proof.
(1) $Cl_R(\emptyset)(x) = 1 - N_x^R(X - \emptyset) = 1 - 1 = 0$.
(2) For any $A \in P(X)$ and $x \in X$, if $x \notin A$, then $N_x^R(A) = 0$.
If $x \in A$, then $Cl_R(A)(x) = 1 - N_x^R(X - A) = 1 - 0 = 1$. Then $[A \subseteq Cl_R(A)] = 1$.
(3), (4) are immediate from Lemma 2.4.
(5) From Theorem 3.1 and (4) above we have

$$\models A \in F_R \longrightarrow d_R(A) \subseteq A \longleftrightarrow A \equiv A \cup d_R(A) \longleftrightarrow A \equiv Cl_R(A).$$

(6) It is similar to the proof of Theorem 2.1 [22]. $\square$

Definition 3.2. Let (X, τ) be a fuzzifying topological space and $A \subseteq X$. The fuzzifying regular interior of $A \in P(X)$, denoted by $Int_R(A) \in \Im(X)$, is given as $x \in Int_R(A) := A \in N_x^R$, i.e., $Int_R(A)(x) = N_x^R(A)$.

Theorem 3.3. *Let (X, τ) be a fuzzifying topological space. Then for any x, A, B, we have*

(1) $\models Int_R(A) \equiv X - Cl_R(X - A)$,
(2) $\models Int_R(X) \equiv X$,
(3) $\models Int_R(A) \subseteq A$,
(4) $\models B \doteq Int_R(A) \longrightarrow B \in \tau_R$,
(5) $\models B \in \tau_R \wedge B \subseteq A \longrightarrow B \subseteq Int_R(A)$,
(6) $\models A \in \tau_R \longrightarrow A \equiv Int_R(A)$,
(7) $\models x \in Int_R(A) \longleftrightarrow (x \in A) \wedge (x \in (X - d_R(X - A)))$.

Proof.
(1) Follows from Lemma 2.4.
(2) $Int_R(X) = N_x^R(X) = 1$, because N_x^R is normal.
(3) Using Theorem 3.2 (2) and (1) above we have

$$Int_R(A)(x) \equiv (X - Cl_R(X - A))(x) \leqslant (X - (X - A))(x) = A(x).$$

(4) and (5) are similar to the proof of Theorem 2.2 (1) and (2) [22], respectively.
(6) From (3) above we have

$$\begin{aligned}[A \equiv Int_R(A)] &= \min\left(\inf_{x\in A} Int_R(A)(x), \inf_{x\in X-A} (1 - Int_R(A)(x))\right) \\ &= \inf_{x\in A} Int_R(A)(x) = \inf_{x\in A} N_x^R(A) \geqslant \tau_R(A) = [A \in \tau_R].\end{aligned}$$

(7) If $x \notin A$, then by Theorem 2.1 (1) $N_x^R(A) = 0$. Hence $[x \in Int_R(A)] = 0 = [(x \in A) \wedge (x \in (X - d_R(X - A)))]$. If $x \in A$, then $[x \in A] = 1$ So,

$$\begin{aligned}[(x \in A) \wedge (x \in (X - d_R(X - A)))] &= [1 - d_R(X - A)(x)] = [1 - (1 - N_x^R(A \cup \{x\}))] \\ &= [N_x^R(A)] = [x \in Int_R(A)]. \qquad \square\end{aligned}$$

Definition 3.3. For any $A \subseteq X$, the fuzzifying regular boundary set of A, denoted $b_R(A) \in \Im(X)$, is defined as $x \in b_R(A) := x \in Cl_R(A) \wedge x \in Cl_R(X - A)$, i.e.,

$$b_R(A)(x) = \min(Cl_R(A)(x), Cl_R(X - A)(x)).$$

Lemma 3.3. *For any x, A,*

$$\models x \in b_R(A) \longleftrightarrow (\forall B)(B \in N_x^R \longrightarrow (B \cap A \neq \emptyset) \wedge (B \cap (X - A)) \neq \emptyset).$$

Proof. It is similar to the proof of Lemma 2.1 [22]. $\square$

Theorem 3.4. *For any A,*

(1) $\models Cl_R(A) \equiv A \cup b_R(A)$ *and so* $\models A \in F_R \longrightarrow b_R(A) \subseteq A$,
(2) $\models Int_R(A) \equiv A \cap (X - b_R(A))$ *and so* $\models A \in \tau_R \longrightarrow b_R(A) \cap A \equiv \emptyset$.

Proof. (1) If $x \in A$, then from Theorem 3.2 (2) we have $[Cl_R(A)(x)] = [(A \cup b_R(A))(x)] = 1$.
If $x \notin A$, then

$$[(A \cup b_R(A))(x)] = [b_R(A)(x)] = \min(Cl_R(A)(x), Cl_R(X - A)(x)) = [Cl_R(A)(x)].$$

Therefore, from Theorems 3.1 and 3.2(4) we have

$$\begin{aligned}&\models A \in F_R \longrightarrow d_R(A) \subseteq A \longleftrightarrow (A \subseteq A) \wedge (d_R(A) \subseteq A) \longleftrightarrow (A \cup d_R(A) \subseteq A) \longleftrightarrow Cl_R(A) \subseteq A \\ &\qquad \longleftrightarrow (A \cup b_R(A)) \subseteq A \longleftrightarrow (A \subseteq A) \wedge (b_R(A) \subseteq A) \longleftrightarrow b_R(A) \subseteq A.\end{aligned}$$

(2) From Theorem 3.3(1) and (1) above we have

$$Int_R(A) = X - ((X - A) \cup b_R(X - A)) = A \cap (X - b_R(A)).$$

Also, from Theorem 3.3(6) we obtain

$$\models A \in \tau_R \longrightarrow Int_R(A) \equiv A \longleftrightarrow A \cap (X - b_R(A)) \equiv A \longleftrightarrow A \subseteq X - b_R(A) \longleftrightarrow b_R(A) \cap A \equiv \emptyset. \qquad \square$$

4. Regular convergence

Definition 4.1. Let (X, τ) be a fuzzifying topological space. The class of all nets in X is denoted by $N(X) = \{S | S : D \longrightarrow X$, where $(D, \geqslant)$ is a directed set$\}$.

Definition 4.2. The binary fuzzy predicates $\rhd^R, \propto^R \in \Im(N(X) \times X)$, are defined as follows:

$$S \rhd^R x := \forall A(A \in N_x^R \longrightarrow S \subsetsim A), \quad S \propto^R x := \forall A(A \in N_x^R \longrightarrow S \sqsubsetsim A), \quad S \in N(X),$$

where $[S \rhd^R x]$ stands for the degree to which S regular converges to x and $[S \propto^R x]$ stands for the degree to which x is a regular accumulation point of S.
Also, $\subsetsim$ and $\sqsubsetsim$ are the binary crisp predicates "almost in'' and "often in'', respectively.

Definition 4.3. The fuzzy sets,

$$\lim_R S(x) = [S \rhd^R x] \quad \text{and} \quad \operatorname*{adh}_R S(x) = [S \propto^R x]$$

are called regular limit and regular adherence sets of S, respectively.

Theorem 4.1. *For any x, A, S,*

(1) $\models \exists S((S \subseteq A - \{x\}) \wedge (S \rhd^R x)) \longrightarrow x \in d_R(A)$,
(2) $\models \exists S((S \subseteq A) \wedge (S \rhd^R x)) \longrightarrow x \in Cl_R(A)$,
(3) $\models A \in F_R \longrightarrow \forall S(S \subseteq A \longrightarrow \lim_R S \subseteq A)$,
(4) $\models \exists T((T < S) \wedge (T \rhd^R x)) \longrightarrow S \propto^R x$, *where $S \subseteq A$ and $T < S$ stand for S is all in A and T is a subnet of S, respectively.*

Proof. (1) We know that $[S \rhd^R x] = \inf_{S \not\gtrsim A}(1 - N_x^R(A))$. Also,

$$[\exists S((S \subseteq A - \{x\}) \wedge (S \rhd^R x))] = \sup_{S \in N(X),\ S \subseteq A - \{x\}} \ \inf_{B \in P(X),\ S \not\gtrsim B} (1 - N_x^R(B)).$$

Since for any $S \in N(X)$ such that $S \subseteq A - \{x\}$, one can prove that $S \not\gtrsim (X - A) \cup \{x\}$, as follows: suppose $S \gtrsim (X - A) \cup \{x\}$. Then there exist $m \in D$ and $n \in D$ such that $n \geqslant m$ and $S(n) \in (X - A) \cup \{x\}$. So, $S(n) \notin X - ((X - A) \cup \{x\}) = A - \{x\}$. Thus $S \not\subseteq A - \{x\}$. Therefore,

$$\begin{aligned}\sup_{S \in N(X),\ S \subseteq A - \{x\}} \ \inf_{B \in P(X),\ S \not\gtrsim B} (1 - N_x^R(B)) &\leqslant \sup_{S \in N(X),\ S \subseteq A - \{x\}} (1 - N_x^R((X - A) \cup \{x\})) \\ &= 1 - N_x^R((X - A) \cup \{x\}) = [x \in d_R(A)].\end{aligned}$$

(2) If $x \in A$, then the result holds. If $x \notin A$, then from (1) above and Theorem 3.2 (4) we have

$$\begin{aligned}[Cl_R(A)(x)] = [d_R(A)(x)] &\geqslant [\exists S((S \subseteq A - \{x\}) \wedge (S \rhd^R x))] \\ &= [\exists S((S \subseteq A) \wedge (S \rhd^R x))],\end{aligned}$$

because $A = A - \{x\}$.

(3) $\left[\forall S(S \subseteq A \longrightarrow \lim\limits_R S \subseteq A)\right] = \inf\limits_{S \subseteq A} \inf\limits_{x \in X - A} \left(1 - \inf\limits_{S \not\gtrsim B} (1 - N_x^R(B))\right) = \inf\limits_{S \subseteq A} \inf\limits_{x \in X - A} \sup\limits_{S \not\gtrsim B} N_x^R(B).$

So, from (2) above and Theorem 3.2 (5) we have

$$\begin{aligned}[A \in F_R] &\leqslant [A \equiv Cl_R(A)] = [Cl_R(A) \subseteq A] \\ &= [X - A \subseteq X - Cl_R(A)] = \inf_{x \in X - A} (1 - Cl_R(A)(x)) \\ &\leqslant \inf_{x \in X - A} \left(1 - \sup_{S \subseteq A} \inf_{S \not\gtrsim B} (1 - N_x^R(B))\right) \\ &= \inf_{x \in X - A} \inf_{S \subseteq A} \sup_{S \not\gtrsim B} N_x^R(B) \\ &= \left[\forall S\left(S \subseteq A \longrightarrow \lim_R S \subseteq A\right)\right].\end{aligned}$$

(4) Set $\Re_S = \{A : S \not\gtrsim A\}$ and $\beta_T = \{A : T \not\gtrsim A\}$. Then for any $T < S$ (for the definition of the subnet see [11]), one can deduce that $\Re_S \subseteq \beta_T$ as follows. Suppose $T = S \circ K$. If $S \not\gtrsim A$, then there exists $m \in D$ such that $S(n) \notin A$ when $n \geqslant m$, where $\geqslant$ directs the domain D of S. Now, we will show that $T \not\gtrsim A$. If not, then there exists $p \in E$ such that $T(q) \in A$ when $q \geqslant p$, where $\geqslant$ directs the domain E of T. Now, for $p \in E$ and $q \geqslant p$ we have $K(q) \geqslant m$, because

$T < S$. Moreover, since $S \not\gtrsim A$ and $K(q) \geqslant m$, we have $S(K(q)) \notin A$. But $S(K(q)) = T(q) \in A$. They are contrary. Hence, $\Re_S \subseteq \beta_T$. Therefore

$$\begin{aligned}[\exists T((T < S) \wedge (T \rhd^R x))] &= \sup_{T<S} \inf_{T \not\gtrsim A} (1 - N_x^R(A)) \\ &= \sup_{T<S} \inf_{A \in \beta_T} (1 - N_x^R(A)) \leqslant \inf_{A \in \Re_S} (1 - N_x^R(A)) \\ &= \inf_{S \not\gtrsim A} (1 - N_x^R(A)) = [S \propto^R x]. \qquad \square\end{aligned}$$

Lemma 4.1. $\models (S \rhd^R x) \longleftrightarrow \forall A(x \in A \in \tau_R \longrightarrow S \lessapprox A)$.

Proof. If $B \subseteq A$ and $S \not\gtrsim A$ then $S \not\gtrsim B$. Therefore

$$\begin{aligned}[S \rhd^R x] &= \inf_{S \not\gtrsim A} (1 - N_x^R(A)) = 1 - \sup_{S \not\gtrsim A} N_x^R(A) \\ &= 1 - \sup_{S \not\gtrsim A} \sup_{x \in B \subseteq A} \tau_R(B) \geqslant 1 - \sup_{S \not\gtrsim B,\, x \in B} \tau_R(B) \\ &= \inf_{S \not\gtrsim B,\, x \in B} (1 - \tau_R(B)) = [\forall B(x \in B \in \tau_R \longrightarrow S \lessapprox B)] \\ &= [\forall A(x \in A \in \tau_R \longrightarrow S \lessapprox A)].\end{aligned}$$

Conversely, since $N_x^R(A) \geqslant \tau_R(A)$, then we have

$$[\forall A(x \in A \in \tau_R \longrightarrow S \lessapprox A)] = \inf_{S \not\gtrsim A,\, x \in A} (1 - \tau_R(A)) \geqslant \inf_{S \not\gtrsim A} (1 - N_x^R(A)) = [S \rhd^R x]. \qquad \square$$

In the following theorem we prove that a universal net in a fuzzifying topological space regular converges to each of its fuzzifying regular accumulation points.

Theorem 4.2. *If S is a universal net, then*

$$\models \lim_R S \equiv \operatorname{adh}_R S.$$

Proof. For any net $S \in N(X)$ and any $A \in P(X)$ one can obtain that if $S \not\gtrsim A$, then $S \not\gtrsim A$. Suppose S is a universal net in X and $S \not\gtrsim A$. Then, $S \lessapprox X - A$. So, one can deduce that $S \not\gtrsim A$ because $S \lessapprox X - A$ if and only if there exists $m_1 \in D$ such that for every $n \in D, n \geqslant m_1$, $S(n) \in X - A$ if and only if there exists $m_1 \in D$ such that for every $n \in D, n \geqslant m_1$, $S(n) \notin A$ if and only if $S \not\gtrsim A$. Hence for any universal net S in X, we have

$$\lim_R S(x) = \inf_{S \not\gtrsim A} (1 - N_x^R(A)) = \inf_{S \not\gtrsim A} (1 - N_x^R(A)) = \operatorname{adh}_R S(x). \qquad \square$$

Theorem 4.3. *Let D and E_m be directed sets for each $m \in D$. Consider the directed set $H = D \times \prod_{m \in D} E_m$. Suppose that $\bar{S} = \{\bar{s}(m) : m \in D\} \in N(X)$, $S^{(m)} = \{s(m,n) : n \in E_m\} \in N(X)$ and $S \circ R(m,f) = \{S(m, f(m)) : (m,f) \in H\} \in N(X)$. Then,*

$$\models \forall m((m \in D) \longrightarrow (S^{(m)} \rhd^R \bar{S}(m))) \wedge (\bar{S} \rhd^R x) \longrightarrow S \circ R \rhd^R x.$$

Proof. From Lemma 4.1, we have

$$\begin{aligned}&[\forall m((m \in D) \longrightarrow (S^{(m)} \rhd^R \bar{S}(m))) \wedge (\bar{S} \rhd^R x)] \\ &\quad = \left(1 - \sup_{m \in D} \sup_{S^{(m)} \not\gtrsim A,\, \bar{S}(m) \in A} \tau_R(A)\right) \wedge \left(1 - \sup_{\bar{S} \not\gtrsim A,\, x \in A} \tau_R(A)\right) \\ &\quad = 1 - \left(\left(\sup_{m \in D} \sup_{S^{(m)} \not\gtrsim A,\, \bar{S}(m) \in A} \tau_R(A)\right) \vee \left(\sup_{\bar{S} \not\gtrsim A,\, x \in A} \tau_R(A)\right)\right).\end{aligned}$$

Also, we have $[S \circ R \rhd^R x] = 1 - \sup_{S\circ R \not\lesssim A,\, x\in A} \tau_R(A)$. Therefore, the proof is obtained if we can show that

$$1 - \left(\left(\sup_{m\in D} \sup_{S^{(m)}\not\lesssim A,\ \bar{S}(m)\in A} \tau_R(A)\right) \vee \left(\sup_{\bar{S}\not\lesssim A,\ x\in A} \tau_R(A)\right)\right) \leqslant 1 - \sup_{S\circ R \not\lesssim A,\ x\in A} \tau_R(A),$$

i.e.,

$$\left(\sup_{m\in D} \sup_{S^{(m)}\not\lesssim A,\ \bar{S}(m)\in A} \tau_R(A)\right) \vee \left(\sup_{\bar{S}\not\lesssim A,\ x\in A} \tau_R(A)\right) \geqslant \sup_{S\circ R \not\lesssim A,\ x\in A} \tau_R(A).$$

Suppose $\sup_{S\circ R \not\lesssim A,\, x\in A} \tau_R(A) > t$. Then there exists $A_\circ$ such that $x \in A_\circ,\ S \circ R \not\lesssim A_\circ$ and $\tau_R(A_\circ) > t$. Hence, for any $(m, f) \in H$ there exists $(n, g) \in H$ such that $(n, g) \geqslant (m, f)$ and $S \circ R(n, g) = S(n, g(n)) \notin A_\circ$. So, for any $m \in D,\ S^{(m)} \not\lesssim A_\circ$ and we have

Case 1. If there exists $m_\circ \in D$ such that $\bar{S}(m_\circ) \in A_\circ$, then

$$\left(\sup_{m\in D} \sup_{S^{(m)}\not\lesssim A,\ \bar{S}(m)\in A} \tau_R(A)\right) \vee \left(\sup_{\bar{S}\not\lesssim A,\ x\in A} \tau_R(A)\right) \geqslant \sup_{S^{(m_\circ)} \not\lesssim A,\ \bar{S}(m_\circ)\in A} \tau_R(A) \geqslant \tau_R(A_\circ) > t.$$

Case 2. If for any $m \in D,\ \bar{S}(m) \notin A_\circ$, then $\bar{S} \not\lesssim A_\circ$. Also,

$\sup_{\bar{S} \not\lesssim A,\ x\in A} \tau_R(A) \geqslant \tau_R(A_\circ) > t.$ Hence, we always have

$$\left(\sup_{m\in D} \sup_{S^{(m)}\not\lesssim A,\ \bar{S}(m)\in A} \tau_R(A)\right) \vee \left(\sup_{\bar{S}\not\lesssim A,\ x\in A} \tau_R(A)\right) > t. \qquad \square$$

5. Regular filter convergence

Definition 5.1. Let $F(X)$ be the set of all filters on X. The binary fuzzy predicates $\rhd^R, \propto^R \in \Im(F(X) \times X)$, are, respectively, defined as follows:

$K \rhd^R x := \forall A(A \in N_x^R \longrightarrow A \in K)$,
$K \propto^{\mathrm{R}} x := \forall A(A \in K \longrightarrow x \in Cl_R(A)), \quad K \in F(X)$.

Definition 5.2. The fuzzy sets,

$$\lim_R K(x) = [K \rhd^R x] \quad \text{and} \quad \operatorname*{adh}_R K(x) = [K \propto^R x].$$

are called regular limit and regular adherence sets of K, respectively.

Theorem 5.1. (1) *If $S \in N(X)$ and K^S is the filter corresponding to S, i.e., $K^S = \{A : S \lesssim A\}$, then*

(a) $\models \lim_R K^S = \lim_R S$, *and*
(b) $\models \mathrm{adh}_R K^S = \mathrm{adh}_R S$.

(2) *If $K \in F(X)$ and S^K is the net corresponding to K, i.e., $S^K : D \longrightarrow X,\ (x, A) \longmapsto x,\ (x, A) \in D$, where $D = \{(x, A) : x \in A \in K\}$, $(x, A) \geqslant (y, B)$ if and only if $A \subseteq B$, then*

(a) $\models \lim_R S^K = \lim_R K$, *and*
(b) $\models \mathrm{adh}_R S^K = \mathrm{adh}_R K$.

Proof. (1) For any $x \in X$, we have

(a)

$$\lim_R K^S(x) = \inf_{A\notin K^S} (1 - N_x^R(A)) = \inf_{S \not\lesssim A} (1 - N_x^R(A)) = \lim_R S(x).$$

(b)

$$\operatorname*{adh}_R K^S(x) = \inf_{A\in K^S} Cl_R(A)(x) = \inf_{S \precsim A} (1 - N_x^R(X-A))$$
$$= \inf_{S \not\precsim (X-A)} (1 - N_x^R(X-A)) = \operatorname*{adh}_R S(x).$$

(2) (a) First we prove that $S^K \precsim A$ if and only if $A \in K$. If $A \in K$, then $A \neq \emptyset$ and so there exists at least an element $x \in A$. So for $(x, A) \in D$ and $(y, B) \in D$ such that $(y, B) \geqslant (x, A)$, $B \subseteq A$ and so $S^K(y, B) = y \in B \subseteq A$. Thus $S^K \precsim A$.

Conversely, suppose $S^K \precsim A$. Then there exists $(y, B) \in D$ such that $(z, C) \geqslant (y, B)$ and we have $S^K(z, C) \in A$. So for every $z \in B$, $(z, B) \geqslant (y, B)$ and $S^K(z, B) = z \in A$ implies $B \subseteq A$. Then $A \in K$. Thus, $A \notin K$ if and only if $S^K \precsim A$. Now,

$$\operatorname*{lim}_R S^K(x) = [S^K \rhd^R x] = \inf_{S \not\precsim A} (1 - N_x^R(A)) = \inf_{A\notin K} (1 - N_x^R(A)) = \operatorname*{lim}_R K(x).$$

(b) First we prove that $X - A \in K$ if and only if $S^K \not\precsim A$. Suppose $S^K \not\precsim A$. Then there exists $(z, B) \in D$ such that for every $(y, C) \in D$ with $(y, C) \geqslant (z, B)$, $S^K(y, C) \notin A$. Now, for every $x \in B$, $(x, B) \geqslant (z, B)$ and $S^K(x, B) = x \notin A$, i.e., $B \cap A = \emptyset$ so $B \subseteq X - A$ and then $X - A \in K$.

Conversely, suppose $X - A \in K$ then $X - A \neq \emptyset$ and thus it contains at least an element x. Now, for any $(z, C) \in D$ such that $(z, C) \geqslant (x, X - A)$, one can have that $S^K(z, C) = z \notin A$. Hence, $S^K \not\precsim A$. Now,

$$\operatorname*{adh}_R S^K(x) = [S^K \propto^R x] = \inf_{S^K \not\precsim A} (1 - N_x^R(A)) = \inf_{X-A\in K} Cl_R(X - A) = \inf_{B\in K} Cl_R(B)(x) = \operatorname*{adh}_R K(x).$$

□

6. Completely continuous functions and R-map

Definition 6.1. Let (X, τ), (Y, σ) be two fuzzifying topological spaces. A unary fuzzy predicate $C_C \in \Im(Y^X)$ called fuzzifying completely continuous functions, is given as follows:

$$C_C(f) := (\forall U)(U \in \sigma \longrightarrow f^{-1}(U) \in \tau_R).$$

Intuitively, the degree to which *f* is fuzzifying completely continuous function is

$$[C_C(f)] = \inf_{U\subseteq Y} \min(1, 1 - \sigma(U) + \tau_R(f^{-1}(U))).$$

Definition 6.2. Let (X, τ), (Y, σ) be two fuzzifying topological spaces. For any $f \in Y^X$, we set

(1) $\alpha_1(f) = \forall B(B \in F_\sigma^Y \longrightarrow f^{-1}(B) \in F_R^X)$, where F_σ^Y is the set of all fuzzifying closed subset of Y and F_R^X is the set of all fuzzifying regular closed subset of X.

(2) $\alpha_2(f) = (\forall x)(\forall U)(U \in N_{f(x)} \longrightarrow f^{-1}(U) \in N_x^R)$, where $N_{f(x)}$ is the fuzzifying neighborhood system of $f(x)$ of Y and N_x^R is the fuzzifying regular neighborhood system of x of X.

(3) $\alpha_3(f) = (\forall x)(\forall U)(U \in N_{f(x)} \longrightarrow \exists V((f(V) \subseteq U) \wedge (V \in N_x^R)))$,

(4) $\alpha_4(f) = (\forall A)(f(Cl_R^X(A)) \subseteq Cl^Y(f(A)))$,

(5) $\alpha_5(f) = (\forall B)(Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B)))$,

(6) $\alpha_6(f) = (\forall A)(f^{-1}(Int(A)) \subseteq Int_R(f^{-1}(A)))$,

(7) $\alpha_7(f) = (\forall A)(f(b_R^X(A)) \subseteq f(A) \cup b(f(A)))$,

(8) $\alpha_8(f) = (\forall x)(\forall S)((S \in N(X)) \wedge (S \rhd^R x) \longrightarrow f \circ S \rhd f(x))$.

Theorem 6.1.

(1) $\models f \in C_C \longleftrightarrow f \in \alpha_1$,

(2) $\models f \in C_C \longrightarrow f \in \alpha_2$,

(3) $\models f \in \alpha_2 \longleftrightarrow f \in \alpha_i, i = 3, \ldots, 6$,

(4) $\models f \in \alpha_4 \longrightarrow f \in \alpha_7$,
(5) $\models f \in \alpha_2 \longrightarrow f \in \alpha_8$.

Proof. (1) We prove that $\models f \in C_C \longleftrightarrow f \in \alpha_1$

$$\begin{aligned}[f \in \alpha_1] &= \inf_{B \in P(Y)} \min(1, 1 - F_\sigma^Y(B) + F_R^X(f^{-1}(B)))\\ &= \inf_{B \in P(Y)} \min(1, 1 - \sigma(Y - B) + \tau_R(X - f^{-1}(B)))\\ &= \inf_{B \in P(Y)} \min(1, 1 - \sigma(Y - B) + \tau_R(f^{-1}(Y - B)))\\ &= \inf_{U \in P(Y)} \min(1, 1 - \sigma(U) + \tau_R(f^{-1}(U))) = [f \in C_C].\end{aligned}$$

(2) To prove that $f \in C_C \longrightarrow f \in \alpha_2$. Suppose that $N_{f(x)}(U) \leqslant N_x^R(f^{-1}(U))$. Then we obtain that $\min(1, 1 - N_{f(x)}(U) + N_x^R(f^{-1}(U))) = 1$. Therefore, the result holds. Now, suppose that $N_{f(x)}(U) > N_x^R(f^{-1}(U))$. We prove that

$$\min(1, 1 - N_{f(x)}(U) + N_x^R(f^{-1}(U))) \geqslant C_C(f).$$

If $f(x) \in A \subseteq U$, then $x \in f^{-1}(A) \subseteq f^{-1}(U)$. So

$$\begin{aligned}N_{f(x)}(U) - N_x^R(f^{-1}(U)) &= \sup_{f(x)\in A\subseteq U} \sigma(A) - \sup_{x\in B\subseteq f^{-1}(U)} \tau_R(B) \leqslant \sup_{f(x)\in A\subseteq U} \sigma(A) - \sup_{f(x)\in A\subseteq U} \tau_R(f^{-1}(A))\\ &\leqslant \sup_{f(x)\in A\subseteq U} (\sigma(A) - \tau_R(f^{-1}(A))).\end{aligned}$$

Then $1 - N_{f(x)}(U) + N_x^R(f^{-1}(U)) \geqslant \inf_{f(x)\in A\subseteq U} (1 - \sigma(A) + \tau_R(f^{-1}(A)))$.
So

$$\begin{aligned}\min(1, 1 - N_{f(x)}(U) + N_x^R(f^{-1}(U))) &\geqslant \inf_{f(x)\in A\subseteq U} \min(1, 1 - \sigma(A) + \tau_R(f^{-1}(A)))\\ &\geqslant \inf_{V \in P(Y)} \min(1, 1 - \sigma(V) + \tau_R(f^{-1}(V))) = C_C(f).\end{aligned}$$

Hence,

$$\inf_{x\in X} \inf_{U \in P(Y)} \min(1, 1 - N_{f(x)}(U) + N_x^R(f^{-1}(U))) \geqslant [f \in C_C].$$

(3) We will prove that $\models f \in \alpha_2 \longleftrightarrow f \in \alpha_3$. From Theorem 2.1 (2) we have

$$\sup_{V\in P(X), f(V)\subseteq U} N_x^R(V) = \sup_{V \in P(X), V\subseteq f^{-1}(U)} N_x^R(V) = N_x^R(f^{-1}(U)).$$

Then,

$$\begin{aligned}\alpha_3(f) &= \inf_{x\in X} \inf_{U\in P(Y)} \min(1, 1 - N_{f(x)}(U) + \sup_{V\in P(X), f(V)\subseteq U} N_x^R(V))\\ &= \inf_{x\in X} \inf_{U\in P(Y)} \min(1, 1 - N_{f(x)}(U) + N_x^R(f^{-1}(U)) = \alpha_2(f).\end{aligned}$$

(4) We will prove that $\models f \in \alpha_4 \longleftrightarrow f \in \alpha_5$. First, for any $B \subseteq Y$ one can deduce that

$$[f^{-1}(f(Cl_R^X(f^{-1}(B)))) \supseteq Cl_R^X(f^{-1}(B))] = 1, \quad [Cl^Y(f(f^{-1}(B))) \subseteq Cl^Y(B)] = 1$$

and $[f^{-1}(Cl^Y(f(f^{-1}(B)))) \subseteq f^{-1}(Cl^Y(B))] = 1$.
So, from Lemma 2.6 we have

$$\begin{aligned}[Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] &\geqslant [f^{-1}(f(Cl_R^X(f^{-1}(B)))) \subseteq f^{-1}(Cl^Y(B))]\\ &\geqslant [f^{-1}(f(Cl_R^X(f^{-1}(B)))) \subseteq f^{-1}(Cl^Y(f(f^{-1}(B))))]\\ &\geqslant [f(Cl_R^X(f^{-1}(B))) \subseteq Cl^Y(f(f^{-1}(B)))],\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_5(f) &= \inf_{B\in P(Y)} [Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))]\\ &\geqslant \inf_{B\in P(Y)} [f(Cl_R^X(f^{-1}(B))) \subseteq Cl^Y(f(f^{-1}(B)))]\\ &\geqslant \inf_{A\in P(X)} [f(Cl_R^X(A)) \subseteq Cl^Y(f(A))] = \alpha_4(f).\end{aligned}$$

Second, for each $A \subseteq X$, there exists $B \subseteq Y$, such that $f(A) = B$, and $f^{-1}(B) \supseteq A$. Hence, $[Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] \leqslant [Cl_R^X(A) \subseteq f^{-1}(Cl^Y(f(A)))]$. So,

$$\begin{aligned}\alpha_4(f) &= \inf_{A\in P(X)} [Cl_R^X(A) \subseteq f^{-1}(Cl^Y(f(A)))]\\ &\geqslant \inf_{B\in P(Y), B=f(A)} [Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))]\\ &\geqslant \inf_{B\in P(Y)} [Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))] = \alpha_5(f).\end{aligned}$$

(5) We prove that $\models f \in \alpha_5 \longleftrightarrow f \in \alpha_2$.

$$\begin{aligned}\alpha_5(f) &= \inf_{B\in P(Y)} [Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl^Y(B))]\\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-N_x^R(X-f^{-1}(B))) + (1-N_{f(x)}(Y-B)))\\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(Y-B) + N_x^R(f^{-1}(Y-B)))\\ &= \inf_{U\in P(Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(U) + N_x^R(f^{-1}(U))) = \alpha_2(f).\end{aligned}$$

(6) We prove that $\models f \in \alpha_6 \longleftrightarrow f \in \alpha_2$.

$$\begin{aligned}\alpha_6(f) &= \inf_{A\subseteq Y} \inf_{x\in X} \min(1, 1-Int(A)(f(x)) + Int_R(f^{-1}(A))(x))\\ &= \inf_{A\subseteq Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(A) + N_x^R(f^{-1}(A))) = \alpha_2(f).\end{aligned}$$

(7) From Lemma 2.5 we have

$$\begin{aligned}\alpha_7(f) &= [(\forall A)(f(b_R^X(A)) \subseteq f(A) \cup b(f(A)))]\\ &= [(\forall A)(f(Cl_R^X(A) \cap Cl_R^X(X-A)) \subseteq Cl^Y(f(A)))]\\ &\geqslant [(\forall A)(f(Cl_R^X(A)) \subseteq Cl^Y(f(A)))] = \alpha_4(f).\end{aligned}$$

(8) We prove that $[\alpha_2(f)] \leqslant [\alpha_8(f)]$, it suffices to show that for any $x \in X$ and $S \in N(X)$,

$$\min(1, 1-[S \rhd^R x] + [f \circ S \rhd f(x)]) \geqslant [\alpha_2(f)],$$

because

$$[\alpha_8(f)] = \inf_{x\in X} \inf_{S\in N(X)} \min(1, 1-[S \rhd^R x] + [f \circ S \rhd f(x)]).$$

In fact, if $[S \rhd^R x] \leqslant [f \circ S \rhd f(x)]$, it is obvious. Assume $[S \rhd^R x] > [f \circ S \rhd f(x)]$. Since $f \circ S \not\supseteq B$ implies $S \not\supseteq f^{-1}(B)$, then

$$\begin{aligned}[S \rhd^R x] - [f \circ S \rhd f(x)] &= \inf_{A\in P(X),\ S\not\supseteq A} (1-N_x^R(A)) - \inf_{B\in P(Y),\ f\circ S\not\supseteq B} (1-N_{f(x)}(B))\\ &\leqslant \inf_{B\in P(Y),\ f\circ S\not\supseteq B} (1-N_x^R(f^{-1}(B))) - \inf_{B\in P(Y),\ f\circ S\not\supseteq B} (1-N_{f(x)}(B))\\ &\leqslant \sup_{B\in P(Y),\ f\circ S\not\supseteq B} (N_{f(x)}(B) - N_x^R(f^{-1}(B))),\end{aligned}$$

So,

$$\min(1, 1-[S \rhd^R x]+[f\circ S \rhd f(x)]) \geqslant \inf_{B\in P(Y),\, f\circ S \subsetneq B} \min(1, 1-N_{f(x)}(B)+N_x^R(f^{-1}(B)))$$
$$\geqslant \inf_{x\in X}\ \inf_{U\in P(Y)} \min(1, 1-N_{f(x)}(U)+N_x^R(f^{-1}(U))) = [\alpha_2(f)].$$

□

Remark 6.1. In crisp setting, one can have $\models f \in C_C \longrightarrow f \in C$.

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 6.1. Let $X = \{a,b,c\}$ and τ be a fuzzifying topology on X defined as $\tau(X) = \tau(\emptyset) = \tau(\{a\}) = \tau(\{a,c\}) = 1$, $\tau(\{b\}) = \tau(\{a,b\}) = 0$ and $\tau(\{c\}) = \tau(\{b,c\}) = \frac{1}{8}$. Consider the identity function f from (X,τ) onto (X,σ) where σ is a fuzzifying topology on X defined as follows:

$$\sigma(A) = \begin{cases} 1, & A \in \{X, \emptyset, \{a,b\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

So, we have $[C_C(f)] = \frac{1}{8} > 0 = [C(f)]$.

Definition 6.3. Let (X,τ), (Y,σ) be two fuzzifying topological spaces. A unary fuzzy predicate $C_R \in \Im(Y^X)$ called fuzzifying R-map is given as follows:

$$C_R(f) := (\forall U)(U \in \sigma_R \longrightarrow f^{-1}(U) \in \tau_R).$$

Intuitively, the degree to which f is fuzzifying R_map is

$$[C_R(f)] = \inf_{U\subseteq Y} \min(1, 1-\sigma_R(U)+\tau_R(f^{-1}(U))).$$

Definition 6.4. Let (X,τ), (Y,σ) be two fuzzifying topological spaces. For any $f \in Y^X$, we set

(1) $\alpha_1(f) = \forall B(B \in F_R^Y \longrightarrow f^{-1}(B) \in F_R^X)$, where F_R^X and F_R^Y is the set of all fuzzifying regular closed subset of X and Y, respectively.

(2) $\alpha_2(f) = (\forall x)(\forall U)(U \in N_{f(x)}^R \longrightarrow f^{-1}(U) \in N_x^R)$, where N_x^R is fuzzifying regular neighborhood system of x of X and $N_{f(x)}^R$ is fuzzifying regular neighborhood system of $f(x)$ of Y.

(3) $\alpha_3(f) = (\forall x)(\forall U)(U \in N_{f(x)}^R \longrightarrow \exists V((f(V) \subseteq U) \wedge (V \in N_x^R)))$,

(4) $\alpha_4(f) = (\forall A)(f(Cl_R^X(A)) \subseteq Cl_R^Y(f(A)))$,

(5) $\alpha_5(f) = (\forall B)(Cl_R^X(f^{-1}(B)) \subseteq f^{-1}(Cl_R^Y(B)))$,

(6) $\alpha_6(f) = (\forall A)(f^{-1}(Int_R(A)) \subseteq Int_R(f^{-1}(A)))$,

(7) $\alpha_7(f) = (\forall A)(f(b_R^X(A)) \subseteq f(A) \cup b_R(f(A)))$,

(8) $\alpha_8(f) = (\forall x)(\forall S)((S \in N(X)) \wedge (S \rhd^R x) \longrightarrow f\circ S \rhd^R f(x))$.

Theorem 6.2.

(1) $\models f \in C_R \longleftrightarrow f \in \alpha_1$,

(2) $\models f \in C_R \longrightarrow f \in \alpha_2$,

(3) $\models f \in \alpha_2 \longleftrightarrow f \in \alpha_i$, $i = 3, \ldots, 6$,

(4) $\models f \in \alpha_4 \longrightarrow f \in \alpha_7$,

(5) $\models f \in \alpha_2 \longrightarrow f \in \alpha_8$.

Proof. It is similar to the proof of Theorem 6.1. □

Remark 6.2. In crisp setting, one can have $\models f \in C_C \longrightarrow f \in C_R$.

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 6.2. Let $X=\{a,b,c\}$ and τ be a fuzzifying topology on X defined as $\tau(X)=\tau(\emptyset)=1$, $\tau(\{a\})=\tau(\{a,b\})=\frac{1}{8}$, $\tau(\{b\})=\frac{1}{3}$ and $=\tau(\{c\})=\tau(\{a,c\})=\tau(\{b,c\})=\frac{1}{4}$. Consider the function f from (X,τ) onto (X,σ), where σ is a fuzzifying topology on X defined as $\sigma(X)=\sigma(\emptyset)=1$, $\sigma(\{a\})=\frac{1}{3}$, $\sigma(\{b\})=\sigma(\{c\})=\sigma(\{a,c\})=\sigma(\{b,c\})=\frac{1}{4}$ and $\sigma(\{a,b\})=\frac{1}{2}$. And the function f is defined as $f(a)=b,\ f(b)=c$ and $f(c)=a$.

So, we have $[C_C(f)]=\frac{5}{6}>\frac{3}{4}=[C_R(f)]$.

7. Decompositions of fuzzy continuity in fuzzifying topology

Theorem 7.1. *Let (X,τ), (Y,σ), (Z,ξ) be three fuzzifying topological spaces. For any $f\in Y^X$, $g\in Z^Y$,*

(1) $\models C_C(f)\longrightarrow(C(g)\longrightarrow C_C(g\circ f))$, *and*
(2) $\models C(g)\longrightarrow(C_C(f)\longrightarrow C_C(g\circ f))$.

Proof.
(1) If $[C(g)]\leqslant[C_C(g\circ f)]$, the result holds, if $[C(g)]>[C_C(g\circ f)]$, then

$$\begin{aligned}[C(g)]-[C_C(g\circ f)]&=\inf_{v\in P(Z)}\min(1,1-\xi(v)+\sigma(g^{-1}(v)))\\&\quad-\inf_{v\in P(Z)}\min(1,1-\xi(v)+\tau_R((g\circ f)^{-1}(v)))\\&\leqslant\sup_{v\in P(Z)}(\sigma(g^{-1}(v))-\tau_R((g\circ f)^{-1}(v)))\\&=\sup_{v\in P(Z)}(\sigma(g^{-1}(v))-\tau_R(f^{-1}(g^{-1}(v))))\\&\leqslant\sup_{u\in P(Y)}(\sigma(u)-\tau_R(f^{-1}(u)))\end{aligned}$$

Therefore,

$$\begin{aligned}[C(g)\longrightarrow C_C(g\circ f)]&=\min(1,1-[C(g)]+[C_C(g\circ f)])\\&\geqslant\inf_{u\in P(Y)}\min(1,1-\sigma(u)+\tau_R(f^{-1}(u)))=[C_C(f)].\end{aligned}$$

(2)

$$\begin{aligned}[C(g)\longrightarrow(C_C(f)\longrightarrow C_C(g\circ f))]&=[C(g)\longrightarrow\neg(C_C(f)\dot\wedge\neg(C_C(g\circ f)))]\\&=[\neg(C(g)\dot\wedge\neg\neg(C_C(f)\dot\wedge\neg(C_C(g\circ f))))]\\&=[\neg(C(g)\dot\wedge C_C(f)\dot\wedge\neg(C_C(g\circ f)))]\\&=[\neg(C_C(f)\dot\wedge C(g)\dot\wedge\neg(C_C(g\circ f)))]\\&=[\neg(C_C(f)\dot\wedge\neg\neg(C(g)\dot\wedge\neg(C_C(g\circ f))))]\\&=[C_C(f)\longrightarrow\neg(C(g)\dot\wedge\neg(C_C(g\circ f)))]\\&=[C_C(f)\longrightarrow(C(g)\longrightarrow C_C(g\circ f))].\qquad\square\end{aligned}$$

Theorem 7.2. *Let (X,τ), (Y,σ), (Z,ξ) be three fuzzifying topological spaces. For any $f\in Y^X$, $g\in Z^Y$,*

(1) $\models C_R(f)\longrightarrow(C(g)\longrightarrow C_R(g\circ f))$, *and*
(2) $\models C(g)\longrightarrow(C_R(f)\longrightarrow C_R(g\circ f))$.

Proof. It is similar to the proof of Theorem 7.1. □

8. Concluding remarks

This paper is a continuation of [18,24]. In the framework of fuzzifying topology, we introduced the concepts of regular derived set, regular interior, regular boundary, regular convergence, completely continuous function and R-map, studied some of their properties and some fundamental results in classical topology were generalized. In Theorem 4.3, we considered the class of all functions S such that $S(m, n)$ is defined whenever m belongs to a directed set D and n belongs to a directed set E_m and supposed that $S(m, n)$ is a member of a fuzzifying topological space, we could find a net R such that $S \circ R$ regular converges to existed iterated limit. We proved by counterexamples that some statements, which are true in classical topology, are not true in fuzzifying topology (counterexamples 6.1 and 6.2). To conclude, we hope to point out that another continuation of this paper is to deal with regular separation axioms and nearly compactness in fuzzifying topological spaces. Further, we hope to study this work and paper [24] in the framework of bifuzzy topological spaces mentioned in [21].

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