Annals of Fuzzy Sets, Fuzzy Logic and Fuzzy Systems
Volume …, Issue …, 2012, Pages …

QUASI TRIANGULAR NORMS BETWEEN BOOLEAN CONJUNCTION AND ANY OTHER QUASI TRIANGULAR NORM

O. R. SAYED

Department of Mathematics
Faculty of Science
Assiut University
Assiut 71516, Egypt
E-mail: o_r_sayed@yahoo.com

Abstract

From quasi t-norms and quasi t-conorms we give methods to generate new quasi t-norms and new quasi-t-conorms. Furthermore, between $\wedge$ (resp. $\vee$) and every quasi t-norm (resp. quasi t-conorm) f (resp. f^*) it is proved the existence of finite sequences of quasi t-norms (resp. quasi t-conorms) lies between $\wedge$ (resp. $\vee$) and f (resp. f^*). Finally, we give some special cases of complete residuated lattice valued-logic.

1. Introduction

In [6], it was showed a number of examples of existing and proposed t-norms and t-conorms and their pictorial representations were made with the aid of a computer. Moreover, averaging operators were summarized and their pictorial representations were made. In [7], the author proposed quasi t-norms and quasi t-conorms which are derived from t-(co)norms and do not necessarily satisfy the associativity. Also, compensatory operators were summarized and generalized compensatory operators were defined which could be obtained from averaging operators. Furthermore, selfdual operators were discussed which could be obtained by using t-norms, t-conorms and averaging operators. We remark that these methods are valid to generate quasi t-norms and quasi t-conorms from old quasi t-norms and quasi t-conorms. Furthermore, we give

2010 Mathematics Subject Classification: Please provide.

Keywords: Please provide.

Received April 2, 2011

methods to generate quasi t-norms from quasi t-norms and to generate quasi t-conorms from quasi t-conorms. We prove that between $\wedge$ and any quasi tnorm $f \neq \wedge,$ there exist denumerable number of finite sequences of quasi t-norms and between f^* and $\vee$, there exist denumerable number of finite sequences of quasi t-conorms. Finally, we point out that for each quasi t-norm f, $\langle I, \wedge, \vee, f, I_f^R \rangle$ is a complete residuated lattice valued-logic, where I_f^R is the residuated fuzzy implication induced by f.

Algebraically speaking, t-norms are binary operations on the closed unit interval $[0, 1]$ such that $([0, 1], T, \leq)$ is commutative, totally ordered semigroup with neutral element [2]. The term triangular norm appeared for the first time (with slightly different axioms) in [5]. The following set of independent axioms for triangular norms goes back to Schweizer and Sklar [12-16].

Definition 1.1. A triangular norm (briefly t-norm) is a binary operation T on the unit interval $[0, 1]$ which is commutative, associative, monotone and has 1 as neutral element, i.e., it is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $x, y, z \in [0, 1]$:

(T1) $T(x, y) = T(y, x)$,

(T2) $T(x, T(y, z)) = T(T(x, y), z)$,

(T3) $T(x, y) \leq T(x, z)$ whenever $y \leq z$,

(T4) $T(x, 1) = x$.

Since a t-norm is an algebraic operation on the unit interval $[0, 1]$, some authors (e.g., in [8]) prefer to use an infix notation like $x * y$ instead of the prefix notation $T(x, y)$. In fact, some of the axioms (T1)-(T4) then look more familiar: for all $x, y, z \in [0, 1]$:

(T1) $x * y = y * x$,

(T2) $x * (y * z) = (x * y) * z$,

(T3) $x * y \leq x * z$ whenever $y \leq z$,

(T4) $x * 1 = x$.

Throughout this paper, we shall consistently use both prefix and infix notations. Since t-norms are obviously extensions of the Boolean conjunction, they are usually used as interpretations of the conjunction $\wedge$ in $[0, 1]$-valued and fuzzy logics. There exist uncountable many t-norms. In [3, Section 4] some parameterized families of t-norms are presented which are interesting from different points of view. The following are the four basic t-norms [4], namely, the minimum T_M, the product T_P, the Łukasiewicz t-norm T_L, and the drastic product T_D, which are given, respectively:

$$T_M(x, y) = \min(x, y),$$

$$T_P(x, y) = x \cdot y,$$

$$T_L(x, y) = \max(0, x + y - 1), \text{ and}$$

$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[\times [0, 1[, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

These four basic t-norms are remarkable for several reasons. The drastic product T_D and the minimum T_M are the smallest and the largest t-norm, respectively (with respect to the pointwise order). The minimum T_M is the only t-norm where each $x \in [0, 1]$ is an idempotent element, whereas the product T_P and the Łukasiewicz t-norm T_L are prototypical examples of two important subclasses of t-norms, namely, of the classes of strict and nilpotent t-norms, respectively. It should be mentioned that the t-norms T_M, T_p, T_L, and T_D were denoted by M, $\prod$, W and Z, respectively, in [16].

The boundary condition (T4) and monotonicity (T3) were given in their minimal form. Together with (T1) it follows that, for all $x \in [0, 1]$, each t-norm satisfies:

$$T(0, x) = T(x, 0) = 0,$$

$$T(1, x) = x.$$

Therefore, all t-norms coincide on the boundary of the unit square $[0, 1] \times [0, 1]$. The monotonicity of a t-norm T in its second component (T3) is, together with commutativity (T1), equivalent to the (joint) monotonicity in both components, i.e., to $T(x_1, y_1) \leq T(x_2, y_2)$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

Definition 1.2. A function $f : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies, for all $x, y, z \in [0, 1]$, the properties (T1), (T3) and (T4) is called a *quasi-t-norm*.

Clearly, each t-norm is a quasi-t-norm, but not vice versa. For example, $Q(x, y) = xy(x + y - xy)$ is a quasi-t-norm, but it is not t-norm because it does not satisfy (T2). Also, all quasi-t-norms coincide on the boundary of the unit square $[0, 1] \times [0, 1]$. As we stated before, (T4) and (T3) together with (T1) implies $T(0, x) = T(x, 0) = 0, T(1, x) = x$. Furthermore, $T(x_1, y_1) \leq T(x_2, y_2)$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$. The authors in [4] gave the following definition.

Definition 1.3. If for two quasi-t-norms T_1 and T_2, we have $T(x, y) \leq T_2(x, y)$ for all $(x, y) \in [0, 1] \times [0, 1]$, we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1 and we write $T_1 \leq T_2$.

Since quasi-t-norms are just functions from the unit square into the unit interval, the comparison of quasi-t-norms is done in the usual way, i.e., pointwise. In [4] triangular conorms were introduced as dual operations of t-norms. The authors in [6] gave the following an independent axiomatic definition.

Definition 1.4. A triangular conorm (t-conorm for short) is a binary operation $S : [0, 1] \times [0, 1] \to [0, 1]$ satisfies for any $x, y, z \in [0, 1]$:

(S1) $S(x, 0) = x$ (existence of a unit 0);

(S2) $x' \leq x'' \Rightarrow S(x', y) \leq S(x'', y)$ (monotonicity);

(S3) $S(x, y) = S(y, x)$ (commutativity);

(S4) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity);

(S5) $S(1, x) = 1$ (existence of 1).

From an algebraic point of view, a t-conorm defines a semigroup on $[0, 1]$ with a unit 0 and a zero 1 and the semigroup operation is order preserving and commutative.

The following are the four basic t-conorms. The maximum S_M, the probabilistic sum S_P, the Lukasiewicz t-conorm or the bounded sum S_L, and the drastic sum S_D, which are given by, respectively.

$$S_M(x, y) = \max(x, y),$$

$$S_P(x, y) = x + y - x \cdot y,$$

$$S_L(x, y) = \min(x + y, 1), \text{ and}$$

$$S_D(x, y) = \begin{cases} 1 & \text{if } (x, y) \in]0, 1] \times]0, 1], \\ \max(x, y) & \text{otherwise.} \end{cases}$$

The t-conorms S_M, S_P, S_L and S_D were denoted by $M^*, \prod^*, W^*$ and Z^*, respectively, in [16].

Definition 1.5. The binary operations $f, f^* : [0, 1] \times [0, 1] \to [0, 1]$ are dual if and only if for every $(x, y) \in [0, 1] \times [0, 1]$, $f^*(x, y) = 1 - f(1 - x, 1 - y)$.

Obviously, $(T_M, S_M), (T_p, S_P), (T_L, S_L)$ and (T_D, S_D) are pairs of t-norms and t-conorms which are mutually dual to each other.

In fuzzy logics, t-conorms are usually used as an interpretation of the disjunction $\vee$.

Definition 1.6. A function $f^* : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies for any $x, y, z \in [0, 1]$, the properties (S1)-(S3) and (S5) is called a *quasi t-conorm*.

For convenience, we make some explanations and conventions of notation for complete residuated lattice-valued logic. For details, we refer to [1, 9-11].

Definition 1.7. A residuated lattice is an algebric structure $= \langle L, \leq, \vee, \wedge, \otimes, \rho, 0, 1\rangle$ such that:

(i) $\langle L, \leq, \vee, \wedge, 0, 1\rangle$ is a lattice with smallest element 0 and greatest element 1;

(ii) $\langle L, \otimes, 1\rangle$ is an abelian monoid;

(iii) The binary operation $\otimes$ is isotone in both variables. That is, for any $a_1, a_2 \in L$, if $a_1 \leq a_2$, then $a_1 \otimes b \leq a_2 \otimes b$, $b \otimes a_1 \leq b \otimes a_2$, $b \in L$.

(iv) The binary operation ρ is antitone in the first variable and isotone in the second one. That is, for any $a_1, a_2 \in L$ with $a_1 \leq a_2$, then $a_2 \rho b \leq a_1 \rho b$ and $b \rho a_1 \leq b \rho a_2$, $b \in L$.

(v) The adjunction condition $a \otimes b \leq c$ if and only if $a \leq b \rho c$ holds for every $a, b, c \in L$.

Note that:

(1) The binary operation $\otimes$ is interpreted as the product in L. Sometimes this operation is also called *many-valued conjunction*, strong conjunction or bold conjunction (to differentiate it from the lattice g. l. b. $\wedge$).

(2) The operation ρ is called *residum* (with respect to $\otimes$). From a logical point of view ρ denotes the implication connective.

(3) The pair $(\otimes, \rho)$ satisfying the adjunction property (v) is said to be an adjoin couple.

(4) Consider the structure $\langle L, \leq, \vee, \wedge, \otimes, 0, 1\rangle$, where $\otimes$ is the product in L. Define $I(a, b) = \vee\{x \mid a \otimes x \leq b\}$ for every $a, b \in L$. $I(a, b)$ is the residuated implication generated by the product $\otimes$. The pair $(\otimes, I)$ is an adjion couple.

2. Generated Quasi *t*-Norms and Quasi *t*-Conorms

The author in [6] gave many examples of t-norms and t-conorms of which we list the most interesting ones in the following table.

t-norm	t-conorm
(1) (Logical product) $x \wedge y = \min(x, y)$	(1)′ (Logical sum) $x \vee y = \max(x, y)$
(2) (Hamacher product) $x \boxdot = \dfrac{xy}{x + y - xy}$	(2)′ (Hamacher sum) $x \boxplus y = \dfrac{x + y - 2xy}{1 - xy}$
(3) (Algebraic product) $x \cdot y = xy$	(3)′ (Algebraic sum) $x + y = x + y - xy$
(4) (Einstein product) $x \mid\cdot\mid y = \dfrac{xy}{x + (1-x)(1-y)}$	(4)′ (Einstein sum) $x \mid+\mid y = \dfrac{x+y}{1+xy}$
(5) (bounded product) $x \odot y = 0 \vee (x + y - 1)$	(5)′ (bounded sum) $x \oplus y = 1 \wedge (x + y)$
(6) (drastic product) $x \wedge y = \begin{cases} x, & y = 1 \\ y, & x = 1 \\ 0, & x, y < 1 \end{cases}$	(6)′ (drastic sum) $x \dot{\vee} y = \begin{cases} x, & y = 0 \\ y, & x = 0 \\ 0, & x, y > 0 \end{cases}$
(7) (strict t-norm) $xT_7 y = \dfrac{2}{\pi}\cot^{-1}\left[\cot\dfrac{1}{2}\pi x + \cot\dfrac{1}{2}\pi y\right]$	(7)′ (strict t-conorm) $xS_7 y = \dfrac{2}{\pi}\tan^{-1}\left[\tan\dfrac{1}{2}\pi x + \tan\dfrac{1}{2}\pi y\right]$
(8) (nilpotent t-norm) $xT_8 y = \dfrac{2}{\pi}\sin^{-1}\left\{\left[\sin\dfrac{1}{2}\pi x + \sin\dfrac{1}{2}\pi y - 1\right] \vee 0\right\}$	(8)′ (nilpotent t-conorm) $xS_8 y = \dfrac{2}{\pi}\cos^{-1}\left\{\left[\cos\dfrac{1}{2}\pi x + \cos\dfrac{1}{2}\pi y - 1\right] \vee 0\right\}$
(9) (nilpotent t-conorm) $xT_9 y = \dfrac{2}{\pi}\cos^{-1}\left\{\left[\cos\dfrac{1}{2}\pi x + \cos\dfrac{1}{2}\pi y\right] \wedge 1\right\}$	(9)′ (nilpotent t-norm) $xS_9 y = \dfrac{2}{\pi}\sin^{-1}\left\{\left[\sin\dfrac{1}{2}\pi x + \sin\dfrac{1}{2}\pi y\right] \wedge 1\right\}$

In the following f^*, g^*, h^* are the dual of f, g, h.

Definition 2.1. For any $f, g, h : [0, 1] \times [0, 1] \to [0, 1]$, the binary operations $Q^f_{g,h}$, $Q^{*f}_{g,h}$: $[0, 1] \times [0, 1] \to [0, 1]$ are defined as follows:

$$Q^f_{g,h}(x, y) = f(g(x, y), h(x, y)), \; Q^{*f}_{g,h}(x, y) = f^*(g^*(x, y), h^*(x, y)).$$

Theorem 2.1. (1) $f \le g$ *if and only if* $f^* \ge g^*$.

(2) $Q^f_{g,h}$ *and* $Q^{*f}_{g,h}$ *are dual.*

(3) *If g and h are quasi t-norms, then* $Q^{\wedge}_{g,h}$ *and* $Q^{\vee}_{g,h}$ *are quasi t-norms.*

(4) *If g and h are quasi t-conorms, then* $Q^{\wedge}_{g,h}$ *and* $Q^{\vee}_{g,h}$ *are quasi t-conorms.*

(5) *If* f, g *and h are quasi t-norms, then* $Q^f_{g,h}$ *is quasi t-norm and* $Q^{*f}_{g,h}$ *is quasi t-conorms.*

Proof. (1) Suppose that $(x, y) \in [0, 1] \times [0, 1]$. Then $f \le g$ if and only if $f(x, y) \le g(x, y)$ if and only if $f(1 - x, 1 - y) \le g(1 - x, 1 - y)$ if and only if $1 - f(1 - x, 1 - y) \ge 1 - g(1 - x, 1 - y)$ if and only if $f^*(x, y) \ge g^*(x, y)$ if and only if $f^* \ge g^*$.

(2) $Q^f_{g,h}(x, y) = f(g(x, y), h(x, y))$

$$= 1 - f^*(1 - g(x, y), 1 - h(x, y))$$

$$= 1 - f^*(g^*(1 - x, 1 - y), h^*(1 - x, 1 - y))$$

$$= 1 - Q^{*f}_{g,h}(1 - x, 1 - y).$$

(3) We have $Q^{\wedge}_{g,h}(x, y) = g(x, y) \wedge h(x, y) = g(y, x) \wedge h(y, x) = Q^{\wedge}_{g,h}(x, y)$. Also, if $x_1 \le x_2$, $y_1 \le y_2$, then $Q^{\wedge}_{g,h}(x_1, y_1) = g(x_1, y_1) \wedge h(x_1, y_1) \le g(x_2, y_2) \wedge h(x_2, y_2) = Q^{\wedge}_{g,h}(x_2, y_2)$. Furthermore, $Q^{\wedge}_{g,h}(x, 1) = g(x, 1) \wedge h(x, 1) = x \wedge x = x$ and $Q^{\wedge}_{g,h}(x, 0) = g(x, 0) \wedge h(x, 0) = 0 \wedge 0 = 0$. Hence, $Q^{\wedge}_{g,h}$ is quasi t-norm. By a similar way, we prove that $Q^{\vee}_{g,h}$ is quasi t-norm.

(4) The commutativity and monotonicity of $Q^{\wedge}_{g,h}$ and $Q^{\vee}_{g,h}$ are easily shown. Now, $Q^{\wedge}_{g,h}(x, 1) = g(x, 1) \wedge h(x, 1) = 1 \wedge 1 = 1$; $Q^{\wedge}_{g,h}(x, 0) = g(x, 0) \wedge h(x, 0) = 0 \wedge 0 = 0$; $Q^{\vee}_{g,h}(x, 1) = g(x, 1) \vee h(x, 1) = 1 \vee 1 = 1$; $Q^{\vee}_{g,h}(x, 0) = g(x, 0) \vee h(x, 0) = 0 \vee 0 = 0$.

(5) (a) Since g and h are commutative and isotone one can have that $Q^{f}_{g,h}$ has the same properties. Now, $Q^{f}_{g,h}(x, 1) = f(g(x, 1), h(x, 1)) = f(x, 1) = x$ and $Q^{f}_{g,h}(x, 0) = f(g(x, 0), h(x, 1)) = f(0, x) = 0$.

(b) By a similar procedure to (a) we can prove that $Q^{*f}_{g,h}$ is a quasi t-conorm.

Remark 2.1. (1) Theorem 2.1(4) strengthen M. Mizumoto result which is given as follows:

For each t-norms f and g and for each t-conorm h, $Q^{f}_{g,h}$ is a quasi t-norm and $Q^{*f}_{g,h}$ is a quasi t-conorm.

(2) Theorem 2.1(2), (3) has a significant meaning if g and h are not comparable.

Theorem 2.2. (1) *For each commutative binary operation* $f : [0, 1] \times [0, 1] \to [0, 1]$ *we have,*

(a) $Q^{f}_{\wedge,\vee} = f$, (b) $Q^{*f}_{\wedge,\vee} = f^{*}$.

(2) *For each* $g, h : [0, 1] \times [0, 1] \to [0, 1]$ *such that* $g \leq h$ *we have,*

(a) $Q^{\wedge}_{g,h} = g$, (b) $Q^{*\wedge}_{g,h} = g^{*}$.

(3) *For each* $f, h : [0, 1] \times [0, 1] \to [0, 1]$ *satisfies the condition*: $f(x, 1) = x$; $f(0, x) = 0$; $g(x, 0) = g(0, x) = 0$, *then we have*

(a) $Q^{f}_{g,\dot{\vee}} \leq g$, (b) $Q^{*f}_{g,\dot{\vee}} \geq g^{*}$.

(4) *If* $f, h : [0, 1] \times [0, 1] \to [0, 1]$, *satisfies the condition*: $f(x, 1) = x$; $f(0, x) = 0$; $h(1, x) = h(x, 1) = 1$, *then we have*

(a) $Q^{f}_{\underset{\cdot}{\wedge},h} = \underset{\cdot}{\wedge}$, (b) $Q^{*f}_{\underset{\cdot}{\wedge},h} = \dot{\vee}$.

(5) *If* $f : [0, 1] \times [0, 1] \to [0, 1]$ *such that* $f(x, y) \leq x$, *then for every* $g, h \in I^{I \times I}$ *we have*

(a) $Q^{f}_{g,h} \leq g$; (b) $Q^{*f}_{g,h} \geq g^{*}$.

(6) *If* $f, h : [0, 1] \times [0, 1] \to [0, 1]$, *such that* $h(x, y) \geq y$, $h(x, y) \geq x$, f *is commutative and isotone in the second variable, then we have*

(a) $Q^{f}_{\wedge,h} \leq f$; (b) $Q^{*f}_{\wedge,h} = f^{*}$.

Proof. Since (b) in every statement follow from (a) it suffices to prove (a).

(1) (a) $Q^{f}_{\wedge,\vee}(x, y) = f(x \wedge y, x \vee y)$. If $x \leq y$, then $Q^{f}_{\wedge,\vee}(x, y) = f(x, y)$ and if $x \geq y$, then $Q^{f}_{\wedge,\vee}(x, y) = f(y, x) = f(x, y)$.

(2) (a) $Q^{\wedge}_{g,h}(x, y) = g(x, y) \wedge h(x, y) = g(x, y)$.

(3) (a) If $x, y > 0$, then $Q^{f}_{g,\dot{\vee}}(x, y) = f(g(x, y), x \dot{\vee} y) = f(g(x, y), 1) = g(x, y)$. If $x = 0$ or $y = 0$, then $Q^{f}_{g,\dot{\vee}}(x, y) = f(g(x, y), x \dot{\vee} y) = f(0, x \dot{\vee} y) = 0 = g(x, y)$.

(4) (a) If $x, y < 1$, then $Q^{f}_{\underset{\cdot}{\wedge},h}(x, y) = f(x \underset{\cdot}{\wedge} y, h(x, y)) = f(0, h(x, y)) = 0 = x \underset{\cdot}{\wedge} y$. If $x = 1$ or $y = 1$, then $Q^{f}_{\underset{\cdot}{\wedge},h}(x, y) = (x \underset{\cdot}{\wedge} y, 1) = x \underset{\cdot}{\wedge} y$.

(5) (a) $Q^{f}_{g,h}(x, y) = f(g(x, y), h(x, y)) \leq g(x, y)$.

(6) (a) If $x \leq y$, then $Q^{f}_{\wedge,h}(x, y) = f(x, h(x, y)) \geq f(x, y)$. If $x \geq y$, then $Q^{f}_{\wedge,h}(x, y) = f(y, h(x, y)) \geq f(y, x) = f(x, y)$.

Corollary 2.1 (1) *For each quasi t-norm* f *we have* (a) $Q^{f}_{\wedge,\vee} = f$; (b) $Q^{*f}_{\wedge,\vee} = f^{*}$.

(2) *For each quasi t-norm* g *and for each quasi t-conorm* h *we have* (a) $Q^{\wedge}_{g,h} = g$; (b) $Q^{*\wedge}_{g,h} = g^{*}$.

(3) *For any quasi t-norms* f, g *we have* (a) $Q^f_{g,\dot{\vee}} = g$; (b) $Q^f_{g,\dot{\vee}} = g^*$.

(4) *For each quasi t-norm f and for each quasi t-conorm h we have* (a) $Q^f_{\wedge,h} = \wedge$; (b) $Q^f_{\dot{\wedge},h} = \dot{\vee}$.

(5) *For each quasi t-norms* f, g *and for each quasi t-conorm h we have* (a) $Q^f_{g,h} \leq g$; (b) $Q^{*f}_{g,h} \geq g^*$.

(6) *For each quasi t-norm f and for each quasi t-conorm h we have* (a) $Q^f_{\wedge,h} \geq f$; (b) $Q^{*f}_{\wedge,h} = f^*$.

Proof. The proof is obtained from Theorem 2.2 (for (2) we remark that: $h(x, y) \geq x \vee y \geq x \wedge y \geq g(x, y)$, i.e. $g \leq h$).

3. Finite Sequences of Quasi *t*-Norms Lies Between $\wedge$ and any Quasi *t*-Norms $f \neq \wedge$

Theorem 3.1. *Let f be any quasi t-norm with* $f \neq \wedge$. *Then*

(a) $f = Q^f_{\wedge,\vee} \leq Q^f_{\wedge,\boxplus} \leq Q^f_{\wedge,S_7} \leq Q^f_{\wedge,\dot{+}} \leq Q^f_{\wedge,1+1} \leq Q^f_{\wedge,\oplus} \leq Q^f_{\wedge,S_9} \leq Q^f_{\wedge,\dot{\vee}} = \wedge \leq \vee$
$= Q^f_{\wedge,\dot{\vee}} \leq Q^{*f}_{\wedge,S_9} \leq Q^{*f}_{\wedge,\oplus} \leq Q^{*f}_{\wedge,1+1} \leq Q^{*f}_{\wedge,\dot{+}} \leq Q^{*f}_{\wedge,S_7} \leq Q^{*f}_{\wedge,\boxplus} \leq Q^{*f}_{\wedge,\vee} = f^*$.

(b) $f = Q^f_{\wedge,\vee} \leq Q^f_{\wedge,S_8} \leq Q^f_{\wedge,\oplus} \leq Q^{*f}_{\wedge,\otimes} \leq Q^{*f}_{\wedge,S_8} \leq Q^{*f}_{\wedge,\vee} = f^*$.

Proof. (a) It follows from Corollary 2.1(1), (3), Theorem 2.1(2) and since $\dot{\vee} \geq S_9 \geq \oplus \geq 1 + 1 \geq \dot{+} \geq S_7 \geq \boxplus \geq \vee$ (see [6]).

(b) It follows from Corollary 2.1(1), Theorem 2.1(2) and since $\oplus \geq S_8 \geq \vee$ (see [6]).

Theorem 3.2. *There exist denumerable number of finite sequences of quasi t-norms lies between* $\wedge$ *and any quasi t-norm* $f \neq \wedge$ *and there exist denumerable number of finite sequences of quasi t-conorms lies between any quasi t-conorm* $f^* \neq \vee$ and $\vee$.

Proof. The first finite sequence H_1 of quasi t-norms lies between $\wedge$ and f, and the first finite sequence M_1 of quasi t-conorms lies between $\vee$ and f^*

are indicated from Theorem 3.1(a) as follows: $H_1 = \langle Q^f_{\wedge, S_9}, Q^f_{\wedge,\oplus}, Q^f_{\wedge,1+1}, Q^f_{\wedge,\dot{+}}, Q^f_{\wedge, S_7}, Q^f_{\wedge,\boxplus}\rangle$ and $M_1 = \langle Q^{*f}_{\wedge,\boxplus}, Q^{*f}_{\wedge, S_7}, Q^{*f}_{\wedge,\dot{+}}, Q^{*f}_{\wedge,1+1}, Q^{*f}_{\wedge,\oplus}, Q^{*f}_{\wedge, S_9}\rangle$. It is clear that we use H_1 to construct M_1. Now, we use M_1 to construct H_2 and use H_2 to construct M_2 as follows: applying Corollary 2.1(5) (a), we obtain $f = Q^f_{\wedge,\vee} \leq Q^f_{\wedge, Q^{*f}_{\wedge, S_9}} \leq Q^f_{\wedge, Q^{*f}_{\wedge,\oplus}} \leq Q^f_{\wedge, Q^{*f}_{\wedge,1+1}} \leq Q^f_{\wedge, Q^{*f}_{\wedge,\dot{+}}} \leq Q^f_{\wedge, Q^{*f}_{\wedge, S_7}} \leq Q^f_{\wedge, Q^{*f}_{\wedge,\boxplus}} \leq Q^f_{\wedge, f^*} \leq \wedge \leq \vee \leq Q^{*f}_{\wedge, f^*} \leq Q^{*f}_{\wedge, Q^{*f}_{\wedge,\boxplus}} \leq Q^{*f}_{\wedge, Q^{*f}_{\wedge, S_7}} \leq Q^{*f}_{\wedge, Q^{*f}_{\wedge,\dot{+}}} \leq Q^{*f}_{\wedge, Q^{*f}_{\wedge,1+1}} \leq Q^{*f}_{\wedge, Q^{*f}_{\wedge,\oplus}} \leq Q^{*f}_{\wedge, Q^{*f}_{\wedge, S_9}} \leq Q^{*f}_{\wedge,\vee} = f^*$. So, $H_2 = \left\langle Q^f_{\wedge, f^*}, Q^f_{\wedge, Q^{*f}_{\wedge,\boxplus}}, Q^f_{\wedge, Q^{*f}_{\wedge, S_7}}, Q^f_{\wedge, Q^{*f}_{\wedge,\dot{+}}}, Q^f_{\wedge, Q^{*f}_{\wedge,1+1}},\right.$ and $M_2 = \left\langle Q^{*f}_{\wedge, Q^{*f}_{\wedge, S_9}}, Q^{*f}_{\wedge, Q^{*f}_{\wedge,\oplus}}, Q^{*f}_{\wedge, Q^{*f}_{\wedge,1+1}}, Q^{*f}_{\wedge, Q^{*f}_{\wedge,\dot{+}}}, Q^{*f}_{\wedge, Q^{*f}_{\wedge, S_7}}, Q^{*f}_{\wedge, Q^{*f}_{\wedge,\boxplus}}, Q^{*f}_{\wedge, *f}\right\rangle$.

Now, for each $n \in N$, there exists $n + 5$ quasi t-norms lies between $\wedge$ and f, and $n + 5$ quasi t-conorms lies between and f^* and $\vee$. It is clear that this statement is true if $n = 1$. Suppose that this statement is true for any n and we prove that this statement is true for $n + 1$. Suppose $H_n = \langle f_1, \ldots, f_{n+5}\rangle$ and $M_n = \langle f^*_{n+5}, \ldots, f^*_1\rangle$. Thus $f = Q^f_{\wedge,\vee} \leq Q^f_{\wedge, f^*_1} \leq \ldots \leq Q^f_{\wedge, f^*_{n+5}} \leq Q^f_{\wedge, f^*} \leq \wedge \leq \vee \leq Q^{*f}_{\wedge, f^*} \leq Q^{*f}_{\wedge, f^*_{n+5}} \leq \ldots \leq Q^{*f}_{\wedge, f^*_1} \leq f^*$. Then $H_{n+1} = \langle Q^{*f}_{\wedge, f^*}, Q^{*f}_{\wedge, f^*_{n+5}}, \ldots, Q^{*f}_{\wedge, f^*_1}\rangle$, $\operatorname{card}(M_{n+1}) = (n+1) + 5$, $M_{n+1} = \langle Q^{*f}_{\wedge, f^*_1}, \ldots, Q^{*f}_{\wedge, f^*_{n+5}}, \ldots, Q^{*f}_{\wedge, f^*}\rangle$.

Theorem 3.3. *For any quasi t-norms* f, $\langle I, \vee, \wedge, f, L^R_f\rangle$ *is a complete residuated lattice valued logic, where* L^R_f *is the residuated fuzzy implication induced by f and defined by*: $L^R_f(x, y) = \sup_{z \in I,\, f(x, z) \leq y} z$.

Proof. (1) $\langle I, \vee, \wedge\rangle$ is a complete lattice whose least and greatest element are 0 and 1, respectively.

(2) f is isotone.

(3) If $y_1 \leq y_2$, then $L^R_f(x, y_1) = \sup_{z \in I,\, f(x, z) \leq y_1} z \leq \sup_{z \in I,\, f(x, z) \leq y_2} z = L^R_f(x, y_2)$.

Thus L_f^R is isotone in the second variable.

(4) $\langle I, f, 1\rangle$ is a commutative monoid because f is commutative and for every $x \in I$, $f(x, 1) = x$.

(5) f and L_f^R are couple as $f(x, y) \leq z$ if and only if $x \leq L_f^R(y, x)$ because $f(x, y) \leq z$ if and only if $f(y, x) \leq z$ if and only if $x \leq L_f^R(y, x)$.

Conclusion

This paper is concerned with triangular norms (t-norms), an important notion in the semantics of fuzzy logic. Essentially, a triangular norm provides a semantic for fuzzy conjunction from which the other operators can be derived.

The paper shows the following results:

(1) Certain simple operations can be used to obtain new t-norms from existing ones.

(2) There are quasi t-norms between any quasi t-norm and the trivial norm $\min(x, y)$.

(3) A certain construction yields a complete residuated lattice valued logic.

References

[1] J. C. Foder, Contrapositive symmetry of fuzzy implications, Fuzzy Sets and Systems 69 (1995), 141-156.

[2] K. H. Hofmann and J. D. Lawson, Clifford's influence, in: K. H. Hofmann and M. W. Mislove, Editors, Semigroup Theory and its Applications, London Mathematics Society Lecture Notes, Volume 231, Cambridge University Press, Cambridge, 1966, pp. 15-39.

[3] E. P. Klement, R. Mesiar and E. Pap, Triangular norms, Kluwer Academic Publishers, Dordrecht, 2000.

[4] E. P. Klement, R. Mesiar and E. Pap, Triangular norms, position paper I: basic analytical and algebric properties, Fuzzy Sets and Systems 143 (2004), 5-26.

[5] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA 8 (1942), 535-537.

[6] M. Mizumoto, Pictorial representations of fuzzy connectives, part I, cases of t-norms, t-conorms and averaging operators, Fuzzy Sets and Systems 31 (1989), 217-242.

[7] M. Mizumoto, Pictorial representations of fuzzy connectives, part II, cases of compensatory operators and self-dual operators, Fuzzy Sets and Systems 32 (1989), 45-79.

[8] H. T. Nguyen and E. Walker, A first Course in Fuzzy Logic, CRC Press, Boca Raton, FL, 1997.

[9] J. Pavelka, On fuzzy logic I: Many-valued rules of inferences, Zeitschr. f. Math. Logik und Grundlagen d. Math. 25 (1979), 45-52.

[10] J. Pavelka, On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi, Zeitschr, f. Math. Logik und Grundlagen d. Math. 25 (1979), 119-134.

[11] J. Pavelka, On fuzzy logic III: Semantical completeness of some many-valued propositional calculi, Zeitschr, f. Math. Logik und Grundlagen d. Math. 25 (1979), 447-464.

[12] B. Schweizer and A. Sklar, Espaces me'triques ale'atoires, C. R. Acad. Sci. Paris Se'r. A 247 (1958), 2092-2094.

[13] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.

[14] B. Schweizer and A. Sklar, Associative functions and statistical triangle inqualities, Publ. Math. Debrecen 8(1961), 169-168.

[15] B. Schweizer and A. Sklar, Associative functions and abstract semigroups, Publ. Math. Debrecen 10 (1963), 69-81.

[16] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, 1983.

[17] M. S. Ying, Fuzzifying topology based on complete residuated lattice-valued logic(I), Fuzzy Sets and Systems 56 (1993), 337-373.