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Some covering properties in semantic method of continuous valued logic[1]

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Abstract. In this paper, some characterizations of fuzzifying semi-compactness are given, including characterizations in terms of nets and semi-subbases. Lastly, several characterizations of locally semi-compactness in the framework of fuzzifying topology are introduced and the mapping theorems are obtained.

Keywords: Łukasiewicz logic, semantics, fuzzifying topology, fuzzifying compactness, semi-compactness, fuzzifying locally compactness

1. Introduction and preliminaries

In 1952, Rosser and Turquette [12] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicate calculus, then what are the detail of such theories? As an attempt to give a partial answer to this problem in the case of point-set topology, M. S. Ying in 1991–1993 [16–18] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set X assigns to each crisp subset of X a certain degree of being open, other than being definitely open or not. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. There are already more than 100 papers in fuzzifying topology published in the last two decades, we guess. But only a few papers can properly use the semantic method introduced in the original papers of Ying, which we strongly believe, can provide more delicate characterization of fuzzifying topological structure. So far, there has been significant research on fuzzifying topologies [1, 6, 7, 13–15]. For example, Ying [19] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [15] the concept of local compactness in fuzzifying topology is introduced and some of its properties are established. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis is the study of variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of semi-open [9] sets introduced by N. Levine. The introduction of semi-open sets raised many basic general topological questions, which has thus far led to a productive study in which many new mathematical tools have been added to the

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general topology tool box, many new properties have been defined and examined, many new gems have been discovered for old properties, additional associated sets and associated topologies have been introduced, examined, and utilized, and, very importantly, additional basic general topological questions continue to arise. Semi-compactness (or s-compactness [11]) in General Topology was studied in [2–5]. We have no references for locally semi-compactness. In [6] the concepts of fuzzifying semi-open sets and fuzzifying semi-continuity were introduced and studied. Furthermore, two extensions of semi-open sets and semi-continuity in fuzzifying topology were introduced in [1]. Depending on these types of semi-open sets, four types of irresolute functions are introduced and studied in fuzzifying topology. Also, the authors in [7] introduced some concepts of fuzzifying semi-separation axioms and clarified the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms.

Based in the concept of semi-open set of N. Levine [9], this paper introduces its generalization for fuzzifying topology of M. S. Ying [16–18], and studies respective concepts of semi-(sub)base and (local) semi-compactness. Additionally, it provides the notion of fuzzifying irresolute map. Thus we fill a gap in the existing literature on fuzzifying topology. All of the contributions in General Topology in this paper which are not referenced may be original.

For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1] and the only designated value is 1. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. Also, $\Im(X)$ is the family of all fuzzy sets in X. The truth valuation rules for primary fuzzy logical formulae and corresponding set theoretical notations are:

(a) (i) $[\alpha] = \alpha(\alpha \in [0, 1])$;
 (ii) $[\varphi \wedge \psi] = \min([\varphi], [\psi])$;
 (iii) $[\varphi \to \psi] = \min(1, 1 - [\varphi] + [\psi])$.
(b) If $\widetilde{A} \in \Im(X)$, then $[x \in \widetilde{A}] := \widetilde{A}(x)$.
(c) If X is the universe of discourse, then $[\forall x \varphi(x)] := \bigwedge_{x \in X} [\varphi(x)]$.
 In addition the truth valuation rules for derived formulae are:

(a) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi]$;
(b) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi], [\psi])$;
(c) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;
(d) $[\varphi \otimes \psi] := [\neg(\varphi \to \neg\psi)] = \max(0, [\varphi] + [\psi] - 1)$.
 This means that $[\alpha] \leq [\varphi \to \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi]$;
(e) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \bigvee_{x \in X} [\varphi(x)]$;
(f) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then
 (i) $[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] = \bigwedge_{x \in X} \min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x))$;
 (ii) $[\widetilde{A} \equiv \widetilde{B}] := [\widetilde{A} \subseteq \widetilde{B}] \wedge [\widetilde{B} \subseteq \widetilde{A}]$.

We now give some definitions and results in fuzzifying topology, which are useful in the rest of the present paper.

Definition 1.1. [16]. Let X be a universe of discourse, and $\tau \in \Im(P(X))$ satisfy the following conditions:

(1) $\tau(X) = \tau(\emptyset) = 1$;
(2) $\forall A, B \in P(X), \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) $\forall \{A_\lambda \in P(X) : \lambda \in \Lambda\}, \tau(\bigcup_{\lambda \in \Lambda} A_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then τ is a fuzzifying topology and (X, τ) is a fuzzifying topological space.

Definition 1.2. [16]. (1) The family of all fuzzifying closed sets, denoted by $\digamma \in \Im(P(X))$, is defined as $A \in \digamma := (X - A) \in \tau$, where $X - A = A^c$ is the complement of A.

(2) The neighborhood system $N_x \in \Im(P(X))$ of $x \in X$ is defined as $N_x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$.

(3) The interior A° or $Int(A)$ of $A \subseteq X$ is defined as $Int(A)(x) = N_x(A)$.

(4) The closure $Cl(A)$ or $\bar{A}$ of A is defined as $Cl(A)(x) = 1 - N_x(X - A)$.

Definition 1.3. [6] (1) The family of all fuzzifying semi-open sets, denoted by $\tau_S \in \Im(P(X))$, is defined as follows: $A \in \tau_S := \forall x(x \in A \to x \in Cl(Int(A)))$, i.e., $\tau_S(A) = \bigwedge_{x \in A} Cl(Int(A))(x)$.

(2) The family of all fuzzifying semi-closed sets, denoted by $\digamma_S \in \Im(P(X))$, is defined as $A \in \digamma_S := X - A \in \tau_S$.

(3) The fuzzifying semi-neighborhood system of a point $x \in X$ is denoted by $N_x^{S^X}$ (or N_x^S) $\in \Im(P(X))$ and defined as $N_x^S(A) = \bigvee_{x \in B \subseteq A} \tau_S(B)$.

(4) The fuzzifying semi-closure of a set $A \subseteq X$, denoted by $Cl_S \in \Im(X)$, is defined as $Cl_S(A)(x) = 1 - N_x^S(X - A)$.

(5) If (X, τ) and (Y, σ) are two fuzzifying topological spaces and $f \in Y^X$, the unary fuzzy predicate $C_S \in \Im(Y^X)$, called fuzzifying semi-continuity, is given as $C_S(f) := \forall B(B \in \sigma \to f^{-1}(B) \in \tau_S)$. Intuitively, the

degree to which f is semi-continuous is $[C_S(f)] = \bigwedge_{B\subseteq X} \min(1, 1 - \sigma(B) + \tau_S(f^{-1}(B)))$.

Definition 1.4. [12] If (X, τ) is a fuzzifying topological space and $N(X)$ is the class of all nets in X, then the binary fuzzy predicates $\rhd^S$, $\propto^S \in \Im(N(X) \times X)$ are defined as $T \rhd^S x := \forall A(A \in N_x^{S^X} \rightarrow T \tilde{\subset} A)$, $T \propto^S x := \forall A(A \in N_x^{S^X} \rightarrow T \tilde{\sqsubset} A)$, where "$T \rhd^S x$", "$T \propto^S x$" stand for "$T$ semi-converges to x", "x is a semi-accumulation point of T", respectively; and "$\tilde{\subset}$", "$\tilde{\sqsubset}$" are the binary crisp predicates "almost in","often in", respectively. The degree to which x is a semi-adherence point of T is $adh_S T(x) = [T \propto^S x]$.

In the following, we always assume Ω be the class of all fuzzifying topological spaces.

Definition 1.5. [7] A unary fuzzy predicate $T_2^S \in \Im(\Omega)$, called fuzzifying semi-Hausdorffness, is given as follows:

$$T_2^S(X, \tau) = \forall x \forall y((x \in X \wedge y \in X \wedge x \neq y) \rightarrow$$
$$\exists B \exists C(B \in N_x^S \wedge C \in N_y^S \wedge B \cap C = \phi)), i.e.,$$
$$[T_2^S(X, \tau)] = \bigwedge_{x,y\in X, x\neq y} \bigvee_{B,C\in P(X), B\cap C=\phi} (N_x^S(B), N_y^S(C)).$$

Definition 1.6. [19] (1) A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness, is given as follows: $\Gamma(X, \tau) := (\forall \Re)(K_\circ(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \otimes FF(\wp)))$ and if $A \subseteq X$, then $\Gamma(A) := \Gamma(A, \tau/A)$. For K, $K_\circ$ (resp. $\leq$ and FF) see [16, Definition 4.4] (resp. [16, Theorem 4.3] and [19, Definition 1.1 and Lemma 1.1]).

(2) A unary fuzzy predicate $fI \in \Im(\Im(P(X)))$, called fuzzy finite intersection property, is given as $fI(\Re) := \forall \wp((\wp \leq \Re) \wedge FF(\wp) \rightarrow \exists x \forall B(B \in \wp \rightarrow x \in B))$.

Definition 1.7. [14]. (1) A fuzzifying topological space (X, τ) is said to a be fuzzifying S-topological space if $\tau_S(A \cap B) \geq \tau_S(A) \wedge \tau_S(B)$.

(2) A binary fuzzy predicate $K_S \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying semi-open covering, is given as $K_S(\Re, A) := K(\Re, A) \otimes (\Re \subseteq \tau_S)$.

(3) A unary fuzzy predicate $\Gamma_S \in \Im(\Omega)$, called fuzzifying semi-compactness, is given as follows: $(X, \tau) \in \Gamma_S := (\forall \Re)(K_S(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \otimes FF(\wp)))$ and if $A \subseteq X$, then $\Gamma_S(A) := \Gamma_S(A, \tau/A)$.

Definition 1.8. [14]. A unary fuzzy predicate $LC \in \Im(\Omega)$, called fuzzifying local compactness, is given as follows: $(X, \tau) \in LC := (\forall x)(\exists B)((x \in Int(B) \otimes \Gamma(B, \tau/B))$, i.e.,

$$LC(X, \tau) = \bigwedge_{x\in X} \bigvee_{B\subseteq X} \max(0, N_x^X(B) + \Gamma(B, \tau/B) - 1).$$

2. Fuzzifying semi-base and semi-subbase

Definition 2.1. Let (X, τ) be a fuzzifying topological space and $\beta_S \subseteq \tau_S$. Then β_S is called a semi-base of τ_S if β_S fulfils the condition: $\models A \in N_x^{S^X} \rightarrow \exists B((B \in \beta_S) \wedge (x \in B \subseteq A))$.

Example 2.2. Let $X = \{a, b, c\}$, and $I = [0, 1]$. Define a mapping $\tau : P(X) \longrightarrow I$ on X as follows: $\tau(\emptyset) = \tau(X) = 1$, $\tau(\{a, c\}) = 0$, $\tau(\{a, b\}) = \frac{1}{5}$, $\tau(\{b, c\}) = \frac{1}{2}$, $\tau(\{a\}) = 0$, $\tau(\{b\}) = \frac{3}{4}$, $\tau(\{c\}) = \frac{1}{2}$. Then we can easily verify that τ is a fuzzifying topology. By calculating, $\tau_S(\emptyset) = \tau_S(X) = 1$, $\tau_S(\{a, c\}) = \frac{1}{2}$, $\tau_S(\{a, b\}) = \frac{3}{4}$, $\tau_S(\{b, c\}) = \frac{1}{2}$, $\tau_S(\{a\}) = 0$, $\tau_S(\{b\}) = \frac{3}{4}$, $\tau_S(\{c\}) = \frac{1}{2}$. If we set $\beta_S = \tau_S$, then $N_x^S(A) = \bigvee_{x\in B\subseteq A} \tau_S(B) = \bigvee_{x\in B\subseteq A} \beta_S(B)$ by Definition 1.6(3). Obviously, β_S is a semi-base of τ_S by Definition 2.1.

The proof of the following two theorems is easy, and we omit it.

Theorem 2.3. *Let (X, τ) be a fuzzifying topological space and $\beta_S \subseteq \tau_S$, then β_S is a semi-base of τ_S if and only if $\tau_S = \beta_S^{(\cup)}$, where $\beta_S^{(\cup)}(A) = \bigvee_{\bigcup_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \beta_S(B_\lambda)$, and Λ is an index set.*

Theorem 2.4. *Let $\beta_S \in \Im(P(X))$. Then β_S is a semi-base for some fuzzifying S-topology τ_S if and only if it has the following properties:*

(1) $\beta_S^{(\cup)}(X) = 1$;
(2) $\models (A \in \beta_S) \wedge (B \in \beta_S) \wedge (x \in A \cap B) \rightarrow \exists C((C \in \beta_S) \wedge (x \in C \subseteq A \cap B))$.

Definition 2.5. $\varphi_S \in \Im(P(X))$ is called a semi-subbase of τ_S if $\varphi_S^{(\cap)}$ is a semi-base of τ_S, where $\varphi_S^{(\cap)}$ is the finite intersection-extension of φ_S in Zadeh's sense, i.e.,
$$\varphi_S^{(\cap)}(A) = \bigvee_{\bigcap_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \varphi_S(B_\lambda), \quad \{B_\lambda : \lambda \in \Lambda\} \Subset P(X),$$
with " $\Subset$ " standing for "a finite subset of".

Theorem 2.6. *$\varphi_S \in \Im(P(X))$ is a semi-subbase of some fuzzifying S-topology if and only if $\varphi_S^{(\cap)}(X) = 1$.*

Proof. We easily demonstrate that $\varphi_S^{\Cap}$ satisfies the second condition of Theorem 2.4, and others are obvious. □

3. Fuzzifying irresolute mappings

The purpose of this section is to introduce and study the concept of irresolute mappings in fuzzifying topological spaces.

Definition 3.1. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I \in \Im(Y^X)$, called fuzzifying irresoluteness, is given as follows:

$I(f) := \forall B(B \in \sigma_S \to f^{-1}(B) \in \tau_S)$.

From [6, Theorem 3.3 (1) (a)] we have $\sigma(B) \leq \sigma_S(B)$, and so we have $\models f \in I \to f \in C_S$.

Example 3.2. Let (X, τ) and τ_S are defined just as in Example 2.2, we have that $\tau(B) \leq \tau_S(B)$ $(\forall B \in P(X))$. Let $Y = \{d\}$, and $I = [0, 1]$. Define a mapping $\varsigma : P(Y) \longrightarrow I$ on Y as follows: $\varsigma(\emptyset) = \varsigma(Y) = 1$, then ς is a fuzzifying topology and $\varsigma_S(\emptyset) = \varsigma_S(Y) = 1$. Now, define a mapping $f \in X^Y$ by $f(d) = a$. Clearly, $\varsigma_S(f^{-1}(B)) = 1 \geq \tau_S(B) \geq \tau(B)$ $(\forall B \in P(X))$, so we have $\models f \in I \to f \in C_S$.

Theorem 3.3. *Let (X, τ), (Y, σ) and (Z, ν) be three fuzzifying topological spaces and let $f \in Y^X$ and $g \in Z^Y$. Then*

(1) $\models I(f) \to (C_S(g) \to C_S(g \circ f))$, (2) $\models C_S(g) \to (I(f) \to C_S(g \circ f))$.

Proof. (1) It suffices to show that $[I(f)] \leq [C_S(g) \to C_S(g \circ f)]$. If $[C_S(g)] \leq [C_S(g \circ f)]$, the results holds. If $[C_S(g)] \geq [C_S(g \circ f)]$, then

$$\begin{aligned}
&[C_S(g)] - [C_S(g \circ f)] \\
&= \bigwedge_{V \in P(Z)} \min\left(1, 1 - \nu(V) + \sigma_S(g^{-1}(V))\right) \\
&\quad - \bigwedge_{V \in P(Z)} \min\left(1, 1 - \nu(V) + \tau_S(f^{-1}(g^{-1}(V)))\right) \\
&\leq \bigvee_{V \in P(Z)} (\sigma_S(g^{-1}(V)) - \tau_S(f^{-1}(g^{-1}(V)))).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[C_S(g) \to C_S(g \circ f)] \\
&= \min(1, 1 - [C_S(g)] + [C_S(g \circ f)]) \\
&\geq \bigwedge_{U \in P(Y)} \min(1, 1 - \sigma_S(U) + \tau_S(f^{-1}(U))) \\
&= [I(f)].
\end{aligned}$$

(2) Follows from (1) and the fact that $[\alpha] \leq [\varphi \to \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi]$.

Definition 3.4. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predicates $\omega_k \in \Im(Y^X)$, where $k = 1, ..., 5$, as follows:

(1) $f \in \omega_1 = \forall B\left(B \in F_S^Y \to f^{-1}(B) \in F_S^X\right)$, where F_S^X and F_S^Y are the fuzzifying semi-closed subsets of X and Y, respectively;

(2) $f \in \omega_2 = \forall x \forall U(U \in N_{f(x)}^{S^Y} \to f^{-1}(U) \in N_x^{S^X})$, where N^{S^X} and N^{S^Y} are the family of fuzzifying semi-neighborhood systems of X and Y, respectively;

(3) $f \in \omega_3 = \forall x \forall U(U \in N_{f(x)}^{S^Y} \to \exists V(f(V) \subseteq U \to V \in N_x^{S^X}))$;

(4) $f \in \omega_4 = \forall A\left(f\left(Cl_S^X(A)\right) \subseteq Cl_S^Y(f(A))\right)$;

(5) $f \in \omega_5 = \forall B(Cl_S^X(f^{-1}(B)) \subseteq f^{-1}(Cl_S^Y(B)))$.

Theorem 3.5. $\models f \in I \leftrightarrow f \in \omega_k,\ k = 1, ..., 5$.

Proof. The proof is similar to that of Theorem 7.2 in [1].

Theorem 3.6. $\models f \in I \to \forall x \forall T(T \in N(X) \wedge T \rhd^S x \to f \circ T \rhd^S f(x))$.

Proof. From Theorem 3.5, the result holds if we prove that $[\forall x \forall T(T \in N(X) \wedge T \rhd^S x \to f \circ T \rhd^S f(x))] \geq \omega_3(f)$. So, it suffices to show that for any $x \in X$ and $T \in N(X)$,

$$\min(1, 1 - [T \rhd^S x] + [f \circ T \rhd^S f(x)]) \geq \omega_3(f).$$

The concrete proof is easy, and we omit it. □

4. Fuzzifying semi-compact spaces

Theorem 4.1. *Let (X, τ) and (Y, σ) be any two fuzzifying topological spaces and let $f \in Y^X$ be a surjection. Then*

(1) $\models \Gamma_S(X, \tau) \otimes C_S(f) \longrightarrow \Gamma(f(X))$,

(2) $\models \Gamma_S(X, \tau) \otimes I(f) \longrightarrow \Gamma_S(f(X))$.

Proof. The proofs of (1) and (2) are the same as for Theorems 4.2 and 4.3 in [14], respectively. □

The above theorem is a generalization of the following corollary [2, Corollary 3.1].

Corollary 4.2. *Let $(X, \tau), (Y, \sigma)$ be two topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a surjective mapping. If f is semi-continuous (resp. irresolute) and X is semi-compact, then Y is compact (resp. semi-compact).*

Definition 4.3. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces. A unary fuzzy predicate $Q_S \in \Im(Y^X)$, called fuzzifying semi-closedness [14], is given as follows:

$$Q_S(f) := \forall B(B \in F_S^Y \to f^{-1}(B) \in F_S^X),$$

where F_S^X and F_S^Y are the fuzzy families of τ, σ-semi-closed sets in X and Y, respectively.

Theorem 4.4. *Let (X, τ) be a fuzzifying topological space, (Y, σ) be a fuzzifying S-topological space and $f \in Y^X$. Then $\models \Gamma_S(X, \tau) \otimes T_2^S(Y, \sigma) \otimes I(f) \longrightarrow Q_S(f)$.*

Proof. Similar to the proof of Theorem 4.5 in [14]. □

Theorem 4.5. *Let (X, τ) be a fuzzifying topological space, φ_S be a semi-subbase of τ_S, and*

$\beta_1 := (\forall \Re)(K_{\varphi_S}(\Re, X) \to \exists \wp((\wp \le \Re) \wedge K(\wp, X) \otimes FF(\wp)))$,

where $K_{\varphi_S}(\Re, X) := K(\Re, X) \otimes (\Re \subseteq \varphi_S)$;

$\beta_2 := (\forall S)((S \text{ is a universal net in } X) \to \exists x((x \in X) \wedge (S \rhd^S x))$;

$\beta_3 := (\forall S)((S \in N(X) \to (\exists T)(\exists x)((T < S) \wedge (x \in X) \wedge (T \rhd^S x))$,

where "$T < S$" stands for "T is a subnet of S";

$\beta_4 := (\forall S)((S \in N(X) \to \neg(adh_S S \equiv \phi))$;

$\beta_5 := (\forall \Re)(\Re \in \Im(P(X)) \wedge \Re \subseteq F_S \otimes fI(\Re) \to \exists x \forall A(A \in \Re \to x \in A))$.

Then $\models (X, \tau) \in \Gamma_S \leftrightarrow \beta_i$, $i = 1, 2, ..., 5$.

Proof. (1) Since $\varphi_S \subseteq \tau_S$, $[\Re \subseteq \varphi_S] \le [\Re \subseteq \tau_S]$ for any $\Re \in \Im(P(X))$. Then $[K_{\varphi_S}(\Re, X)] \le [K_S(\Re, X)]$. Therefore $\Gamma_S(X, \tau) \le [\beta_1]$.

(2) $[\beta_2] = \bigwedge\{\bigvee_{x \in X} [S \rhd^S x] : S \text{ is a universal net in } X\}$.

(2.1) Assume X is finite. We set $X = \{x_1, ..., x_m\}$. For any universal net S in X, there exists $i_\circ \in \{1, ..., m\}$ with $S \widetilde{\subset} \{x_{i_\circ}\}$. In fact, if not, then for any $i \in \{1, ..., m\}$, $S \widetilde{\not\subset} \{x_i\}$, $S \widetilde{\subset} X - \{x_i\}$ and $S \widetilde{\subset} \bigcap_{i=1}^m (X - \{x_i\}) = \phi$, a contradiction. Therefore $x_{i_\circ} \notin A$ and $N^S_{x_{i_\circ}}(A) = 0$ (see [6], Theorem 7.2 (1)) provided $S \widetilde{\not\subset} A$, and furthermore $[S \rhd^S x_{i_\circ}] = \bigwedge_{S \widetilde{\not\subset} A} \left(1 - N^S_{x_{i_\circ}}(A)\right) = 1$. Therefore, $[\beta_2] = 1 \ge [\beta_1]$.

(2.2) In general, to prove that $[\beta_1] \le [\beta_2]$, we prove that for any $\lambda \in [0, 1]$, if $[\beta_2] < \lambda$, then $[\beta_1] < \lambda$. Assume for any $\lambda \in [0, 1]$, $[\beta_2] < \lambda$. Then there exists a universal net S in X such that $\bigvee_{x \in X} [S \rhd^S x] < \lambda$ and for any $x \in X$, $[S \rhd^S x] = \bigwedge_{S \widetilde{\not\subset} A} \left(1 - N^S_x(A)\right) < \lambda$, i.e., there exists $A \subseteq X$ with $S \widetilde{\not\subset} A$ and $N^S_x(A) > 1 - \lambda$. Since φ_S is a semi-subbase of τ_S, $\varphi_S^{\Cap}$ is a semi-base of τ_S and from Definition 2.1, we have $\bigvee_{x \in B \subseteq A} \varphi_S^{\Cap}(B) \ge N^S_x(A) > 1 - \lambda$, i.e., there exists $B \subseteq A$ such that $x \in B \subseteq A$ and $\bigvee \left\{ \bigwedge_{\lambda \in \Lambda} \varphi_S(B_\lambda) : \bigcap_{\lambda \in \Lambda} B_\lambda = B, B_\lambda \subseteq X, \lambda \in \Lambda \right\} = \varphi_S^{\Cap}(B) > 1 - \lambda$, where Λ is finite. Therefore, there exists a finite set Λ and $B_\lambda \subseteq X (\lambda \in \Lambda)$ such that $\bigcap_{\lambda \in \Lambda} B_\lambda = B$ and for any $\lambda \in \Lambda, \varphi_S(B_\lambda) > 1 - \lambda$. Since $S \widetilde{\not\subset} A$ and Λ is finite, there exists $\lambda(x) \in \Lambda$ such that $S \widetilde{\not\subset} B_{\lambda(x)}$. We set $\Re_\circ(B_{\lambda(x)}) = \bigvee_{x \in X} \varphi_S(B_{\lambda(x)})$. If $\wp \le \Re_\circ$, then for any $\delta > 0$, $\wp_\delta \subseteq \{B_{\lambda(x)} : x \in X\}$. Consequently, for any $B \in \wp_\delta$, $S \widetilde{\not\subset} B$ and $S \widetilde{\subset} B^c$ since S is a universal net. If $[FF(\wp)] = 1 - \inf\{\delta \in [0, 1] : F(\wp_\delta)\} = t$, then for any $n \in w$ (the non-negative integer), $\inf\{\delta \in [0, 1] : F(\wp_\delta)\} < 1 - t + \frac{1}{n}$, and there exists $\delta_\circ < 1 - t + \frac{1}{n}$ such that $F(\wp_{\delta_\circ})$. If $\delta_\circ = 0$, then $P(X) = \wp_{\delta_\circ}$ is finite and it is proved in (2.1). If $\delta_\circ > 0$, then for any $B \in \wp_{\delta_\circ}$, $S \widetilde{\subset} B^c$. Since $F(\wp_{\delta_\circ})$, we have $S \widetilde{\subset} \bigcap\{B^c : B \in \wp_{\delta_\circ}\} \ne \phi$, i.e., $\bigcup \wp_{\delta_\circ} \ne X$ and there exist $x_\circ \in X$ such that for any $B \in \wp_{\delta_\circ}, x_\circ \notin B$. Therefore, if $x_\circ \in B$, then $B \notin \wp_{\delta_\circ}$, i.e., $\wp(B) < \delta_\circ$, $K(\wp, X) = \bigwedge_{x \in X} \bigvee_{x \in B} \wp(B) \le \bigvee_{x_\circ \in B} \wp(B) \le \delta_\circ < 1 - t + \frac{1}{n}$. Let $n \to \infty$. We obtain $K(\wp, X) \le 1 - t$ and $[K(\wp, X) \otimes FF(\wp)] = 0$. In addition, $[K_{\varphi_S}(\Re_\circ, X)] \ge 1 - \lambda$. In fact, $[\Re_\circ \subseteq \varphi_S] = 1$ and $[K(\Re_\circ, X)] = \bigwedge_{x \in X} \bigvee_{x \in B} \Re_\circ(B) \ge \bigwedge_{x \in X} \Re_\circ(B_{\lambda(x)}) \ge \bigwedge_{x \in X} \varphi_S(B_{\lambda(x)}) \ge 1 - \lambda$ since $x \in B_{\lambda(x)}$.

Now, we have $[\beta_1] = (\forall \Re)(K_{\varphi_S}(\Re, X) \to \exists \wp((\wp \le \Re) \wedge K(\wp, X) \otimes FF(\wp))) \le K_{\varphi_S}(\Re_\circ, X) \to \exists \wp((\wp \le \Re_\circ) \wedge K(\wp, X) \otimes FF(\wp)) = \min(1, 1 - K_{\varphi_S}(\Re_\circ, X) + \bigvee_{\wp \le \Re_\circ} [K(\wp, X) \otimes FF(\wp))] \le \lambda$.

By noticing that λ is arbitrary, we have $[\beta_1] \le [\beta_2]$.

(3) It is immediate that $[\beta_2] \le [\beta_3]$.

(4) To prove that $[\beta_3] \le [\beta_4]$, first we prove that $[\exists T((T < S) \wedge (T \rhd^S x))] \le [S \propto^S x]$, where $[\exists T((T < S) \wedge (T \rhd^S x))] = \bigvee_{T < S} \bigwedge_{T \widetilde{\not\subset} A} \left(1 - N^S_x(A)\right)$ and $[S \propto^S x] = \bigwedge_{S \widetilde{\not\sqsubset} A} \left(1 - N^S_x(A)\right)$. Indeed, for any $T < S$ one can deduce $\{A : S \widetilde{\not\sqsubset} A\} \subseteq \{A : T \widetilde{\not\subset} A\}$ as follows. Suppose $T = S \circ K$. If $S \widetilde{\not\sqsubset} A$, then there exists $m \in D$

such that $S(n) \notin A$ when $n \geq m$, where $\geq$ directs the domain D of S. Now, we will show that $T \,\widetilde{\not\subset}\, A$. If not, then there exists $p \in E$ such that $T(q) \in A$ when $q \geq p$, where $\geq$ directs the domain E of T. Moreover, there exists $n_1 \in E$ such that $K(n_1) \geq m$ since $T < S$, and there exists $n_2 \in E$ such that $n_2 \geq n_1, p$ since $(E, \geq)$ is directed. So, $K(n_2) \geq K(n_1) \geq m$, $S \circ K(n_2) \notin A$ and $S \circ K(n_2) = T(n_2) \in A$, which is a contradiction. Hence $\{A : S \,\widetilde{\not\subset}\, A\} \subseteq \{A : T \,\widetilde{\not\subset}\, A\}$. Therefore $[\exists T\,((T < S) \wedge (T \rhd^S x))] = \bigvee_{T<S} \bigwedge_{T \widetilde{\not\subset} A} \left(1 - N_x^S(A)\right)$ $= \bigvee_{T<S} \bigwedge_{\{A:T \widetilde{\not\subset} A\}} \left(1 - N_x^S(A)\right) \leq \bigwedge_{\{A:S \widetilde{\not\subset} A\}} \left(1 - N_x^S(A)\right)$ $= \bigwedge_{S \widetilde{\not\subset} A} \left(1 - N_x^S(A)\right) = [S \propto^S x]$. Therefore for any $x \in X$ and $S \in N(X)$, we have

$$\begin{aligned}
[\beta_3] &= \bigwedge_{S\in N(X)} \bigvee_{x\in X} [\exists T\,((T < S) \wedge (T \rhd^S x))] \\
&\leq \bigwedge_{S\in N(X)} \bigvee_{x\in X} [S \propto^S x] \\
&= \bigwedge_{S\in N(X)} \neg \left(\bigwedge_{x\in X} \left(1 - [S \propto^S x]\right) \right) \\
&= \bigwedge_{S\in N(X)} [\neg(adh_S S \equiv \phi)] = [\beta_4].
\end{aligned}$$

(5) We want to show that $[\beta_4] \leq [\beta_5]$. For any $\Re \in \Im(P(X))$, assume $[fI(\Re)] = \lambda$. Then for any $\delta > 1 - \lambda$, if $A_1, ..., A_n \in \Re_\delta$, $A_1 \cap A_2 \cap ... \cap A_n \neq \phi$. In fact, we set $\wp(A_i) = \bigvee_{i=1}^{n} \Re(A_i)$. Then $\wp \leq \Re$ and $FF(\wp) = 1$. By putting $\varepsilon = \lambda + \delta - 1 > 0$, we obtain $\lambda - \varepsilon < \lambda \leq [FF(\wp) \to (\exists x)(\forall B)(B \in \wp \to x \in B)] = \bigvee_{x\in X} \bigwedge_{x\notin B} (1 - \wp(B))$. There exists $x_\circ \in X$ such that $\lambda - \varepsilon < \bigwedge_{x_\circ \notin B} (1 - \wp(B))$, $x_\circ \notin B$ implies $\wp(B) < 1 - \lambda + \varepsilon = \delta$ and $x_\circ \in \bigcap \wp_\delta = A_1 \cap A_2 \cap ... \cap A_n$. Now, we set $\vartheta_\delta = \{A_1 \cap A_2 \cap ... \cap A_n : n \in N, A_1, ..., A_n \in \Re_\delta\}$ and $S : \vartheta_\delta \to X, B \mapsto x_B \in B, B \in \vartheta_\delta$ and know that $(\vartheta_\delta, \subseteq)$ is a directed set and S is a net in X. Therefore $[\beta_4] \leq [\neg(adh_S S \equiv \phi)] = \bigvee_{x\in X} \bigwedge_{S \widetilde{\not\subset} A} (1 - N_x^S(A))$. Assume $[\Re \subseteq \digamma_S] = \mu$. Then for any $B \in P(X)$, $\Re(B) \leq 1 + \digamma_S(B) - \mu$, and $[\Re \subseteq \digamma_S \otimes fI(\Re) \to (\exists x)(\forall A)((A \in \Re) \to x \in A)]$ $= \min(1, 2 - \mu - \lambda + \bigvee_{x\in X} \bigwedge_{x\notin A} (1 - \Re(A)))$. Therefore it suffices to show that for any $x \in X$, $\bigwedge_{S \widetilde{\not\subset} A} (1 - N_x^S(A)) \leq 2 - \mu - \lambda + \bigwedge_{x\notin A} (1 - \Re(A))$, i.e., $\bigvee_{x\notin A} \Re(A) \leq 2 - \mu - \lambda + \bigvee_{S \widetilde{\not\subset} A} N_x^S(A)$ for some $\delta > 1 - \lambda$. For any $t \in [0, 1]$, if $\bigvee_{x\notin A} \Re(A) > t$, then there exists $A_\circ$ such that $x_\circ \notin A_\circ$ and $\Re(A_\circ) > t$.

Case 1: If $t \leq 1 - \lambda$, then $t \leq 2 - \mu - \lambda + \bigvee_{S \widetilde{\not\subset} A} N_x^S(A)$.

Case 2: Let $t > 1 - \lambda$. Here we set $\delta = \frac{1}{2}(t + 1 - \lambda)$ and have $A_\circ \in \Re_\delta$, $A_\circ \in \vartheta_\delta$. In addition, $t < \Re(A_\circ) \leq 1 + \digamma_S(A_\circ) - \mu$, $\quad t + \mu - 1 \leq \digamma_S(A_\circ) = \tau_S(A_\circ^c)$. Since $A_\circ \in \vartheta_\delta$, we know that $S_B \in A_\circ$, i.e., $S_B \notin A_\circ^c$ when $B \subseteq A_\circ$ and $S \,\widetilde{\not\subset}\, A_\circ^c$. Therefore, $2 - \mu - \lambda + \bigvee_{S \widetilde{\not\subset} A} N_x^S(A) \geq 2 - \mu - \lambda + N_x^S(A_\circ^c) \geq 2 - \mu - \lambda + \tau_S(A_\circ^c) \geq t + (1 - \lambda) \geq t$. By noticing that t is arbitrary, we have completed the proof.

(6) To prove that $[\beta_5] = [(X, \tau) \in \Gamma_S]$ see [14, Theorem 3.3]. □

The above theorem is a generalization of the following corollary [3, Theorems 3.3 and 3.6].

Corollary 4.6. *The following are equivalent for a topological space (X, τ).*

(a) *X is a semi-compact space.*
(b) *Every cover of X by members of a semi-subbase of τ_S has a finite subcover.*
(c) *Every universal net in X semi-converges to a point in X.*
(d) *Each net in X has a subnet that semi-converges to some point in X.*
(e) *Each net in X has a semi-adherent point.*
(f) *Each family of semi-closed sets in X that has the finite intersection property has a non-void intersection.*

Definition 4.7. Let $\{(X_i, \tau_i) : i \in I\}$ be a family of fuzzifying topological spaces, $\prod_{i\in I} X_i$ be the cartesian product of $\{X_i : i \in I\}$ and $\varphi = \{p_i^{-1}(U_i) : i \in I, U_i \in P(X_i)\}$, where $p_t : \prod_{i\in I} X_i \to X_t (t \in I)$ is a projection. For $\Phi \subseteq \varphi$, $I(\Phi)$ stands for the set of indices of elements in Φ. The semi-base $\beta_S \in \Im(\prod_{i\in I} X_i)$ of $\prod_{i\in I} (\tau_S)_i$ is defined as $V \in \beta_S := (\exists \Phi)(\Phi \Subset \varphi \wedge (\bigcap \Phi = V)) \to \forall i (i \in I(\Phi) \to V_i \in (\tau_S)_i)$, i.e., $\beta_S(V) = \bigvee_{\Phi \Subset \varphi, \bigcap \Phi = V} \bigwedge_{i\in I(\Phi)} (\tau_S)_i(V_i)$.

Example 4.8. Let (X, τ) and τ_S are defined just as in Example 2.2. Define a mapping $\varsigma : P(Y) \longrightarrow I$ on Y as follows: $\varsigma(\emptyset) = \varsigma(Y) = 1$, then ς is a fuzzifying topology and $\varsigma_S(\emptyset) = \varsigma_S(Y) = 1$ (see Example 3.2). Hence, $Y \times X = \{(d, a), (d, b), (d, c)\}$, so $\varphi = \{\emptyset, X \times Y, \{(d, a)\}, \{(d, b)\}, \{(d, c)\}, \{(d, a), (d, b)\}, \{(d, b), (d, c)\}, \{(d, a), (d, c)\}\}$.

By calculating, $\beta_S(\emptyset) = 1, \beta_S(X \times Y) = 1$, $\beta_S(\{(d, a)\}) = 0$, $\beta_S(\{(d, b)\}) = \frac{3}{4}$, $\beta_S(\{(d, c)\}) = \frac{1}{2}$, $\beta_S(\{(d, a), (d, c)\}) = \frac{1}{2}$, $\beta_S(\{(d, a), (d, b)\}) = \frac{3}{4}$, $\beta_S(\{(d, b), (d, c)\}) = \frac{1}{2}$. According to Theorem 2.3, we can easily obtain $\beta_S^{(\cup)} = \beta_S$, so $\tau_S \times \varsigma_S = \beta_S$.

Definition 4.9. Let (X, τ), (Y, σ) be two fuzzifying topological spaces. A unary fuzzy predicate $O_S \in \Im(Y^X)$, called fuzzifying semi-openness, is given as: $O_S(f) := \forall U(U \in \tau_S \to f(U) \in \sigma_S)$. Intuitively, the degree to which f is semi-open is $[O_S(f)] = \bigwedge_{U \subseteq X} \min(1, 1 - \tau_S(U) + \sigma_S(f(U)))$.

Example 4.10. We know that (X, τ) is a fuzzifying topological space (see Example 2.2) and $\tau_S(\emptyset) = \tau_S(X) = 1$, $\tau_S(\{a, c\}) = \frac{1}{2}$, $\tau_S(\{a, b\}) = \frac{3}{4}$, $\tau_S(\{b, c\}) = \frac{1}{2}$, $\tau_S(\{a\}) = 0$, $\tau_S(\{b\}) = \frac{3}{4}$, $\tau_S(\{c\}) = \frac{1}{2}$. We set $Y = X$, $\sigma = \tau$ and $f = id_X$, then $[O_S(f)] = \bigwedge\limits_{U \subseteq X} \min(1, 1 - \tau_S(U) + \sigma_S(f(U))) = 1$.

Lemma 4.11. *Let (X, τ) and (Y, σ) be two fuzzifying topological spaces. For any $f \in Y^X$, $O_S(f) := \forall B(B \in \beta_S^X \to f(B) \in \sigma_S)$, where β_S^X is a semi-base of τ_S.*

Lemma 4.12. *For any family $\{(X_i, \tau_i) : i \in I\}$ of fuzzifying topological spaces.*

(1) $\models (\forall i)(i \in I \to p_i \in O_S)$; (2) $\models (\forall i)(i \in I \to p_i \in C_S)$.

Theorem 4.13. *Let $\{(X_i, \tau_i) : i \in I\}$ be a family of fuzzifying topological spaces. Then*
$\models \exists U(U \subseteq \prod_{i\in I} X_i \wedge \Gamma_S(U, \tau/U) \wedge \exists x(x \in Int_S(U)) \to \exists J(J \Subset I \wedge \forall j(j \in I - J \wedge \Gamma_S(X_j, \tau_j)))$.

Proof. It suffices to show that $\bigvee_{U\in P(\prod_{i\in I} X_i)}(\Gamma_S(U, \tau/U) \wedge \bigvee_{x\in\prod_{i\in I} X_i} N_x^S(U)) \leq \bigvee_{J\Subset I}\bigwedge_{j\in I-J}\Gamma_S(X_j, \tau_j)$. Indeed, if $\bigvee_{U\in P(\prod_{i\in I} X_i)}(\Gamma_S(U, \tau/U) \wedge \bigvee_{x\in\prod_{i\in I} X_i} N_x^S(U)) > \mu > 0$, then there exists $U \in P(\prod_{i\in I} X_i)$ such that $\Gamma_S(U, \tau/U) > \mu$ and $\bigvee_{x\in\prod_{i\in I} X_i} N_x^S(U) > \mu$, where $N_x^S(U) = \bigvee_{x\in V\subseteq U} \left(\prod_{i\in I}(\tau_S)_i\right)(V)$. Furthermore, there exists V such that $x \in V \subseteq U$ and $\left(\prod_{i\in I}(\tau_S)_i\right)(V) > \mu$. Since β_S is a semi-base of

$$\prod_{i\in I}(\tau_S)_i, (\prod_{i\in I}(\tau_S)_i)(V)$$

$$= \bigvee_{\bigcup\limits_{\lambda\in\Lambda} B_\lambda = V} \bigwedge_{\lambda\in\Lambda} \beta_S(B_\lambda)$$

$$= \bigvee_{\bigcup\limits_{\lambda\in\Lambda} B_\lambda = V} \bigwedge_{\lambda\in\Lambda} \bigvee_{\Phi_\lambda\Subset\varphi, \bigcap \Phi_\lambda = B_\lambda} \bigwedge_{i\in I(\Phi_\lambda)} (\tau_S)_i(V_i) > \mu,$$

where $\Phi_\lambda = \{p_i^{-1}(V_i) : i \in I(\Phi_\lambda)\}(\lambda \in \Lambda)$. Hence there exists $\{B_\lambda : \lambda \in \Lambda\} \subseteq P(\prod_{i\in I} X_i)$ such that $\bigcup_{\lambda\in\Lambda} B_\lambda = V$. Furthermore, for any $\lambda \in \Lambda$, there exists $\Phi_\lambda \Subset \varphi$ such that $\bigcap \Phi_\lambda = B_\lambda$ and for any $i \in I(\Phi_\lambda)$, we have $(\tau_S)_i(V_i) > \mu$. Since $x \in V$, there exists B_{λ_x} such that $x \in B_{\lambda_x} \subseteq V \subseteq U$. Hence there exists $\Phi_{\lambda_x} \Subset \varphi$ such that $\bigcap \Phi_{\lambda_x} = B_{\lambda_x}$ and $\bigcap_{i\in I(\Phi_\lambda)} p_i^{-1}(V_i) = B_{\lambda_x} \subseteq \prod_{i\in I} X_i$ and for any $i \in I(\Phi_\lambda)$, we have $(\tau_S)_i(V_i) > \mu$. By $\bigcap_{i\in I(\Phi_\lambda)} p_i^{-1}(V_i) = B_{\lambda_x}$, we have $p_\delta(B_{\lambda_x}) = V_\delta \subseteq X_\delta$, if $\delta \in I(\Phi_{\lambda_x})$; $p_\delta(B_{\lambda_x}) = X_\delta$, if $\delta \in I - I(\Phi_{\lambda_x})$. Since $B_{\lambda_x} \subseteq U$, for any $\delta \in I - I(\Phi_{\lambda_x})$, we have $p_\delta(U) \supseteq p_\delta(B_{\lambda_x}) = X_\delta$ and $p_\delta(U) = X_\delta$. On the other hand, since for any $i \in I$ and $U_i \in P(X_i)$, $\left(\prod_{j\in I}(\tau_S)_j\right)\left(p_i^{-1}(U_i)\right) \geq (\tau_S)_i(U_i)$, we have for any $i \in I$, $I(p_i) = \bigwedge_{U_i\in P(X_i)} \min(1, 1 - (\tau_S)_i(U_i) + \prod_{j\in I}(\tau_S)_j(p_i^{-1}(U_i))) = 1$. Furthermore, by Theorem 4.1, $\Gamma_S(U, \tau/U) = \Gamma_S(U, \tau/U) \otimes I(p_\delta) \leq \Gamma_S(P_\delta(U), \tau_\delta) = \Gamma_S(X_\delta, \tau_\delta)$ for each $\delta \in I - I(\Phi_\lambda)$. Therefore, $\bigvee_{J\Subset I}\bigwedge j \in I - J\Gamma_S(X_j, \tau_j) \geq \bigwedge_{\delta\in I-I(\Phi_\lambda)}\Gamma_S(X_\delta, \tau_\delta) \geq \Gamma_S(U, \tau/U) > \mu$. □

The above theorem is a generalization of the following corollary.

Corollary 4.14. *If there exists a coordinate semi-neighborhood semi-compact subset U of some point $x \in X$ of the product space, then all except a finite number of coordinate spaces are semi-compact.*

Lemma 4.15. *For any fuzzifying topological space (X, τ) and $A \subseteq X$ we have*
$\models T_2^S(X, \tau) \to T_2^S(A, \tau/A)$.

Lemma 4.16. *For any fuzzifying S-topological space (X, τ) we have*
$\models T_2^S(X, \tau) \otimes \Gamma_S(X, \tau) \to T_4^S(X, \tau)$ (for the definition of $T_4^S(X, \tau)$ see [7, Definition 3.1]).

The above lemma is a generalization of the following corollary [10, Theorem 3.9].

Corollary 4.17. Every semi-compact semi-T_2 S-topological space is s-normal.

Lemma 4.18. *For any fuzzifying $S-$topological space (X, τ), $\models T_2^S(X, \tau) \otimes \Gamma_S(X, \tau) \to T_3^S(X, \tau)$(For the definition of $T_3^S(X, \tau)$ see [7, Definition 3.1]).*

The above lemma is a generalization of the following corollary [11, Theorem 3.9].

Corollary 4.19. *Every semi-compact semi-T_2 S-topological space is s-regular.*

Theorem 4.20. *For any fuzzifying topological space (X, τ) and $A \subseteq X$ we have*
$\models T_2^S(X, \tau) \otimes \Gamma_S(A) \to A \in F_S$.

Proof. For any $\{x\} \subset A^c$, we have $\{x\} \cap A = \phi$ and $\Gamma_S(\{x\}) = 1$. By Theorem 4.4 in [13] $[T_2^S(X, \tau) \otimes (\Gamma_S(A) \wedge \Gamma_S(\{x\}))] \leq \bigvee_{G \cap H_x = \phi, A \subseteq G, x \in H_x} \min(\tau_S(G), \tau_S(H_x)))$. Assume $\beta_x = \{H_x : A \cap H_x = \phi, x \in H_x\}$, $\bigcup_{x \in A^c} f(x) \supseteq A^c$ and $\bigcup_{x \in A^c} f(x) \cap A = \bigcup_{x \in A^c} (f(x) \cap A) = \phi$. So, $\bigcup_{x \in A^c} f(x) = A^c$ and therefore,

$$\begin{aligned}
&[T_2^S(X, \tau) \otimes \Gamma_S(A)] \\
&\leq \bigvee_{G \cap H_x = \phi, A \subseteq G, x \in H_x} \tau_S(H_x) \\
&\leq \bigwedge_{x \in A^c} \bigvee_{A \cap H_x = \phi, x \in H_x} \tau_S(H_x) \\
&= \bigvee_{f \in \prod_{x \in A^c} \beta_x} \bigwedge_{x \in A^c} \tau_S(f(x)) \\
&\leq \bigvee_{f \in \prod_{x \in A^c} \beta_x} \tau_S(\bigcup_{x \in A^c} f(x)) \\
&= \bigvee_{f \in \prod_{x \in A^c} \beta_x} \tau_S(A^c) = F_S(A). \qquad \square
\end{aligned}$$

The above theorem is a generalization of the following corollary.

Corollary 4.21. *Semi-compact subspace of a semi-Hausdorff topological space is semi-closed.*

Theorem 4.22. $\models (X, \tau) \in \Gamma_S \to (\forall B)(B \in F_S \to (B, \tau/B) \in \Gamma_S)$.

Proof. From Theorem 4.1 [13], we have for any $B \subseteq X$, $[\Gamma_S(X, \tau) \otimes F_S(B)] \leq \Gamma_S(B)$. So $\Gamma_S(X, \tau) \leq [F_S(B) \to \Gamma_S(B)]$. Therefore, $\Gamma_S(X, \tau) \leq [(\forall B)(B \in F_S \to (B, \tau/B) \in \Gamma_S)]$. $\square$

The above theorem is a generalization of Theorem 3.2 in [3].

5. Fuzzifying locally semi-compactness

Definition 5.1. Let Ω be a class of fuzzifying topological spaces. A unary fuzzy predicate $L_S C \in \Im(\Omega)$, called fuzzifying locally semi-compactness, is given as follows:

$(X, \tau) \in L_S C := (\forall x)(\exists B)((x \in Int_S(B) \otimes \Gamma_S(B, \tau/B))$.

Since $[x \in Int_S(X)] = N_x^S(X) = 1$, then $L_S C(X, \tau) \geq \Gamma_S(X, \tau)$. Therefore,

$$\models (X, \tau) \in \Gamma_S \to (X, \tau) \in L_S C.$$

Also, since $\models (X, \tau) \in \Gamma \to (X, \tau) \in LC$ [15] and $\models (X, \tau) \in \Gamma_S \to (X, \tau) \in \Gamma$ [14], $\models (X, \tau) \in \Gamma_S \to (X, \tau) \in LC$.

Theorem 5.2. *For any fuzzifying topological space (X, τ) and $A \subseteq X$,*

$$\models (X, \tau) \in L_S C \otimes A \in F_S \to (A, \tau/A) \in L_S C.$$

Proof. We have

$$L_S C(X, \tau) = \bigwedge_{x \in X} \bigvee_{B \subseteq X} \max(0, N_x^{S^X}(B) + \Gamma_S(B, \tau/B) - 1)$$

and

$$L_S C(A, \tau/A) = \bigwedge_{x \in A} \bigvee_{G \subseteq A} \max(0, N_x^{S^A}(G) + \Gamma_S(G, (\tau/A)/G) - 1).$$

Now, suppose that $[(X, \tau) \in L_S C \otimes A \in F_S] > \lambda > 0$. Then for any $x \in A$, there exists $B \subseteq X$ such that

$$N_x^{S^X}(B) + \Gamma_S(B, \tau/B) + \tau_S(X - A) - 2 > \lambda. \quad (1)$$

Let $E = A \cap B \in P(A)$. Then $N_x^{S^A}(E) = \bigvee_{E = C \cap B} N_x^{S^X}(C) \geq N_x^{S^X}(B)$ and for any $U \in P(E)$, we have

$$\begin{aligned}
&(\tau_S/A)_S/E(U) \\
&= \bigvee_{U = C \cap E} \tau_S/A(C) = \bigvee_{U = C \cap E} \bigvee_{C = D \cap A} \tau_S(D) \\
&= \bigvee_{U = D \cap A \cap E} \tau_S(D) = \bigvee_{U = D \cap E} \tau_S(D).
\end{aligned}$$

Similarly, $(\tau_S/B)_S/E(U) = \bigvee_{U = D \cap E} \tau_S(D)$. Thus, $(\tau_S/B)_S/E = (\tau_S/A)_S/E$ and $\Gamma_S(E, (\tau/A)/E) = \Gamma_S(E, (\tau/B)/E)$. Furthermore, $[E \in F_S/B] = \tau_S/B(B - E) = \tau_S/B(B \cap E^c) = \bigvee_{B \cap E^c = B \cap D} \tau_S(D) \geq \tau_S(X - A) = F_S(A)$. Since $\models (X, \tau) \in \Gamma_S \otimes A \in F_S \to$

$(A, \tau/A) \in \Gamma_S$ (see [14], Theorem 4.1), from (1) we have for any $x \in A$ that

$$\bigvee_{G \subseteq A} \max(0, N_x^{S^A}(G) + \Gamma_S(G, (\tau/A)/G) - 1)$$
$$\geq N_x^{S^A}(E) + \Gamma_S(E, (\tau/A)/E) - 1$$
$$= N_x^{S^A}(E) + \Gamma_S(E, (\tau/B)/E) - 1$$
$$\geq N_x^{S^X}(B) + [\Gamma_S(B, \tau/B) \otimes E \in F_S/B] - 1$$
$$\geq N_x^{S^X}(B) + \Gamma_S(B, \tau/B) + [E \in F_S/B] - 2$$
$$\geq N_x^{S^X}(B) + \Gamma_S(B, \tau/B) + [A \in F_S] - 2 > \lambda.$$

Therefore,

$$L_SC(A, \tau/A)$$
$$= \bigwedge_{x \in A} \bigvee_{G \subseteq A} \max(0, N_x^{S^A}(G)$$
$$+\Gamma_S(G, (\tau/A)/G) - 1) > \lambda.$$

Hence $[(X, \tau) \in L_SC \otimes A \in F_S] \leq L_SC(A, \tau/A)$. $\square$

As a crisp result of the above theorem we have the following Corollary.

Corollary 5.3. *Let A be a semi-closed subset of locally semi-compact space (X, τ). Then A with the relative topology τ/A is locally semi-compact.*

The following theorem is a generalization of the statement "If X is a semi-Hausdorff space and A is a dense locally semi-compact subspace, then A is semi-open", where A is a dense in a topological space X if and only if the semi-closure of A is X.

Theorem 5.4. For any fuzzifying S-topological space (X, τ) and $A \subseteq X$, $\models T_2^S(X, \tau) \otimes L_SC(A, \tau/A) \otimes (Cl_S(A) \equiv X) \to A \in \tau_S$.

Proof. Suppose that $[T_2^S(X, \tau) \otimes L_SC(A, \tau/A) \otimes (Cl_S(A) \equiv X)] > \lambda > 0$. Then $L_SC(A, \tau/A) > \lambda - [T_2^S(X, \tau) \otimes (Cl_S(A) \equiv X)] + 1 = \lambda' > \lambda$, i.e., $\bigwedge_{x \in A} \bigvee_{B \subseteq A} \max(0, N_x^{S^A}(B) + \Gamma_S(B, (\tau/A)/B) - 1) > \lambda'$. Thus for any $x \in A$, there exists $B_x \subseteq A$ such that $N_x^{S^A}(B_x) + \Gamma_S(B_x, (\tau/A)/B_x) - 1 > \lambda'$, i.e., $\bigvee_{H \cap A = B_x} \bigvee_{x \in K \subseteq H} \tau_S(K) + \Gamma_S(B_x, (\tau/A)/B_x) - 1 > \lambda'$. Hence there exists K_x such that $K_x \cap A = B_x$, $\tau_S(K_x) + \Gamma_S(B_x, (\tau/A)/B_x) - 1 > \lambda'$. Therefore, $\tau_S(K_x) > \lambda'$.

(1) If for any $x \in A$, there exists K_x such that $x \in K_x \subseteq B_x \subseteq A$ (thus $K_x = B_x$), then $\bigcup_{x \in A} K_x = A$ and $\tau_S(A) = \tau_S(\bigcup_{x \in A} K_x) \geq \bigwedge_{x \in A} \tau_S(K_x) \geq \lambda' > \lambda$.

(2) If there exists $x_\circ \in A$ such that $K_{x_\circ} \cap (B_{x_\circ}^c) \neq \phi$, $\tau_S(K_{x_\circ}) + \Gamma_S(B_{x_\circ}, (\tau/A)/B_{x_\circ}) - 1 > \lambda'$. From the hypothesis, we have that $[T_2^S(X, \tau) \otimes L_SC(A, \tau/A) \otimes (Cl_S(A) \equiv X)] > \lambda > 0$, we have $[T_2^S(X, \tau) \otimes (Cl_S(A) \equiv X)] \neq 0$. So $\tau_S(K_{x_\circ}) + \Gamma_S(B_{x_\circ}, (\tau/A)/B_{x_\circ}) - 1 + [T_2^S(X, \tau) \otimes (Cl_S(A) \equiv X)] - 1 > \lambda$. Therefore, $\tau_S(K_{x_\circ}) + \Gamma_S(B_{x_\circ}, (\tau/A)/B_{x_\circ}) - 1 + T_2^S(X, \tau) + [(Cl_S(A) \equiv X)] - 1 - 1 > \lambda$. Since

$$(\tau_S/A)_S/B_{x_\circ}(U)$$
$$= \bigvee_{U = C \cap B_{x_\circ}} \tau_S/A(C) = \bigvee_{U = C \cap B_{x_\circ}} \bigvee_{C = D \cap A} \tau_S(D)$$
$$= \bigvee_{U = D \cap B_{x_\circ}} \tau_S(D) = \tau_S/B_{x_\circ}(U), \Gamma_S(B_{x_\circ}, (\tau/A)/B_{x_\circ})$$
$$= \Gamma_S(B_{x_\circ}, \tau/B_{x_\circ}),$$

from Theorem 4.20, we have $\tau_S(B_{x_\circ}^c) \geq T_2^S(X, \tau) \otimes \Gamma_S(B_{x_\circ}, \tau/B_{x_\circ}) \geq T_2^S(X, \tau) + \Gamma_S(B_{x_\circ}, \tau/B_{x_\circ}) - 1$. Hence $\tau_S(K_{x_\circ}) + \tau_S(B_{x_\circ}^c) + [Cl_S(A) \equiv X] - 2 > \lambda$. Now, for any $y \in A^c$ we have $[Cl_S(A) \equiv X] = \bigwedge_{x \in X}(1 - N_x^{S^X}(A^c)) \leq 1 - N_y^{S^X}(A^c)$. Since (X, τ) is a fuzzifying S-topological space, $\tau_S(K_{x_\circ}) + \tau_S(B_{x_\circ}^c) - 1 \leq \tau_S(K_{x_\circ}) \otimes \tau_S(B_{x_\circ}^c) \leq \tau_S(K_{x_\circ}) \wedge \tau_S(B_{x_\circ}^c) \leq \tau_S(K_{x_\circ} \cap B_{x_\circ}^c) \leq N_y^{S^X}(K_{x_\circ} \cap B_{x_\circ}^c) \leq N_y^{S^X}(A^c)$, where $y \in K_{x_\circ} \cap B_{x_\circ}^c \subseteq H_{x_\circ} \cap (H_{x_\circ} \cap A)^c = H_{x_\circ} \cap (H_{x_\circ}^c \cup A^c) = H_{x_\circ} \cap A^c \subseteq A^c$. Therefore, $0 < \lambda < \tau_S(K_{x_\circ}) + \tau_S(B_{x_\circ}^c) + [Cl_S(A) \equiv X] - 2 = \tau_S(K_{x_\circ}) + \tau_S(B_{x_\circ}^c) - 1 + [Cl_S(A) \equiv X] - 1 \leq N_y^{S^X}(A^c) + 1 - N_y^{S^X}(A^c) - 1 = 0$, a contradiction. So, case (2) does not hold. We have completed the proof. $\square$

Theorem 5.5. *For any fuzzifying S-topological space (X, τ), $\models T_2^S(X, \tau) \otimes (L_SC(X, \tau))^2 \to \forall x \forall U(U \in N_x^{S^X} \to \exists V(V \in N_x^{S^X} \wedge Cl_S(V) \subseteq U \wedge \Gamma_S(V)))$, where $(L_SC(X, \tau))^2 := L_SC(X, \tau) \otimes L_SC(X, \tau)$.*

Proof. We need to show that for any x and U, $x \in U$,

$$T_2^S(X, \tau) \otimes (L_SC(X, \tau))^2 \otimes N_x^{S^X}(U)$$
$$\leq \bigvee_{V \subseteq X} (N_x^{S^X}(V) \wedge \bigwedge_{y \in U^c} N_x^{S^X}(V^c) \wedge \Gamma_S(V, \tau/V)).$$

Assume that $T_2^S(X, \tau) \otimes (L_SC(X, \tau))^2 \otimes N_x^{S^X}(U) > \lambda > 0$. Then for any $x \in X$ there exists C such that

$$T_2^S(X, \tau) + N_x^{S^X}(C) + (L_SC(X, \tau))^2$$
$$+N_x^{S^X}(U) - 3 > \lambda. \tag{2}$$

Since (X, τ) is a fuzzifying S-topological space, $N_x^{S^X}(C) + N_x^{S^X}(U) - 1 \le N_x^{S^X}(C) \otimes N_x^{S^X}(U) \le N_x^{S^X}(C) \wedge N_x^{S^X}(U) \le N_x^{S^X}(C \cap U) = \bigvee_{x \in W \subseteq C \cap U} \tau_S(W)$. Therefore there exists W such that $x \in W \subseteq C \cap U$, and $T_2^S(X, \tau) + (L_S C(X, \tau))^2 + \tau_S(W) - 2 > \lambda$. By Lemmas 4.15 and 4.17 we have $T_2^S(X, \tau) \le T_2^S(C, \tau/C)$ and $T_2^S(C, \tau/C) + \Gamma_S(C, \tau/C) - 1 \le T_2^S(C, \tau/C) \otimes \Gamma_S(C, \tau/C) \le T_3^S(C, \tau/C)$. Thus $T_3^S(X, \tau) + \Gamma_S(C, \tau/C) + \tau_S(W) - 2 > \lambda$. Since for any $x \in W \subseteq U$, we have $T_3^S(C, \tau/C) \le 1 - \tau_S/C(W) + \bigvee_{G \subseteq C}((N_x^{S^C}(G) \wedge \bigwedge_{y \in C - W} N_y^{S^C}(C - G)))$ (see [7, Theorem 3.16]), so there exists $G, x \in G \subseteq W$ such that $((N_x^{S^C}(G) \wedge \bigwedge_{y \in C - W} N_y^{S^C}(C - G))) \ge T_3^S(C, \tau/C) + \tau_S/C(W) - 1 \ge T_3^S(C, \tau/C) + \tau_S(W) - 1$, and $((N_x^{S^C}(G) \wedge \bigwedge_{y \in C - W} N_y^{S^C}(C - G))) + \Gamma_S(C, \tau/C) - 1 > \lambda$. Thus $N_x^{S^C}(G) = \bigvee_{D \cap C = G} N_x^{S^X}(D) = N_x^{S^X}(G \cup C^c) > \lambda' = \lambda + 1 - \Gamma_S(C, \tau/C) \ge \lambda$. Furthermore, for any $y \in C - W$, $N_y^{S^C}(C - G) = \bigvee_{D \cap C = C \cap G^c} N_y^{S^X}(G^c \cup C^c) = N_y^{S^X}(G^c) > \lambda'$ and $N_x^{S^X}(G) = N_x^{S^X}((G \cup C^c) \cap C) \ge N_x^{S^X}(G \cup C^c) \wedge N_x^{S^X}(C) > \lambda'$. Since $N_y^{S^X}(G^c) = \bigvee_{x \in B^c \subseteq G^c} \tau_S(B^c) > \lambda'$, for any $y \in C - W$, there exists B_y^c such that $y \in B_y^c \subseteq G^c$ and $\tau_S(B_y^c) > \lambda'$. Set $B^c = \bigcup_{y \in C - W} B_y^c$. Then $C - W \subseteq B^c \subseteq G^c$ and $\tau_S(B^c) \ge \bigwedge_{y \in C - W} \tau_S(B_y^c) \ge \lambda'$. Again, let $V = B \cap C$, then $V \subseteq (C - W)^c \cap C = (C^c \cup W) \cap C = C \cap W = W \subseteq U \cap C$ and $V^c = B^c \cup C^c$. Since (X, τ) is a fuzzifying S-topological space,

$$N_x^{S^X}(V) = N_x^{S^X}(B \cap C) \ge N_x^{S^X}(B) \wedge N_x^{S^X}(C) \ge N_x^{S^X}(G) \wedge N_x^{S^X}(C) > \lambda. \tag{3}$$

By (2) and Theorem 4.20,

$$\tau_S(C^c) \ge T_2^S(X, \tau) \otimes \Gamma_S(C, \tau/C) \ge T_2^S(X, \tau) + \Gamma_S(C, \tau/C) - 1 \ge \lambda'.$$

So $\tau_S(V^c) = \tau_S(B^c \cup C^c) \ge \tau_S(B^c) \wedge \tau_S(C^c) \ge \lambda'$, i.e., $\tau_S(V^c) + \Gamma_S(C, \tau/C) - 1 \ge \lambda$ and

$$\Gamma_S(V, \tau/V) = \Gamma_S(V, (\tau/C)/V) \ge \tau_S/C(C - V) + \Gamma_S(C, \tau/C) - 1 \ge \tau_S(V^c) + \Gamma_S(C, \tau/C) - 1 \ge \lambda. \tag{4}$$

Finally,

$$\bigwedge_{y \in U^c} N_y^{S^X}(V^c) \ge \bigwedge_{y \in V^c} N_y^{S^X}(V^c) = \tau_S(V^c) \ge \lambda \tag{5}$$

Thus by (3), (4) and (5), for any $x \in U$, there exists $V \subseteq U$ such that $N_x^{S^X}(V) > \lambda$, $\bigwedge_{y \in U^c} N_y^{S^X}(V^c) \ge \lambda$ and $\Gamma_S(V, \tau/V) \ge \lambda$.

So $\bigvee_{V \subseteq X} (N_x^{S^X}(V) \wedge \bigwedge_{y \in U^c} N_y^{S^X}(V^c) \wedge \Gamma_S(V, \tau/V)) \ge \lambda$. □

Theorem 5.6. *For any fuzzifying S-topological space (X, τ), $\models T_2^S(X, \tau) \otimes (L_S C(X, \tau))^2 \to T_3^S(X, \tau)$.*

Proof. By Theorem 5.5, for any $x \in U$, we have

$$\bigvee_{x \in V \subseteq U} (N_x^{S^X}(V) \wedge \bigwedge_{y \in U^c} N_y^{S^X}(V^c)) \ge [T_2^S(X, \tau) \otimes (\Gamma_S(C, \tau/C))^2 \otimes N_x^{S^X}(U)].$$

Thus

$$1 - N_x^{S^X}(U) + \bigvee_{x \in V \subseteq U} (N_x^{S^X}(V) \wedge \bigwedge_{y \in U^c} N_y^{S^X}(V^c)) \ge [T_2^S(X, \tau) \otimes (\Gamma_S(C, \tau/C))^2],$$

i.e., $[T_3^S(X, \tau)] \ge [T_2^S(X, \tau) \otimes (\Gamma_S(C, \tau/C))^2]$. □

Theorem 5.7. *For any fuzzifying S-topological space (X, τ),*

$$\models T_3^S(X, \tau) \otimes L_S C(X, \tau) \to \forall A \forall U(U \in N_A^{S^X} \otimes \Gamma_S(A, \tau/A) \to \exists V(V \subseteq U \wedge U \in N_A^{S^X} \wedge \tau_S(V^c) \wedge \Gamma_S(V, \tau/V))),$$

where $U \in N_A^{S^X} := (\forall x)(x \in A \wedge U \in N_x^{S^X})$.

Proof. We only need to show that for any $A, U \in P(X)$, $[T_3^S(X, \tau) \otimes L_S C(X, \tau) \otimes \Gamma_S(A, \tau/A) \otimes N_A^{S^X}(U)] \le \bigvee_{V \subseteq U} (N_A^{S^X}(V) \wedge \tau_S(V^c) \wedge \Gamma_S(V, \tau/V))$.

Indeed, if $[T_3^S(X, \tau) \otimes L_S C(X, \tau) \otimes \Gamma_S(A, \tau/A) \otimes N_A^{S^X}(U)] > \lambda > 0$, then for any $x \in A$, there exists $C \in P(X)$ such that $[T_3^S(X, \tau) \otimes N_x^{S^X}(C) \otimes \Gamma_S(C, \tau/C) \otimes \Gamma_S(A, \tau/A) \otimes N_A^{S^X}(U)] > \lambda$. Since (X, τ) is a fuzzifying S-topological space,

$$\bigvee_{x \in W \subseteq C \cap U} \tau_S(W) = N_x^{S^X}(C \cap U) \ge N_x^{S^X}(C) \wedge N_x^{S^X}(U) \ge N_x^{S^X}(C) \wedge N_A^{S^X}(U) \ge N_x^{S^X}(C) \otimes N_A^{S^X}(U).$$

Then there exists W such that $x \in W \subseteq C \cap U$, and $[T_3^S(X, \tau) \otimes \tau_S(W) \otimes \Gamma_S(C, \tau/C) \otimes \Gamma_S(A, \tau/A)] > \lambda$.

Therefore

$$[T_3^S(X,\tau)]+\tau_S(W)-1>\lambda+2-\Gamma_S(C,\tau/C)$$
$$-\Gamma_S(A,\tau/A)]=\lambda'\geq\lambda. \quad (6)$$

Since for any $x\in W$, $[T_3^S(X,\tau)]\leq 1-\tau_S(W)+\bigvee_{B\subseteq W}(N_x^{S^X}(B)\wedge\bigwedge_{y\in W^c}N_y^{S^X}(B^c))$, we have

$$\bigvee_{B\subseteq W}(N_x^{S^X}(B)\wedge\bigwedge_{y\in W^c}N_y^{S^X}(B^c))>\lambda'.$$

Thus there exists B_x such that $x\in B_x\subseteq W\subseteq C\cap U$ and for any $y\in W^c$, we have $N_y^{S^X}(B_x^c)>\lambda'$, $N_x^{S^X}(B_x)>\lambda'$. Since $N_y^{S^X}(B_x^c)=\bigvee_{x\in G^c\subseteq B_x^c}\tau_S(G^c)>\lambda'$, then for any $y\in W^c$, there exists G_{xy} such that $x\in G_{xy}^c\subseteq B_x^c$ and $\tau_S(G_{xy}^c)>\lambda'$. Set $G_x^c=\bigcup_{y\in W^c}G_{xy}^c$. Then $W^c\subseteq G_{xy}^c\subseteq B_x^c$ and $\tau_S(G_x^c)\geq\bigwedge_{y\in W^c}\tau_S(G_{xy}^c)\geq\lambda'$. Since $G_x\supseteq B_x$, $N_x^{S^X}(G_x)\geq N_x^{S^X}(B_x)>\lambda'$, i.e., $\bigvee_{x\in H\subseteq G_x}\tau_S(H)>\lambda'$. Thus there exists H_x such that $x\in H_x\subseteq G_x$ and $\tau_S(H_x)>\lambda'$. Hence for any $x\in A$, there exists H_x and G_x such that $x\in H_x\subseteq G_x\subseteq U$, $\tau_S(H_x)>\lambda'$ and $W\supseteq\bigcup_{x\in A}G_x\supseteq\bigcup_{x\in A}H_x\supseteq A$. We define $\Re\in\Im(P(A))$ as follows:

$$\Re(D)=\begin{cases}\bigvee_{H_x\cap A=D}\tau_S(H_x), & \text{there exists } H_x \text{ s.t. } H_x\cap A=D,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\Gamma_S(A,\tau/A)=\mu>\mu-\epsilon$ $(\epsilon>0)$. Then

$$1-K_S(\Re,A)+\bigvee_{\wp\leq\Re}[K(\wp,A)\otimes FF(\wp)]>\mu-\epsilon, \quad (7)$$

where

$$[K(\Re,A)]=\bigwedge_{x\in A}\bigvee_{x\in B}\Re(B)=\bigwedge_{x\in A}\bigvee_{x\in D}\Re(D)$$
$$=\bigwedge_{x\in A}\bigvee_{x\in D}\bigvee_{H_{x'}\cap A=D}\tau_S(H_{x'})\geq\lambda'$$

and

$$[\Re\subseteq\tau_S\backslash A]$$
$$=\bigwedge_{B\subseteq X}\min(1,1-\Re(B)+\tau_S\backslash A(B))$$
$$=\bigwedge_{B\subseteq X}\min(1,1-\bigvee_{H_x\cap A=B}\tau_S(H_x)+\bigvee_{H\cap A=B}\tau_S(H))=1.$$

So, $K_S(\Re,A)=[K(\Re,A)]\geq\lambda'$. By (7), $[K(\wp,A)\otimes FF(\wp)]>\mu-\epsilon-1+K_S(\Re,A)\geq\mu-\epsilon-1+\lambda'\geq\lambda-\epsilon$. Thus

$$\bigwedge_{x\in A}\bigvee_{x\in E}\wp(E)+1-\bigwedge\{\delta:F(\wp_\delta)\}-1>\lambda-\epsilon,$$

and

$$\bigwedge_{x\in A}\bigvee_{x\in E}\wp(E)>\lambda-\epsilon+\bigwedge\{\delta:F(\wp_\delta)\}.$$

Hence there exists $\beta>0$ such that $F(\wp_\beta)$ and $\bigwedge_{x\in A}\bigvee_{x\in D}\wp(D)>\lambda-\epsilon+\beta$. Therefore for any $x\in A$, there exists $D_x\subseteq A$ such that $\wp(D_x)>\lambda-\epsilon+\beta$ and $\bigcup_{x\in A}D_x\subseteq A$. Suitably chosen ϵ such that $\lambda-\epsilon>0$ provides $\wp(D_x)>\beta>0$. Since $\Re(D_x)\geq\wp(D_x)>0$, $D_x=H_{x'}\cap A$, i.e., $H_{x'}\cap A\in\wp_\beta$. By $F(\wp_\beta)$, there exists finite $H_{x'_1},H_{x'_2},\ldots,H_{x'_n}$ such that $\bigcup_{i=1}^n H_{x'_i}\supseteq A$ and $\bigcup_{i=1}^n H_{x'_i}\subseteq\bigcup_{i=1}^n G_{x'_i}$. Set $V=\bigcup_{i=1}^n G_{x'_i}$, and $V^c=\bigcap_{i=1}^n G_{x'_i}^c$, $A\subseteq V\subseteq U$, and $\tau_S(V^c)\geq\bigwedge_{1\leq i\leq n}\tau_S(G_{x'_i}^c)\geq\lambda'>\lambda$. Since for any $x\in A$, $G_x\subseteq W\subseteq C\cap U\subseteq C$, we have $V=\bigcup_{i=1}^n G_{x'_i}\subseteq W\subseteq C$ by $\tau_S\backslash C(C-V)=\bigvee_{D\cap C=C\cap V^c}\tau_S(D)\geq\tau_S(V^c)\geq\lambda'$. Thus by (6), $\tau_S\backslash C(C-V)+\Gamma_S(C,\tau/C)-1>\lambda$. By Theorem 4.1 in [14], $\Gamma_S(V,\tau/V)=\Gamma_S(V,\tau/C/V)\geq[\Gamma_S(C,\tau/C)\otimes\tau_S\backslash C(C-V)]>\lambda$. Finally, we have for any $x\in A$, $N_x^{S^X}(V)=N_x^{S^X}(\bigcup_{i=1}^n G_{x'_i})\geq N_x^{S^X}(\bigcup_{i=1}^n H_{x'_i})\geq\tau_S(\bigcup_{i=1}^n H_{x'_i})\geq\bigwedge_{1\leq i\leq n}\tau_S(H_{x'_i})\geq\lambda'>\lambda$. So $N_A^{S^X}(V)=\bigwedge_{x\in A}N_x^{S^X}(V)\geq\lambda$. Therefore, $N_A^{S^X}(V)\wedge\tau_S(V^c)\wedge\Gamma_S(V,\tau/V)\geq\lambda$. Thus $\bigvee_{V\subseteq U}(N_A^{S^X}(V)\wedge\tau_S(V^c)\wedge\Gamma_S(V,\tau/V))\geq\lambda$. □

Theorem 5.8. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and $f\in Y^X$ be surjective. Then $\models L_SC(X,\tau)\otimes C_S(f)\otimes O(f)\to LC(Y,\sigma)$. For the definition of $O(f)$, see [17].*

Proof. If $[L_SC(X,\tau)\otimes C_S(f)\otimes O(f)]>\lambda>0$, then for any $x\in X$, there exists $U\subseteq X$, such that

$$[N_x^{S^X}(U)\otimes\Gamma_S(U,\tau/U)\otimes C_S(f)\otimes O(f)]>\lambda.$$

Since $N_x^{S^X}(U)=\bigvee_{x\in V\subseteq U}\tau_S(V)$, there exists $V'\subseteq X$ such that $x\in V'\subseteq U$ and $[\tau_S(V')\otimes\Gamma_S(U,\tau/U)\otimes C_S(f)\otimes O(f)]>\lambda$. By Theorem 4.1 (1),

$$[\Gamma_S(U,\tau/U)\otimes C_S(f)]\leq[\Gamma(f(U),\sigma/f(U))]$$

and

$$[\tau(V') \otimes O(f)] = \max(0, \tau(V') + O(f) - 1)$$
$$= \max(0, \tau(V') + \bigwedge_{V \subseteq X} \min(1, 1 - \tau(V') + \sigma(f(V))) - 1)$$
$$\leq \max(0, \tau(V') + 1 - \tau(V') + \sigma(f(V)) - 1)$$
$$= \sigma(f(V)) \leq N^{Y}_{f(x)}(f(V')) \leq N^{Y}_{f(x)}(f(U)).$$

Since f is surjective,

$$LC(Y, \sigma) = LC(f(X), \sigma)$$
$$= \bigwedge_{y \in f(x) \subseteq f(X)} \bigvee_{U' = f(U) \subseteq f(X)} [N^{Y}_{y}(U') \otimes [\Gamma(U', \sigma/U')]$$
$$\geq \bigwedge_{y \in f(x) \subseteq f(X)} [N^{Y}_{f(x)}(f(U)) \otimes [\Gamma(f(U), \sigma/f(U))]$$
$$\geq \bigwedge_{y \in f(x) \subseteq f(X)} [\tau(V') \otimes O(f)$$
$$\otimes \Gamma_S(U, \tau/U) \otimes C_S(f)] \geq \lambda. \qquad \square$$

Theorem 5.9. *Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and $f \in Y^X$ be surjective. Then* $\models L_SC(X, \tau) \otimes I(f) \otimes O_S(f) \to L_SC(Y, \sigma)$.

Proof. By Theorem 4.4, the proof is similar to that of Theorem 5.8. $\square$

Theorems 5.8 and 5.9 are generalizations of the following corollary.

Corollary 5.10. *Let (X, τ) and (Y, σ) be two topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a surjective mapping. If f is semi-continuous (resp. irresolute), open (resp. semi-open) and X is a locally semi-compact, then Y is locally compact (resp. locally semi-compact) space.*

Theorem 5.11. *Let $\{(X_i, \tau_i) : i \in I\}$ be a family of fuzzifying topological spaces. Then*

$$\models L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i) \to$$
$$\forall i (i \in I \wedge L_SC(X_i, (\tau_S)_i) \wedge \exists J (J \Subset I$$
$$\wedge \forall j (j \in I - J \wedge \Gamma_S(X_j, \tau_j))).$$

Proof. It suffices to show that

$$L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i) \leq [\bigwedge_{i \in I} L_SC(X_i, (\tau_S)_i)$$
$$\wedge \bigvee_{J \Subset I} \bigwedge_{j \in I - J} \Gamma_S(X_j, \tau_j)].$$

From Theorem 5.8 and Lemma 4.12 we have for any $j \in I$, $L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i) = [L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i) \otimes C_S(p_j) \otimes O_S(p_j)] \leq L_SC(X_j, \tau_j)$.

So, $\bigwedge_{j \in I} L_SC(X_j, \tau_j) \geq L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i)$.

By Theorem 4.13, we have

$$\bigvee_{J \Subset I} \bigwedge_{j \in I - J} \Gamma_S(X_j, \tau_j)$$
$$\geq [\bigvee_{U \subseteq \prod_{i \in I} X_i} \Gamma_S(U, \prod_{i \in I} (\tau_S)_i / U) \otimes \bigvee_{X \subseteq \prod_{i \in I} X_i} N^{S^X}_{x}(U))]$$
$$\geq \bigvee_{U \subseteq \prod_{i \in I} X_i} \bigvee_{X \subseteq \prod_{i \in I} X_i} [\Gamma_S(U, \prod_{i \in I} (\tau_S)_i / U) \otimes N^{S^X}_{x}(U))]$$
$$\geq \bigwedge_{X \subseteq \prod_{i \in I} X_i} \bigvee_{U \subseteq \prod_{i \in I} X_i} [\Gamma_S(U, \prod_{i \in I} (\tau_S)_i / U) \otimes N^{S^X}_{x}(U))]$$
$$= L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i).$$

Therefore,

$$L_SC(\prod_{i \in I} X_i, \prod_{i \in I} (\tau_S)_i)$$
$$\leq [\bigwedge_{i \in I} L_SC(X_j, \tau_j) \wedge \bigvee_{J \Subset I} \bigwedge_{j \in I - J} \Gamma_S(X_j, \tau_j)].$$

$\square$

We can obtain the following corollary in crisp setting.

Corollary 5.12. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of non-empty topological spaces. If $\prod_{\lambda \in \Lambda} X_\lambda$ is locally semi-compact, then each X_λ is locally semi-compact and all but finitely many X_λ are semi-compact.

6. Conclusion

The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in semantic method of continuous-valued-logic). It continues various investigations into fuzzy topology in a legitimate way and extends some fundamental results in General Topology to fuzzifying topology. An important virtue of our approach (in which we follow Ying) is that we define topological notions as fuzzy predicates (by formulae of Łukasiewicz fuzzy logic) and prove the validity of fuzzy implications (or equivalences). Unlike the (more wide-spread) style of defining notions in fuzzy mathematics as crisp predicates of fuzzy sets, fuzzy predicates of fuzzy sets provide a more genuine fuzzification; furthermore the theorems in the form of valid fuzzy implications are more general than the corresponding theorems on crisp predicates of fuzzy sets. The main contribution of the present paper is to give characterizations of fuzzifying semi-compactness. Also, we define the concept of locally semi-compactness of fuzzifying topological spaces and obtain some basic properties of such spaces. There is a problem for further study.

Our results are derived in the Łukasiewicz continuous logic. Is it possible to generalize them to a more general logical setting, like residuated lattice-valued logic considered in [20–21]?

It's worth to say, there already exists a version in the setting of (2, L) topologies (see [10]). Our obtained results will be still valid in the setting of [10], in which the author only introduced L-continuity degree, L-openness degree, L-closedness degree of mappings between L-fuzzifying topological spaces. Moreover, he showed that most of elementary results related to continuous mappings, open mappings, and closed mappings in general topology could be extended to L-fuzzifying topological spaces by means of the graded concepts.

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