Fuzzy preuniform structure based on way below relation

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Abstract. In this paper, we will define an L-fuzzifying preuniform structure based on way below relation (or L-fuzzifying preuniform structure) and study their properties. Also, the concept of interior and closure operators in L-fuzzifying setting were established. Furthermore,the relation between L-fuzzifying preuniform and L-fuzzifying topology were explained.

1 Introduction and Preliminaries

In the last few years fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [5, 6, 7, 14, 16]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [6], the kind of topologies were defined by Chang [1] and Goguen [3] is called the topologies of fuzzy subsets, and further is naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L-topological space) is a family s of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and satisfy the basic conditions of classical topologies [13]. On the other hand, Höhle in [8] proposed the terminology L-fuzzy topology to be an L-valued mapping on the traditional powerset P(X) of X. The authors in [9, 14, 16, 22] defined an L-fuzzy topology to be an L-valued mapping on L^X of X. In 1952, Rosser and Turquette [23] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories? As an attempt to give a partial answer to this problem in the case of point set topology. Also, In [16], Ming introduced the concept of a fuzzifying uniform space and established some of its fundamental properties. In 2003, [15] the authors introduced the concept of a strong fuzzifying uniformity. Also, they established the relations between fuzzifying proximities, strong fuzzifying uniformities and corresponding fuzzifying topologies. The fuzzy quasi-uniformities were introduced by Hutton in [11]. Two other notions of fuzzy uniformities were given by Lowen in [15] and by Höhle in [10]. Some properties of the fuzzy quasi-uniformity due to Hutton were investigated in [12] by Katsaras for the lattice $L = [0, 1]$. In this paper was organized as follow: In section 2, the notion of uniform space was established and some of its properties were studied. In section 3, the uniform topology was studied. Furthermore, the concepts of interior and closure relative to uniform topology were investigated. In section 4, the uniform continuity was studied.

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Throughout this work $L = (L, \leq, \wedge, \vee, ')$ is a completely distributive complete lattice with an order reversing involution $'$, i.e,$(L, \leq, \wedge, \vee, ')$ is a complete lattice, for every $i \in I$ and for $A_i \subseteq L$,
$\bigwedge\limits_{i\in I} \bigvee A_i = \bigvee\limits_{\Psi \in \prod\limits_{i\in I} A_i} \bigwedge\limits_{i\in I} \Psi(i)$. and $' : L \longrightarrow L$ is a function such that for every $\alpha, \beta \in L$, $(\alpha')' = \alpha$ and if $\alpha \leq \beta$, then $\alpha' \geq \beta'$. The upper(resp.lower) universal element of L will denoted by $\top$ (resp.$\perp$).

Definition 1.1. [2] Let L be a complete lattice. We say that x is way below y, in symbols $x \ll y$, if for any directed subset $D \subseteq L$ the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$.

Proposition 1.2 [2] In a complete lattice L one has the following statements for all $u, x, y, z \in L$:
(i) $x \ll y$ implies $x \leq y$;
(ii) $u \leq x \ll y \leq z$ implies $u \ll z$;
(iii) $x \ll z$ and $y \ll z$ together imply $x \vee y \ll z$;
(iv) $\perp \ll x$.
(v) $x \ll y$ and $z \leq y$ implies $x \ll z$.
(vi) If $\top \not\ll \top$, then $\bigvee\limits_{\alpha < \top} \alpha = \top$.

Definition 1.3. [4] Let X be a nonempty set, L be a complete lattice and $\tau : 2^X \longrightarrow L$ be a function that satisfies the following conditions:
(O1) $\tau(X) = \tau(\phi) = 1$,
(O2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, for all $A, B \subseteq 2^X$;
(O3) for each $\{A_j : j \in J\} \subseteq 2^X$, $\tau(\bigcup\limits_{j \in J} A_j) \geq \bigwedge\limits_{j \in J} \tau(A_j)$.
Then τ is called an L-fuzzifying topology on X and the pair (X, τ) is called an L-fuzzifying topological space.

Definition 1.4. [4] Let (X, τ_1) and (Y, τ_2) be two L-fuzzifying topological spaces.
A function $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ is called an L-fuzzifying continuous if for all $B \in 2^Y$, $\tau_2(B) \leq \tau_1(f^{-1}(B))$.

2 L-fuzzifying preuniform structure

In this section, the notion of L-fuzzifying preuniform structure was established and some of its properties were studied. Furthermore, the concepts of L-fuzzifying preuniform interior and L-fuzzifying preuniform closure were investigated.

Definition 2.1. A function $\mathcal{U} : 2^{X \times X} \longrightarrow L$ is called an L-fuzzifying preuniform structure on X if it satisfies the following axioms:
PU1 : For any $u \in 2^{X \times X}$, if $\mathcal{U}(u) \neq \perp$, then $\Delta \subseteq u$.
PU2 : If $\mathcal{U}(u) \ll r$ and $u \subseteq v$, then $\mathcal{U}(v) \ll r$. Where $r \in L - \{\perp\}$.
The pair $(X, \mathcal{U})$ is called an L-fuzzifying preuniform space.
An L-fuzzifying preuniform is called of type **D** for any $u_1, u_2 \in 2^{X \times X}$.
PU3 : If $(\mathcal{U}(u_1) \wedge \mathcal{U}(u_2)) \ll r$, then $\mathcal{U}(u_1 \cap u_2) \ll r$.
An L-fuzzifying preuniform is called symmetrical for any $u \in 2^{X \times X}$.
PU4 : If $\mathcal{U}(u) \ll r$, then $\mathcal{U}(u^{-1}) \ll r$.

Proposition 2.2. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ two L-fuzzifying pre-uniform structure. Then satisfies the following:-
(1) $\mathcal{U}_1 \wedge \mathcal{U}_2$ and $\mathcal{U}_1 \vee \mathcal{U}_2$ are L-fuzzifying preuniform structures.
(2) When $\mathcal{U}_1$, $\mathcal{U}_2$ are symmetrical, then so are $\mathcal{U}_1 \wedge \mathcal{U}_2$ and $\mathcal{U}_1 \vee \mathcal{U}_2$.
(3) When $\mathcal{U}_1$, $\mathcal{U}_2$ are of type **D**, then so is $\mathcal{U}_1 \wedge \mathcal{U}_2$, but $\mathcal{U}_1 \vee \mathcal{U}_2$ is not of type **D**.
Proof.
(1) First, we prove that $\mathcal{U}_1 \wedge \mathcal{U}_2$ is an L-fuzzifying preuniform structure.
PU1 : If $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u) \neq \perp$, then $\mathcal{U}_1(u) \wedge \mathcal{U}_2(u) \neq \perp$. So, $\mathcal{U}_1(u) \neq \perp$ and $\mathcal{U}_2(u) \neq \perp$. Hence $\Delta \subseteq u$ for any $u \in 2^{X \times X}$.
PU2: Suppose $u \subseteq v$. If $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u) \ll r$, then $\mathcal{U}_1(u) \wedge \mathcal{U}_2(u) \ll r$.
So, $\mathcal{U}_1(u) \ll r$ or $\mathcal{U}_2(u) \ll r$. Hence $\mathcal{U}_1(v) \ll r$ or $\mathcal{U}_2(v) \ll r$. Thus $(\mathcal{U}_1 \wedge \mathcal{U}_2)(v) = \mathcal{U}_1(v) \wedge \mathcal{U}_2(v) \ll r$. Therefore $\mathcal{U}_1 \wedge \mathcal{U}_2$ is an L-fuzzifying preuniform structure.
Second, we prove that $\mathcal{U}_1 \vee \mathcal{U}_2$ is an L-fuzzifying pre-uniform structure.
PU1 : If $(\mathcal{U}_1 \vee \mathcal{U}_2)(u) \neq \perp$, then $\mathcal{U}_1(u) \vee \mathcal{U}_2(u) \neq \perp$. So, $\mathcal{U}_1(u) \neq \perp$ or $\mathcal{U}_2(u) \neq \perp$. Hence $\Delta \subseteq u$ for any $u \in 2^{X \times X}$
PU2: Suppose $u \subseteq v$. If $(\mathcal{U}_1 \vee \mathcal{U}_2)(u) \ll r$, then $\mathcal{U}_1(u) \vee \mathcal{U}_2(u) \ll r$. So, $\mathcal{U}_1(u) \ll r$ and $\mathcal{U}_2(u) \ll r$. Hence

$\mathcal{U}_1(v) \ll r$ and $\mathcal{U}_2(v) \ll r$. Thus $(\mathcal{U}_1 \vee \mathcal{U}_2)(v) = \mathcal{U}_1(v) \vee \mathcal{U}_2(v) \ll r$. Therefore $\mathcal{U}_1 \vee \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

(2) Suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are symmetrical.
If $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u) = \mathcal{U}_1(u) \wedge \mathcal{U}_2(u) \ll r$, then $\mathcal{U}_1(u) \ll r$ or $\mathcal{U}_2(u) \ll r$. So, $\mathcal{U}_1(u^{-1}) \ll r$ or $\mathcal{U}_2(u^{-1}) \ll r$ which implies $(\mathcal{U}_1(u^{-1}) \wedge \mathcal{U}_2(u^{-1})) = (\mathcal{U}_1 \wedge \mathcal{U}_2)(u^{-1}) \ll r$. Hence $\mathcal{U}_1 \wedge \mathcal{U}_2$ is symmetrical.
If $(\mathcal{U}_1 \vee \mathcal{U}_2)(u) = \mathcal{U}_1(u) \vee \mathcal{U}_2(u) \ll r$, then $\mathcal{U}_1(u) \ll r$ and $\mathcal{U}_2(u) \ll r$. So, $\mathcal{U}_1(u^{-1}) \ll r$ and $\mathcal{U}_2(u^{-1}) \ll r$ which implies $(\mathcal{U}_1(u^{-1}) \vee \mathcal{U}_2(u^{-1})) = (\mathcal{U}_1 \vee \mathcal{U}_2)(u^{-1}) \ll r$. Hence $\mathcal{U}_1 \vee \mathcal{U}_2$ is symmetrical.

(3) Suppose $\mathcal{U}_1$, $\mathcal{U}_2$ are of type $\mathbf{D}$.
If $((\mathcal{U}_1 \wedge \mathcal{U}_2)(u) \wedge (\mathcal{U}_1 \wedge \mathcal{U}_2)(v)) \ll r$, then $((\mathcal{U}_1(u) \wedge \mathcal{U}_1(v)) \wedge (\mathcal{U}_2(u) \wedge \mathcal{U}_2(v))) \ll r$. So, $(\mathcal{U}_1(u) \wedge \mathcal{U}_1(v)) \ll r$ or $(\mathcal{U}_2(u) \wedge \mathcal{U}_2(v)) \ll r$. Hence $(\mathcal{U}_1(u \cap v) \wedge \mathcal{U}_2(u \cap v)) \ll r$. Then $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u \cap v) \ll r$. Therefore, $\mathcal{U}_1 \wedge \mathcal{U}_2$ is of type $\mathbf{D}$.

Definition 2.3. Let $\mathcal{U}_1, \mathcal{U}_2$ are two L-fuzzifying preuniform structures on X. We denote $\mathcal{U}_1 \odot \mathcal{U}_2 : 2^{X \times X} \longrightarrow L$ defined by $(\mathcal{U}_1 \odot \mathcal{U}_2)(u) = \bigvee \{\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) \,|v \cap w \subseteq u\}$

Proposition 2.4. Let $\mathcal{U}_1, \mathcal{U}_2$ are two L-fuzzifying preuniform structures on X. Then
(i) $\mathcal{U}_1 \odot \mathcal{U}_2$ is an L-fuzzifying preuniform structure.
(ii) When $\mathcal{U}_1$ and $\mathcal{U}_2$ are symmetrical, then so is $\mathcal{U}_1 \odot \mathcal{U}_2$.
(iii) When $\mathcal{U}_1$ and $\mathcal{U}_2$ are of type $\mathbf{D}$, then so is $\mathcal{U}_1 \odot \mathcal{U}_2$.
Proof.
(i) Suppose $\mathcal{U}_1, \mathcal{U}_2$ are two L-fuzzifying preuniform structures on X.
PU1 : If $\mathcal{U}_1 \odot \mathcal{U}_2(u) = \bigvee \{\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) \,|v \cap w \subseteq u\} \neq \perp$, then there exist $v, w \in 2^{X \times X}$ such that $\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) \neq \perp$ and $v \cap w \subseteq u$. So, $\mathcal{U}_1(v) \neq \perp$ and $\mathcal{U}_2(w) \neq \perp$. Hence $\Delta \subseteq v \cap w \subseteq u$.
PU2 : is an L-fuzzifying preuniform structure.

(ii) Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are symmetrical.
If $(\mathcal{U}_1 \odot \mathcal{U}_2)(u) = \bigvee \{\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) \,|v \cap w \subseteq u\} \ll r$, then $(\mathcal{U}_1(v) \wedge \mathcal{U}_2(w)) \ll r$ for all $v \cap w \subseteq u$. So, $\mathcal{U}_1(v) \ll r$ or $\mathcal{U}(w) \ll r$ which implies $\mathcal{U}_1(v^{-1}) \ll r$ or $\mathcal{U}(w^{-1}) \ll r$. Thus $(\mathcal{U}_1(v^{-1}) \wedge \mathcal{U}_2(w^{-1})) \ll r$ for all $v^{-1} \cap w^{-1} \subseteq u^{-1}$.
Then $\bigvee \left\{\mathcal{U}_1(v^{-1}) \wedge \mathcal{U}_2(w^{-1}) \,\middle|v^{-1} \cap w^{-1} \subseteq u^{-1}\right\} = (\mathcal{U}_1 \odot \mathcal{U}_2)(u^{-1}) \ll r$. Therefore $\mathcal{U}_1 \odot \mathcal{U}_2$ is symmetrical.

(iii) Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are $\mathbf{D}$ and
$(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1 \cap u_2) < t < (\mathcal{U}_1 \odot \mathcal{U}_2)(u_1) \wedge (\mathcal{U}_1 \odot \mathcal{U}_2)(u_2)$, $t \in (0, 1)$. So
$(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1) \wedge (\mathcal{U}_1 \odot \mathcal{U}_2)(u_2) > t$ which implies $(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1) > t$ and $(\mathcal{U}_1 \odot \mathcal{U}_2)(u_2) > t$. Then
$\sup \{\mathcal{U}_1(x_1) \wedge \mathcal{U}_2(y_1) \,|x_1 \cap y_1 \subseteq u_1\} > t$ and $\sup \{ \mathcal{U}_1(x_2) \wedge \mathcal{U}_2(y_2) \,|x_2 \cap y_2 \subseteq u_2\} > t$.
Suppose $\alpha = \sup \{ \mathcal{U}_1(x_1) \wedge \mathcal{U}_2(y_1) \,|x_1 \cap y_1 \subseteq u_1\}$, then $\mathcal{U}_1(x_1) \wedge \mathcal{U}_2(y_1) \leq \alpha$ and $\beta = \sup \{\mathcal{U}_1(x_2) \wedge \mathcal{U}_2(y_2) \,|x_2 \cap y_2 \subseteq u_2\}$, then $\mathcal{U}_1(x_2) \wedge \mathcal{U}_2(y_2) \leq \beta$. So, $(\mathcal{U}_1(x_1) \wedge \mathcal{U}_2(y_1)) \wedge (\mathcal{U}_1(x_2) \wedge \mathcal{U}_2(y_2)) \leq \alpha \wedge \beta$.
Hence $(\mathcal{U}_1(x_1) \wedge \mathcal{U}_1(x_2)) \wedge (\mathcal{U}_2(y_1) \wedge \mathcal{U}_2(y_2)) \leq \alpha \wedge \beta$.
Since $\mathcal{U}_1(x_1 \cap x_2) \wedge \mathcal{U}_2(y_1 \cap y_2) \geq (\mathcal{U}_1(x_1) \wedge \mathcal{U}_1(x_2)) \wedge (\mathcal{U}_2(y_1) \wedge \mathcal{U}_2(y_2))$.
So, $\mathcal{U}_1(x_1 \cap x_2) \wedge \mathcal{U}_2(y_1 \cap y_2) \geq \alpha \wedge \beta$, then
$\sup \{\mathcal{U}_1(x_1 \cap x_2) \wedge \mathcal{U}_2(y_1 \cap y_2) \,|(x_1 \cap x_2) \cap (y_1 \cap y_2) \subseteq u \cap v\} \geq \alpha \wedge \beta > t$. Thus $\sup \{\mathcal{U}_1(x) \wedge \mathcal{U}_2(y) \,|x \cap y \subseteq u \cap v\} > t$.
Hence $(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1 \cap u_2) > t$. It is contradiction, then $(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1 \cap u_2) \geq (\mathcal{U}_1 \odot \mathcal{U}_2)(u_1) \wedge (\mathcal{U}_1 \odot \mathcal{U}_2)(u_2)$.
If $(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1 \cap u_2) \ll r$, then $(\mathcal{U}_1 \odot \mathcal{U}_2)(u_1) \wedge (\mathcal{U}_1 \odot \mathcal{U}_2)(u_2) \ll r$. Wherefore, $(\mathcal{U}_1 \odot \mathcal{U}_2)$ is of type $\mathbf{D}$.

Theorem 2.5. Let $(X, \mathcal{U})$ be an L-fuzzifying preuniform space. Define the function $\mathcal{I}_{\mathcal{U}}(A, r) : 2^X \times (L - \{\top\}) \longrightarrow 2^X$ as follows:-

$$\mathcal{I}_{\mathcal{U}}(A, r) = \bigcup \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A} (\mathcal{U}(w))' \right) \ll r' \right\}.$$

satisfies the following:
(1) $\mathcal{I}_{\mathcal{U}}(X, r) = X$; $\mathcal{I}_{\mathcal{U}}(\phi, r) = \phi$.

(2) $\mathcal{I}_{\mathcal{U}}(A,r) \subseteq A$.
(3) If $A \subseteq B$, then $\mathcal{I}_{\mathcal{U}}(A,r) \subseteq \mathcal{I}_{\mathcal{U}}(B,r)$.
(4) $\mathcal{I}_{\mathcal{U}}(A \cap B,r) \subseteq \mathcal{I}_{\mathcal{U}}(A,r) \cap \mathcal{I}_{\mathcal{U}}(B,r)$, but if $\mathcal{U}$ is of type **D** the equality holds.
(5) If $r_1 \leq r_2$, then $\mathcal{I}_{\mathcal{U}}(A,r_1) \supseteq \mathcal{I}_{\mathcal{U}}(A,r_2)$.
The function $\mathcal{I}_{\mathcal{U}}$ is called an L-fuzzifying preuniform interior.
Proof.
(1) Since $w[D] \subseteq X$, then $\mathcal{I}_{\mathcal{U}}(X,r) = \bigcup \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq X} (\mathcal{U}(w))' \right) \ll r' \right\} = X$. So, $\mathcal{I}_{\mathcal{U}}(X,r) = X$.
Since $w[D] \subseteq \phi$, then $w[D] = D = \phi$ for all r. So, $\mathcal{I}_{\mathcal{U}}(\phi,r) = \bigcup \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq \phi} (\mathcal{U}(w))' \right) \ll r' \right\} = \bigcup \phi = \phi$

(2) suppose $x \in \mathcal{I}_{\mathcal{U}}(A,r)$, then there exist $D \in 2^X$ s.t.
$x \in D$, $\left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A} (\mathcal{U}(w))' \right) \ll r'$, where $x \in D \subseteq w[D] \subseteq A$, then $x \in A$. So $\mathcal{I}_{\mathcal{U}}(A,r) \subseteq A$.

(3) Suppose $A \subseteq B$, then $\left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A} (\mathcal{U}(w))' \right) \geq \left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq B} (\mathcal{U}(w))' \right)$.
When $\left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A} (\mathcal{U}(w))' \right) \ll r'$, then $\left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq B} (\mathcal{U}(w))' \right) \ll r'$. Hence $\mathcal{I}_{\mathcal{U}}(A,r) \subseteq \mathcal{I}_{\mathcal{U}}(B,r)$.

(4) It is clear from (3) when $A \cap B \subseteq A$, then $\mathcal{I}_{\mathcal{U}}(A \cap B,r) \subseteq \mathcal{I}_{\mathcal{U}}(A,r)$ and when $A \cap B \subseteq B$, then $\mathcal{I}_{\mathcal{U}}(A \cap B,r) \subseteq \mathcal{I}_{\mathcal{U}}(B,r)$. Hence $\mathcal{I}_{\mathcal{U}}(A \cap B,r) \subseteq \mathcal{I}_{\mathcal{U}}(A,r) \cap \mathcal{I}_{\mathcal{U}}(B,r)$.
Let $x \in \mathcal{I}_{\mathcal{U}}(A,r) \cap \mathcal{I}_{\mathcal{U}}(B,r)$, then $x \in \mathcal{I}_{\mathcal{U}}(A,r)$ and $x \in \mathcal{I}_{\mathcal{U}}(B,r)$. Then there exist $D_1, D_2 \in 2^X$ such that $x \in D_1$, $\left(\bigwedge_{w_1 \in 2^{X \times X}, w_1[D_1] \subseteq A} (\mathcal{U}(w_1))' \right) \ll r'$ and $x \in D_2$, $\left(\bigwedge_{w_2 \in 2^{X \times X}, w_2[D_2] \subseteq B} (\mathcal{U}(w_2))' \right) \ll r'$. So, $x \in D_1 \cap D_2$ such that $\left(\left(\bigwedge_{w_1 \in 2^{X \times X}, w_1[D_1] \subseteq A} (\mathcal{U}(w_1))' \right) \ll r' \right) \wedge \left(\left(\bigwedge_{w_2 \in 2^{X \times X}, w_2[D_2] \subseteq B} (\mathcal{U}(w_2))' \right) \ll r' \right)$. Then $x \in D_1 \cap D_2$ such that $\left(\bigwedge_{w_1 \in 2^{X \times X}, w_1[D_1] \subseteq A} (\mathcal{U}(w_1))' \right) \vee \left(\bigwedge_{w_2 \in 2^{X \times X}, w_2[D_2] \subseteq B} (\mathcal{U}(w_2))' \right) \ll r'$. Thus
$\left(\bigwedge_{w_1 \in 2^{X \times X}, w_1[D_1] \subseteq A, w_2 \in 2^{X \times X}, w_2[D_2] \subseteq B} (\mathcal{U}(w_1) \wedge \mathcal{U}(w_2))' \ll r' \right)$. Where $\mathcal{U}$ is of type **D**, then
$\left(\bigwedge_{w_1 \cap w_2 \in 2^{X \times X}, (w_1 \cap w_2)[D_1 \cap D_2] \subseteq A \cap B} (\mathcal{U}(w_1 \cap w_2))' \ll r' \right)$. Therefore $x \in D$ such that $\left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A \cap B} (\mathcal{U}(w))' \ll r' \right)$, then $x \in \mathcal{I}_{\mathcal{U}}(A \cap B,r)$. Hence $\mathcal{I}_{\mathcal{U}}(A \cap B,r) = \mathcal{I}_{\mathcal{U}}(A,r) \cap \mathcal{I}_{\mathcal{U}}(B,r)$.

(5) Suppose $r_1 \leq r_2$, $\mathcal{I}_{\mathcal{U}}(A,r_2) = \cup \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A} (\mathcal{U}(w))' \right) \ll r_2' \right\}$. Since $r_2' \leq r_1'$, then $\left(\bigwedge_{w \in 2^{X \times X}, w[D] \subseteq A} (\mathcal{U}(w))' \right) \ll r_1'$. So, $\mathcal{I}_{\mathcal{U}}(A,r_1) \supseteq \mathcal{I}_{\mathcal{U}}(A,r_2)$.

Theorem 2.6. Let $(X,\mathcal{U})$ be an L-fuzzifying preuniform space. Define the function $\mathcal{C}_{\mathcal{U}}(A,r) : 2^X \times (L - \{\top\}) \longrightarrow 2^X$ as follows:-

$$\mathcal{C}_{\mathcal{U}}(A,r) = \bigcap \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X \times X}, A \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r' \right\}.$$

satisfies the following:
(1) $\mathcal{C}_{\mathcal{U}}(\phi,r) = \phi$; $\mathcal{C}_{\mathcal{U}}(X,r) = X$.

(2) $\mathcal{C}_{\mathcal{U}}(A,r) \supseteq A$.
(3) If $A \subseteq B$, then $\mathcal{C}_{\mathcal{U}}(A,r) \subseteq \mathcal{C}_{\mathcal{U}}(B,r)$.
(4) $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r)$, but if $\mathcal{U}$ is of type **D** the equality holds.
(5) If $r_1 \leq r_2$, then $\mathcal{C}_{\mathcal{U}}(A,r_1) \subseteq \mathcal{C}_{\mathcal{U}}(A,r_2)$.
The function $\mathcal{C}_{\mathcal{U}}$ is called an L-fuzzifying preuniform closure.
Proof.
(1) Since $\phi \subseteq w[D]$, then $\mathcal{C}_{\mathcal{U}}(\phi,r) = \bigcap \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X\times X}, \phi \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r' \right\} = \phi$. So, $\mathcal{C}_{\mathcal{U}}(\phi,r) = \phi$.
Since $X \subseteq w[D]$, then $D = X$ for all r. So, $\mathcal{C}_{\mathcal{U}}(X,r) = \bigcap \left\{ D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X\times X}, X \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r' \right\} = \bigcap X = X$

(2) Suppose $x \in \mathcal{C}_{\mathcal{U}}(A,r)$, then for all $D \in 2^X$ such that $x \in D$, $\left(\bigwedge_{w \in 2^{X\times X}, A \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r'$. So, $\mathcal{C}_{\mathcal{U}}(A,r) \supseteq A$.

(3) Suppose $A \subseteq B$, then $\left(\bigwedge_{w \in 2^{X\times X}, B \subseteq w[D]} (\mathcal{U}(w))' \right) \geq \left(\bigwedge_{w \in 2^{X\times X}, A \subseteq w[D]} (\mathcal{U}(w))' \right)$.
When $\left(\bigwedge_{w \in 2^{X\times X}, B \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r'$, then $\left(\bigwedge_{w \in 2^{X\times X}, A \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r'$. Hence $\mathcal{C}_{\mathcal{U}}(A,r) \subseteq \mathcal{C}_{\mathcal{U}}(B,r)$.

(4) It is clear from (3) when $A \cup B \supseteq A$, then $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(A,r)$ and when $A \cup B \supseteq B$, then $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(B,r)$. Hence $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r)$.

$$\mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r) = \bigcap \left\{ D_1 \in 2^X \,\middle|\, \left(\bigwedge_{w_1 \in 2^{X\times X}, A \subseteq w_1[D_1]} (\mathcal{U}(w_1))' \right) \ll r' \right\} \cup \bigcap \left\{ D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_2 \in 2^{X\times X}, B \subseteq w_2[D_2]} (\mathcal{U}(w_2))' \right) \ll r' \right\}.$$
$$= \bigcap \left\{ D_1 \cup D_2 \in 2^X \,\middle|\, \left(\left(\bigwedge_{w_1 \in 2^{X\times X}, A \subseteq w_1[D_1]} (\mathcal{U}(w_1))' \right) \ll r' \right) or \left(\left(\bigwedge_{w_2 \in 2^{X\times X}, B \subseteq w_2[D_2]} (\mathcal{U}(w_2))' \right) \ll r' \right) \right\}$$
$$= \bigcap \left\{ D_1 \cup D_2 \in 2^X \,\middle|\, \left(\left(\bigwedge_{w_1 \in 2^{X\times X}, A \subseteq w_1[D_1]} (\mathcal{U}(w_1))' \right) \wedge \left(\bigwedge_{w_2 \in 2^{X\times X}, B \subseteq w_2[D_2]} (\mathcal{U}(w_2))' \right) \right) \ll r' \right\}$$
$$= \bigcap \left\{ D_1 \cup D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_1 \in 2^{X\times X}, A \subseteq w_1[D_1], w_2 \in 2^{X\times X}, B \subseteq w_2[D_2]} (\mathcal{U}(w_1) \vee \mathcal{U}(w_2))' \ll r' \right) \right\}$$
$$\supseteq \bigcap \left\{ D_1 \cup D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_1 \in 2^{X\times X}, A \subseteq w_1[D_1], w_2 \in 2^{X\times X}, B \subseteq w_2[D_2]} (\mathcal{U}(w_1) \wedge \mathcal{U}(w_2))' \ll r' \right) \right\}$$
$$\supseteq \bigcap \left\{ D_1 \cup D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_1 \cap w_2 \in 2^{X\times X}, A \cup B \subseteq (w_1 \cap w_2)[D_1 \cup D_2]} (\mathcal{U}(w_1 \cap w_2))' \ll r' \right) \right\}$$
$$= \bigcap \left\} D \in 2^X \,\middle|\, \left(\bigwedge_{w \in 2^{X\times X}, A \cup B \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r' \right\} = \mathcal{C}_{\mathcal{U}}(A \cup B,r).$$
Such that $(w_1 \cap w_2)[D_1 \cup D_2] \subseteq w_1[D_1] \cap w_2[D_2]$. Hence $\mathcal{C}_{\mathcal{U}}(A \cup B,r) = \mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r)$.

(5) Suppose $r_1 \leq r_2$, $\mathcal{C}_{\mathcal{U}}(A,r_2) = \cap \left\{ D \in 2^{X\times X} \,\middle|\, \left(\bigwedge_{w \in 2^{X\times X}, A \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r_2' \right\}$. Since $r_2' \leq r_1'$, then $\left(\bigwedge_{w \in 2^{X\times X}, A \subseteq w[D]} (\mathcal{U}(w))' \right) \ll r_1'$. So, $\mathcal{C}_{\mathcal{U}}(A,r_1) \subseteq \mathcal{C}_{\mathcal{U}}(A,r_2)$.

3 The relation between L-fuzzifying preuniform and L-fuzzifying topology

In this section, the relation between L-fuzzifying preuniform structure and L-fuzzifying topologies are established.

Theorem 3.1. Let $(X,\mathcal{U})$ is an L-fuzzifying preuniform space of type **D** and $\top \not\ll \top$. Define a map $\tau_{\mathcal{U}} : 2^X \longrightarrow L$ by

$$\tau_{\mathcal{U}}(A) = \sup\left\{r \in (L - \{\top\})\,|\mathcal{I}_{\mathcal{U}}(A, r) = A\right\}$$

Then $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.
Proof.
(O1) Since $\mathcal{I}_{\mathcal{U}}(X, r) = X$ and $\mathcal{I}_{\mathcal{U}}(\phi, r) = \phi$, for all $r \in (L - \{\top\})$. Then $\tau_{\mathcal{U}}(X) = \tau_{\mathcal{U}}(\phi) = \top$.

(O2) Suppose there exist $A, B \in 2^X$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B) > t > \tau_{\mathcal{U}}(A \cap B)$. Then $\tau_{\mathcal{U}}(A) > t$ and $\tau_{\mathcal{U}}(B) > t$. So there exist $r_1, r_2 > t$ such that $\mathcal{I}_{\mathcal{U}}(A, r_1) = A$ and $\mathcal{I}_{\mathcal{U}}(B, r_2) = B$. Put $r = r_1 \wedge r_2$ and from Theorem 2.5 (4) and (5), we have $\mathcal{I}_{\mathcal{U}}(A \cap B, r) = \mathcal{I}_{\mathcal{U}}(A, r) \cap \mathcal{I}_{\mathcal{U}}(B, r) \supseteq \mathcal{I}_{\mathcal{U}}(A, r_1) \cap \mathcal{I}_{\mathcal{U}}(B, r_2) = A \cap B$. So, $\mathcal{I}_{\mathcal{U}}(A \cap B, r) = A \cap B$. Thus, $\tau_{\mathcal{U}}(A \cap B) \geq r$ and $r > t$ and this is a contradiction. Hence $\tau_{\mathcal{U}}(A \cap B) \geq \tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B)$.

(O3) Suppose there exists a family $\left\{A_i \in 2^X\,|i \in \Gamma\right\}$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}\left(\bigcup\limits_{i\in\Gamma} A_i\right) < t < \bigwedge\limits_{i\in\Gamma} \tau_\eta(A_i)$. Since $\bigwedge\limits_{i\in\Gamma} \tau_{\mathcal{U}}(A_i) > t$ for each $i \in \Gamma$.There exist $r_i > t$ such that $\mathcal{I}_{\mathcal{U}}(A_i, r_i) = A_i$. Put $r = \bigwedge\limits_{i\in\Gamma} r_i$. We have $\mathcal{I}_{\mathcal{U}}(\bigcup\limits_{i\in\Gamma} A_i, r) \supseteq (\bigcup\limits_{i\in\Gamma} \mathcal{I}_{\mathcal{U}}(A_i, r)) \supseteq (\bigcup\limits_{i\in\Gamma} \mathcal{I}_{\mathcal{U}}(A_i, r_i)) = \bigcup\limits_{i\in\Gamma} A_i$.
So, $\mathcal{I}_{\mathcal{U}}(\bigcup\limits_{i\in\Gamma} A_i, r) = \bigcup\limits_{i\in\Gamma} A_i$. Thus, $\tau_{\mathcal{U}}\left(\bigcup\limits_{i\in\Gamma} A_i\right) \geq r$ and $r > t$ and this is a contradiction.
Hence $\tau_{\mathcal{U}}\left(\bigcup\limits_{i\in\Gamma} A_i\right) \geq \bigwedge\limits_{i\in\Gamma} \tau_{\mathcal{U}}(A_i)$. Thus, $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Theorem 3.2. Let $(X,\mathcal{U})$ is an L-fuzzifying preuniform space of type **D** and $\top \not\ll \top$. Define a map $\tau_{\mathcal{U}} : 2^X \longrightarrow L$ by

$$\tau_{\mathcal{U}}(A) = \sup\left\{r \in (L - \{\top\})\,|\mathcal{C}_{\mathcal{U}}(A^c, r) = A^c\right\}$$

Then $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.
proof
(O1) Since $\mathcal{C}_{\mathcal{U}}(X, r) = X$ and $\mathcal{C}_{\mathcal{U}}(\phi, r) = \phi$, for all $r \in (L - \{\top\})$. Then $\tau_{\mathcal{U}}(X) = \tau_{\mathcal{U}}(\phi) = \top$.

(O2) Suppose there exist $A, B \in 2^X$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B) > t > \tau_{\mathcal{U}}(A \cap B)$. Then $\tau_{\mathcal{U}}(A) > t$ and $\tau_{\mathcal{U}}(B) > t$. So there exist $r_1, r_2 > t$ such that $\mathcal{C}_{\mathcal{U}}(A^c, r_1) = A^c$ and $\mathcal{C}_{\mathcal{U}}(B^c, r_2) = B^c$. Put $r = r_1 \wedge r_2$ and from Theorem 2.5 (4) and (5), we have $\mathcal{C}_{\mathcal{U}}(A^c \cup B^c, r) = \mathcal{C}_{\mathcal{U}}(A^c, r) \cup \mathcal{C}_{\mathcal{U}}(B^c, r) \subseteq \mathcal{C}_{\mathcal{U}}(A^c, r_1) \cup \mathcal{C}_{\mathcal{U}}(B^c, r_2) = A^c \cup B^c$. So, $\mathcal{C}_{\mathcal{U}}((A \cap B)^c, r) = (A \cap B)^c$. Thus, $\tau_{\mathcal{U}}(A \cap B) \geq r > t$ and this is a contradiction. Hence $\tau_{\mathcal{U}}(A \cap B) \geq \tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B)$.

(O3) Suppose there exists a family $\left\{A_i \in 2^X\,|i \in \Gamma\right\}$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}\left(\bigcup\limits_{i\in\Gamma} A_i\right) < t < \bigwedge\limits_{i\in\Gamma} \tau_\eta(A_i)$. Since $\bigwedge\limits_{i\in\Gamma} \tau_{\mathcal{U}}(A_i) > t$ for each $i \in \Gamma$.There exist $r_i > t$ such that $\mathcal{C}_{\mathcal{U}}((A_i)^c, r_i) = (A_i)^c$. Put $r = \bigwedge\limits_{i\in\Gamma} r_i$. We have $\mathcal{C}_{\mathcal{U}}(\bigcap\limits_{i\in\Gamma} (A_i)^c, r) \subseteq (\bigcap\limits_{i\in\Gamma} \mathcal{C}_{\mathcal{U}}((A_i)^c, r)) \subseteq (\bigcap\limits_{i\in\Gamma} \mathcal{C}_{\mathcal{U}}((A_i)^c, r_i)) = \bigcap\limits_{i\in\Gamma} (A_i)^c$.
So, $\mathcal{C}_{\mathcal{U}}((\bigcup\limits_{i\in\Gamma} A_i)^c, r) = (\bigcup\limits_{i\in\Gamma} A_i)^c$. Thus, $\tau_{\mathcal{U}}\left(\bigcup\limits_{i\in\Gamma} A_i\right) \geq r > t$ and this is a contradiction.

Hence $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) \geq \bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i)$. Thus, $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Theorem 3.3. Let $(X,\mathcal{U})$ is an L-fuzzifying preuniform space of type **D**. Define a map $\tau_{\mathcal{U}} : 2^X \longrightarrow L$ by

$$\tau_{\mathcal{U}}(A) = \bigwedge_{x\in A} \bigvee_{u[x]\subseteq A} \mathcal{U}(u)$$

Then $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Proof.

(O1) It is clear $\tau_{\mathcal{U}}(X) = \top$.

(O2) $\tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B) = (\bigwedge_{x\in A} \bigvee_{u_1[x]\subseteq A} \mathcal{U}(u_1)) \wedge (\bigwedge_{x\in B} \bigvee_{u_2[x]\subseteq B} \mathcal{U}(u_2)) \leq \bigwedge_{x\in A, x\in B} \bigvee_{u_1[x]\subseteq A, u_2[x]\subseteq B} (\mathcal{U}(u_1) \wedge \mathcal{U}(u_2))$

(O3) $\tau_{\mathcal{U}}(\bigcup_{i\in\Gamma} A_i) = \bigwedge_{x\in \bigcup_{i\in\Gamma} A_i} \bigvee_{u[x]\subseteq \bigcup_{i\in\Gamma} A_i} \mathcal{U}(u) = \bigwedge_{i\in\Gamma}\left(\bigwedge_{x\in A_i} \bigvee_{u[x]\subseteq \bigcup_{i\in\Gamma} A_i} \mathcal{U}(u)\right) \geq \bigwedge_{i\in\Gamma}\left(\bigwedge_{x\in A_i} \bigvee_{u[x]\subseteq A_i} \mathcal{U}(u)\right) = \bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i)$.

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