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A STUDY ON DOUBLE TOPOLOGICAL SPACES

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The purpose of this paper is to introduce a new type of topological spaces called a double topological space defined on a bi-set. This notion is different from the bitopological space due to J. C. Kelly [8]. Also, some fundamental concepts defined on a double topological space are investigated.

1. Introduction

There are many theories concerning different types of sets have appeared in the literature, such as ordinary set theory [6], fuzzy set theory [16], rough set theory [10], vague set theory [4], intuitionistic fuzzy set theory [1], soft set theory [9], etc. With each one of these kinds of sets there is a topological structure, fuzzy topology [2], rough topology [11], soft and fuzzy soft topology [5], [13], [15], intuitionistic fuzzy topology [3], [12], vague topology [14]. The study of bitopological spaces $(\mathcal{X}, \tau_1, \tau_2)$ was studied by J. C. Kelly [8]. We notice that both τ_1 and τ_2 defined on the same set X. This generalization of the topological spaces was noticed, and many papers were written in this field. Now, if we have a set of objects A_1 and to each object, we have attached one or more object from another set A_2, then we have a pair of sets, the original set and the attached set. This pair $A_B = (A_1, A_2)$ of sets, such that $A_1 \neq A_2$, is called a bi-set [7]. So, we can give another generalization of the topological space by introducing the double topological space on a bi-set.

In [7] the concept of a bi-set, equality, cardinality, complement, subsets of a bi-set, union, intersection, and symmetric difference of two bi-sets were defined. Also, the functions on a bi-set were investigated and some of their properties were discussed. This paper continues the study of the theory of bi-set and introduce the notion

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of the double topological space. It is organized as follows. In Section 2 we give some of the basic notions and results concerning the bi-sets. In Section 3, we investigate some new results of the bi-sets. The purpose of Section 4 is to introduce and study the double topology and the double subspace of a double topology. Also, we give and study the definitions of a double base and a double subbase of a double topological space. In section 5, we introduce the concepts of the double limit points, a double closure, and a double interior of a bi-set. Furthermore, the concepts of the double neighborhoods are investigated in Section 6. The goal of the last section is to conclude this paper with a succinct but precise recapitulation of our main findings, and to give some lines for future research.

2. Preliminaries

In this section, we present the basic definitions and results of bi-set theory.

Definition 2.1 [7]. A bi-set is simply a pair of collections of distinct objects. The number of objects contained within a bi-set may be finite, countably infinite or uncountably infinite. It is written as $A_B = (A_1, A_2)$, where A_1 and A_2 are ordinary sets with $A_1 \neq A_2$. That is, a bi-set A_B is defined as a pair of different characteristics of two sets A_1 and A_2 which they are called components of the bi-set. A bi-set is said to be finite if both components are finite, infinite if any one or both the components are infinite, countable if both components are countable, and uncountable if any one or both components are uncountable. The universal bi-set is the set of pair of all possible objects. The universal bi-set is commonly written as $XY_B = (X, Y)$, where X and Y are universal sets of the first and second components, respectively.
The set of all bi-subsets of a bi-set XY_B is called the power bi-set of the bi-set XY_B. We denote it by $\mathcal{P}(XY_B)$. That is, $\mathcal{P}(XY_B) = (\mathcal{P}(X), \mathcal{P}(Y))$.

Example 2.1. (1) Suppose that X denote the set of students in some class and Y be the set of courses that those students are studying. Then (X, Y) is a bi-set.

(2) Suppose we have a set of people X and the set Y of cities in which they live. Then (X, Y) is a bi-set.

Definition 2.2 [7]. Suppose that $A_B = (A_1, A_2)$ is a bi-set , we say that x_B is a bi-point of (belong to or a member of) A_B, denoted by $x_B \in A_B$, if $x_B = (x_1, x_2)$, where $x_1 \in A_1$ and $x_2 \in A_2$. If x_B is not a bi-point of A_B, we write $x_B \notin A_B$.
Two bi-points $a_B = (a_1, a_2)$ and $b_B = (b_1, b_2)$ are said to be bi-different if $a_1 \neq b_1$ and $a_2 \neq b_2$. If $a_1 \neq b_1$ or $a_2 \neq b_2$ but not both, we say that the two bi-points are different.

Definition 2.3 [7]. (1) Two bi-sets A_B and C_B are said to be bi-equal if they have exactly the same components. If $A_B = (A_1, A_2)$ and $C_B = (C_1, C_2)$ be two bi-sets, we say that A_B and C_B are bi-equal and written as $A_B = C_B$ if and only if A_1 and C_1 are identical and A_2 and C_2 are identical. That is, $A_1 = C_1$ and $A_2 = C_2$. Otherwise they are not equal.

(2) For two bi-sets $A_B = (A_1, A_2)$ and $C_B = (C_1, C_2)$, we say that A_B is a bi-subset of C_B if each element of A_1 is also an element of C_1 and each element of A_2 is also an element of C_2. In formal notation we write $A_B \subseteq C_B$ where $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$. If $A_B \subseteq C_B$, then we also say C_B bi-contains (or super bi-set of) A_B.

(3) The bi-complement of a bi-set A_B, denoted by A_B^c, is the collection of pair of all objects in the universal bi-set that are not in A_B. That is, $A_B^c = (A_1^c, A_2^c)$.

In set builder form, $A_B^c = (\{x : x \notin A_1\}, \{y : y \notin A_2\})$.

Definition 2.4 [7]. (i) The bi-union of two bi-sets $A_B = (A_1, A_2)$ and $C_B = (C_1, C_2)$, denoted by $A_B \cup C_B$, is defined as $D_B = A_B \cup C_B$, where $D_B = (D_1, D_2)$, $D_1 = A_1 \cup C_1$ and $D_2 = A_2 \cup C_2$. Using set builder form, $D_B = (D_1, D_2) = (\{x : x \in A_1 \cup C_1\},\{y : y \in A_2 \cup C_2\})$.

(ii) The bi-intersection of two bi-sets $A_B = (A_1, A_2)$ and $C_B = (C_1, C_2)$, denoted by $A_B \cap C_B$, is defined as $D_B = A_B \cap C_B$, where $D_B = (D_1, D_2)$, $D_1 = A_1 \cap C_1$ and $D_2 = A_2 \cap C_2$. Using set builder form, $D_B = (D_1, D_2) = (\{x : x \in A_1 \cap C_1\}, \{y : y \in A_2 \cap C_2\})$.

(iii) If A_B and C_B are bi-sets and $A_B \cap C_B = \phi$, that is, $A_1 \cap C_1 = \phi$ and $A_2 \cap C_2 = \phi$, then we say that A_B and C_B are disjoint bi-sets.

3. Some Remarks on Bi-Sets

In this section we investigate some basic properties of a bi-sets.

Definition 3.1. Let $A_B = (A_1, A_2)$ and $C_B = (C_1, C_2)$ be two bi-sets. The bi-difference between A_B and C_B , denoted by $A_B \backslash C_B$, is defined by the bi-set $A_B \backslash C_B = D_B$, where $D_B = (A_1 \backslash C_1, A_2 \backslash C_2)$.

Example 3.1. Let $A_B = (\{1, 2, 3, 4\}, \{x, y, z\})$ and $C_B = (\{3, 4, 5, 6\}, \{z, w, r, s\})$. Then $A_B \backslash C_B = (\{1, 2\}, \{x, y\})$, $A_B \cap C_B = (\{3, 4\}, \{z\})$ and $A_B \cup C_B = (\{1, 2, 3, 4, 5, 6\}, \{x, y, z, w, r, s\})$.

Remark 3.1. (1) We shall denote the empty bi-set by $\phi_B = (\phi, \phi)$.

(2) By a non-empty bi-set $A_B = (A_1, A_2)$, we mean that $A_1 \neq \phi$ and $A_2 \neq \phi$. Therefore (A_1, ϕ) and (ϕ, A_2) are not non-empty bi-sets, unless $A_1 = \phi$ and $A_2 = \phi$.

Notations . Let $A_B = (A_1, A_2)$ and $B_B = (B_1, B_2)$ be two bi-sets. we shall use the notation $A_B \cap B_B \neq \phi$ to mean that $A_1 \cap B_1 \neq \phi$ and $A_2 \cap B_2 \neq \phi$. The notation $A_B \cap B_B \neq \phi_B$ to mean that $A_1 \cap B_1 \neq \phi$ or $A_2 \cap B_2 \neq \phi$ but not both.

Proving the following theorem forward and thus omitted.

Theorem 3.1. *Let XY_B be a universal bi-set and $A_B, B_B, C_B \subset XY_B$. Then:*

(1) *The associative laws are satisfied, i.e.* $(A_B \cup B_B) \cup C_B = A_B \cup (B_B \cup C_B)$ *and* $(A_B \cap B_B) \cap C_B = A_B \cap (B_B \cap C_B)$

(2) *The distributive laws are satisfied, i.e.* $A_B \cup (B_B \cap C_B) = (A_B \cup B_B) \cap (A_B \cup C_B)$ *and* $A_B \cap (B_B \cup C_B) = (A_B \cap B_B) \cup (A_B \cap C_B)$

(iii) *The commutative laws are satisfied, i.e.* $A_B \cup B_B = B_B \cup A_B$ *and* $A_B \cap B_B = B_B \cap A_B$.

(iv) *The identity laws are satisfied, i.e.* $A_B \cup \phi_B = A_B$, $A_B \cap XY_B = A_B$ *and* $A_B \cup XY_B = XY_B, A_B \cap \phi_B = \phi_B$.

4. Double Topologies

Now, we introduce the concept of a double topology on a bi-set.

Definition 4.1. Let XY_B be a bi-set. A subcollection τ_D of $\mathcal{P}(XY_B)$ is called a double topology on XY_B, if it satisfies the following three conditions:

(1) XY_B, $\phi_B \in \tau_D$;

(2) If $C_{B_\alpha} \in \tau_D$, $\alpha \in \Lambda$, then $\bigcup\{C_{B_\alpha} : \alpha \in \Lambda\} \in \tau_D$;

(3) If $C_{B_1}, C_{B_2} \in \tau_D$, then $C_{B_1} \cap C_{B_2} \in \tau_D$.

The pair (XY_B, τ_D) is called a double topological space. The elements of τ_D are called bi-open sets. The bi-complement of a bi-open set is called bi-closed.

Remark 4.1. It is obvious that in a double topological space (XY_B, τ_D), XY_B and ϕ_B are bi-closed sets, arbitrary bi-intersection of a bi-closed sets is bi-closed and finite bi-union of a bi-closed sets is bi-closed.

Example 4.1. Let XY_B be a bi-set. Then

(a) $\mathcal{J}_B = \{XY_B,\ \phi_B\}$ is a double topology on XY_B called the double indiscrete topology. $(XY_B, \mathcal{J}_B)$ is said to be the double indiscrete space.

(b) $\mathcal{D}_B = \mathcal{P}(XY_B)$ is a double topology on XY_B called the double discrete topology. $(XY_B, \mathcal{D}_B)$ is said to be the double discrete space.

Proposition 4.1. *Let τ_D be a double topology on a bi-set XY_B, τ_1 is the set of all first components of members of τ_D, and τ_2 is the set of all second components of members of τ_D. Then τ_1 is a topology on X and τ_2 is a topology on Y.*

Proof. Since $XY_B = (X, Y)$ and $\phi_B = (\phi, \phi)$ are members of τ_D, $X, \phi \in \tau_1$. Now, assume that $U_i \in \tau_1, i \in \Lambda$. Then for each $i \in \Lambda$, there exists $V_i \subset Y$, such that $(U_i, V_i) \in \tau_D$. Then $\bigcup (U_i, V_i) = (\bigcup U_i, \bigcup V_i) \in \tau_D$, and so $\bigcup U_i \in \tau_1$. Finally, assume that $U_1, U_2 \in \tau_1$. Then there exist $V_1, V_2 \subset Y$ such that $(U_1, V_1), (U_2, V_2) \in \tau_D$. Thus $(U_1, V_1) \cap (U_2, V_2) = (U_1 \cap U_2, V_1 \cap V_2) \in \tau_D$. Therefore, $U_1 \cap U_2 \in \tau_1$. This shows that τ_1 is a topology on X. Similarly, τ_2 is a topology on Y. ■

Remark 4.2. The topologies τ_1 and τ_2 on X and Y, respectively, in Proposition 4.1, are called the topologies induced by the double topology τ_D on XY_B. (X, τ_1) and (Y, τ_2) are called the induced topological spaces.

Proposition 4.2. *Let XY_B be a bi-set and τ_D be a double topology on XY_B and τ_1* and *τ_2 be the topologies on X and Y, respectively, induced by the double topology τ_D. Then $(\tau_1, \tau_2) = \{(U, V) : U \in \tau_1$ and $V \in \tau_2\}$ is a double topology on XY_B finer than τ_D, i.e., $\tau_D \subset (\tau_1, \tau_2)$.*

Proof. Since $X, \phi \in \tau_1$ and $Y, \phi \in \tau_2$, then $XY_B = (X, Y) \in (\tau_1, \tau_2)$ and $\phi_B = (\phi, \phi) \in (\tau_1, \tau_2)$. Now, assume that $\{G_i = (U_i, V_i) : i \in \Lambda\} \subset (\tau_1, \tau_2)$. Then $U_i \in \tau_1$ and $V_i \in \tau_2$ for $i \in \Lambda$. Since τ_1 and τ_2 are topologies, $\bigcup U_i \in \tau_1$ and $\bigcup V_i \in \tau_2$. Thus $\bigcup G_i = \bigcup (U_i, V_i) = (\bigcup U_i, \bigcup V_i) \in (\tau_1, \tau_2)$. Finally, suppose that $G_1 = (U_1, V_1)$, $G_2 = (U_2, V_2) \in (\tau_1, \tau_2)$. Then $U_1, U_2 \in \tau_1$ and $V_1, V_2 \in \tau_2$, and so $U_1 \cap U_2 \in \tau_1$ and $V_1 \cap V_2 \in \tau_2$. Therefore $G_1 \cap G_2 = (U_1, V_1) \cap (U_2, V_2) = (U_1 \cap U_2, V_1 \cap V_2) \in (\tau_1, \tau_2)$. This shows that (τ_1, τ_2) a double topology on XY_B. Finally, since $U_B = (U_1, U_2) \in \tau_D$, $U_1 \in \tau_1$ and $U_2 \in \tau_2$. So $U_B = (U_1, U_2) \in (\tau_1, \tau_2)$. Thus, $\tau_D \subset (\tau_1, \tau_2)$. ■

Remark 4.3. The inclusion relation in Proposition 4.2 may not be equality, as shown by the following example..

Example 4.2. Let XY_B be a bi-set, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$ and $\tau_D = \{XY_B,\ (\{1\}, \{a\}),\ (\{2\}, \{a\}),\ (\{1, 2\}, \{a\}),\ (\phi, \{a\}), \phi_B\}$. It is clear that τ_D is a double topology on XY_B. The topologies on X and Y induced by τ_D are $\tau_1 = \{X, \phi, \{1\},\ \{2\},\ \{1, 2\}\}$ and $\tau_2 = \{Y, \phi, \{a\}\}$, respectively. So $(\tau_1, \tau_2) = \{(X, Y), (X, \phi), (X, \{a\}), (\{1\}, Y), (\{1\}, \phi), (\{1\}, \{a\}), (\{2\}, Y), (\{2\}, \phi), (\{2\}, \{a\}), (\{1, 2\}, Y), (\{1, 2\}, \phi), (\{1, 2\}, \{a\}), (\phi, Y),\ (\phi, \{a\}), (\phi, \phi)\}$. It clear that $\tau_D \neq (\tau_1, \tau_2)$ while the two topologies induced by (τ_1, τ_2) are the same as the two topologies induced by τ_D.

Remark 4.4. The induced topologies of the double indiscrete (respectively double discrete) topology are the indiscrete (respectively discrete) topologies on the respective components of XY_B. But not conversely, as can be shown by the following example.

Example 4.3. Let $XY_B = (X, Y)$, where $X = \{1, 2, 3, 4\}$ and $Y = \{x, y, z\}$. Then

$$\begin{gathered}\tau_D = \{XY_B, \phi_B, (\{1\}, \{x, y\}), (\{2\}, \{x\}), (\{3\}, \{y, z\}), (\{4\}, \{x, z\}),\\ (\{1, 2\}, \{x, y\}), (\phi, \{x\}), (\{1, 3\}, Y), (\phi, \{y\}), (\{1, 4\}, Y), (\{2, 3\}, Y)\\ (\{2, 4\}, \{x, z\}), (\phi, \{x, y\}), (\{1, 2, 3\}, Y), (\{2\}, \{x, y\}), (\{1, 2, 4\}, Y),\\ (\{3, 4\}, Y), (\phi, \{z\}), (\{3\}, Y), (\{1, 3, 4\}, Y), (\phi, \{y, z\}), (\{2, 3, 4\}, Y)\\ (\{4\}, Y), (\{2, 3, 4\}, Y), (\{2, 4\}, Y), (\{1\}, Y), (\{2\}, \{x, z\}), (\{2\}, Y)\\ (\{1, 2\}, Y), (\phi, \{x, z\}), (\phi, Y)\}\end{gathered}$$

is a double topology on XY_B.

The topologies induced by τ_D are $\tau_1 = \{X, \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\},$

$\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$

the discrete topology on X, and

$\tau_2 = \{Y,\ \phi,\ \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ the discrete topology on Y, although, τ_D is not the double discrete topology on XY_B.

Proposition 4.3. *Let (XY_B, τ_D) be a double-topological space, where $G_B = (G_1, G_2)$ be a bi-subset of XY_B. If G_B is bi-open in (XY_B, τ_D) , then G_1 is open in the induced topological space (X, τ_1) and G_2 is open in the induced topological space (Y, τ_2). Also, if G_B is* *bi-closed in* (XY_B, τ_D) , *then* G_1 *is closed in the induced topological space* (X, τ_1) *and* G_2 *is closed in the induced topological space* (Y, τ_2). *The converse may not be true.*

Remark 4.5. In the Example 4.3, note that the induced topologies on X and Y are the discrete topologies. Therefore $\{1,2,3\}$ is open in X and $\{x,z\}$ is open in Y but $B_B = (\{1,2,3\}, \{x,z\})$ is not bi-open in $XY_B = (\{1,2,3,4\}, \{x,y,z\})$. Also, $\{2,3\}$ is closed in X and $\{y\}$ is closed in Y but $C_B = (\{2,3\}, \{y\})$ is not bi-closed in $XY_B = (\{1,2,3,4\}, \{x,y,z\})$.

Theorem 4.1. *Let* (XY_B, τ_D) *be a double topological space on a bi-set* XY_B *and* $B_B = (H_B, G_B) \subset XY_B$. *Then* $(\tau_D)_{B_B} = \{B_B \cap U_B : U_B \in \tau_D\}$ *is a double topology on* B_B.

Definition 4.2. The double topology $(\tau_D)_{B_B}$ on B_B, in Theorem 4.1, is called the relative double topology or subspace double topology of τ_D generated by B_B. $(B_B, (\tau_D)_{B_B})$ is called double subspace of (XY_B, τ_D).

Example 4.4. In Example 4.3, let $B_B = (\{1,2\}, \{x,z\})$ be a bi-subset of XY_B. Then $(\tau_D)_{B_B} = \{B_B, \phi_B, (\{1\}, \{x\}), (\{2\}, \{x\}),$
$(\phi, \{z\}), (\phi, \{x,z\}),$

$(\{1,2\}, \{x\}), (\phi, \{x\}), (\{1\}, \{x,z\}), (\{2\}, \{x,z\})\}$. The induced topologies of $(\tau_D)_{B_B}$ are $(\tau_1)_{\{1,2\}} = \{\phi, \{1\}, \{2\}, \{1,2\}\}$, the relative topology of τ_1 on $\{1,2\}$ and $(\tau_2)_{\{x,z\}} = \{\phi, \{x\}, \{z\}, \{x,z\}\}$, the relative topology of τ_2 on $\{x,z\}$.

Definition 4.3. Let (XY_B, τ_D) be a double topological space on a bi-set XY_B. A class $\mathcal{B}_B$ of bi-open subsets of XY_B , i.e., $\mathcal{B}_B \subset \tau_D$, is a double base for the double topology τ_D if and only if

(i) every bi-open set $U_B \in \tau_D$ is the union of members of $\mathcal{B}_B$.

Equivalently,

(ii) for any bi-point p_B belonging to a bi-open set U_B , there exist $B_B \in \mathcal{B}_B$ such that $p_B \in \mathcal{B}_B \subset U_B$.

Example 4.5. $\mathcal{B}_B = \{(\{1\}, \{x, y\}), (\{2\}, \{x\}), (\{3\}, \{y, z\}), (\{4\}, \{x, z\}), (\phi, \{x\}), (\phi, \{y\}), (\phi, \{z\})\}$ is a double base for the double topology in Example 4.3.

Remark 4.6. Let $\mathcal{B}_B$ be a double base for the double topology τ_D on the bi-set XY_B and τ_1 and τ_2 be the topologies induced by τ_D on X and Y, respectively. We can show that the set of all first (resp. second) components of members of $\mathcal{B}_B$ forms a base for τ_1 (resp. τ_2).

Example 4.6. In Example 4.3, the set $\mathcal{B}_1 = \{\{1\}, \{2\}, \{3\}\}$ of all first components of members of $\mathcal{B}_B$ is a base for τ_1. Also, the set $\mathcal{B}_2 = \{\{x\}, \{y\}, \{z\}, \{x, z\}, \{y, z\}\}$ of all second components of members of $\mathcal{B}_B$ is a base for τ_2.

We now ask the following question: Given a collection $\mathcal{B}_B$ of bi-subsets of a bi-set XY_B, when will the collection $\mathcal{B}_B$ be a double base for some double topology on XY_B? The following theorem gives the necessary and sufficient conditions for a collection of bi-sets to be a double base for some double topology.

Theorem 4.2. *Let $\mathcal{B}_B$ be a class of bi-subsets of a non-empty bi-set XY_B. Then $\mathcal{B}_B$ is a double base for some double topology on XY_B if and only if the following two conditions hold:*

(i) $XY_B = \bigcup\{B_B \in \mathcal{B}_B\}$.

(ii) *For any $B_B, B_B^* \in \mathcal{B}_B$, $B_B \cap B_B^*$ is the bi-union of members of $\mathcal{B}_B$, or*

equivalently, if $p_B \in B_B \cap B_B^*$, then there exists $B_{B_p} \in \mathcal{B}_B$ such that $p_B \in B_{B_p} \subset B_B \cap B_B^*$.

Proof. Suppose $\mathcal{B}_B$ is a double base for a double topology τ_D on XY_B. Since XY_B is bi-open, XY_B is the bi-union of members $\mathcal{B}_B$, i.e. $XY_B = \bigcup\{B_B \in \mathcal{B}_B\}$. Furthermore, if $B_B, B_B^* \in \mathcal{B}_B$, then B_B and B_B^* are bi-open sets. Hence the bi-intersection $B_B \cap B_B^*$ is also bi-open and since $\mathcal{B}_B$ is a double base for τ_D, it is the bi-union of members of $\mathcal{B}_B$. This satisfy (i) and (ii).

On the other hand, suppose $\mathcal{B}_B$ a class of bi-subsets of a non-empty bi-set XY_B which satisfy (i) and (ii) above. Let τ_D be the class of all bi-subsets of XY_B which are bi-unions of members of $\mathcal{B}_B$. We claim that τ_D is a double topology on XY_B. Observe that $\mathcal{B}_B \subset \tau_D$ will be a double base for this double topology. By (i), $XY_B = \bigcup\{B_B \in \mathcal{B}_B\}$, so $XY_B \in \tau_D$. Note that ϕ_B is the bi-union of empty subclass of $\mathcal{B}_B$, i.e. $\phi_B = \bigcup\{B_B \in \phi_B \subset \mathcal{B}_B\}$, hence $\phi_B \in \tau_D$ and τ_D satisfy the first condition of the double topology.

Now, let $\{G_{Bi}\}$ be a class of members of τ_D. By definition of τ_D, each G_{Bi} is the bi-union of members of $\mathcal{B}_B$, hence the bi-union $\bigcup G_{Bi}$ is also the bi-union of members of $\mathcal{B}_B$ and so belongs to τ_D. Thus τ_D satisfy the second condition of bi-topology.

Finally, suppose $G_B, H_B \in \tau_D$. We show that $G_B \cap H_B \in \tau_D$. By definition of τ_D, there exist subclasses $\{B_{Bi} : i \in I\}$ and $\{B_{Bj} : j \in J\}$ of $\mathcal{B}_B$ such that $G_B = \bigcup B_{Bi}$ and $H_B = \bigcup B_{Bj}$. Then

$$[G_B \cap H_B = \bigcup B_{Bi} \cap \bigcup B_{Bj} = \bigcup\{B_{Bi} \cap B_{Bj} : i \in I,\ j \in J\}]$$

But by (ii) $B_{Bi} \cap B_{Bj}$ is the bi-union of members of $\mathcal{B}_B$, hence $G_B \cap H_B = \bigcup\{B_{Bi} \cap B_{Bj} : i \in I,\ j \in J\}$ is also the bi-union of members of $\mathcal{B}_B$ and so belong to τ_D which therefore satisfy the third condition of the double topology. Hence τ_D is a double topology on XY_B with double base $\mathcal{B}_B$. ∎

DEFINITION 4.3. Let (XY_B, τ_D) be a double topological space on a bi-set XY_B. A class $\mathcal{S}_B$ of bi-open subsets of XY_B , i.e., $\mathcal{S}_B \subset \tau_D$, is a double subbase for the double topology τ_D if and only if finite bi-intersections of members of $\mathcal{S}_B$ form a double base for τ_D.

EXAMPLE 4.7. $\mathcal{S}_B = \{(\{1\}, \{x, y\}), (\{2\}, \{x\}), (\{3\}, \{y, z\}), (\{4\}, \{x, z\})\}$ is a double subbase for the double topology in Example 4.3, since finite bi-intersections of members of $\mathcal{S}_B$ induced the double base in Example 4.5.

5. Double Limit Points, Double Closure and Double Interior of Bi-Sets

Definition 5.1. Let (XY_B, τ_D) be a double topological space on a bi-set XY_B and $C_B \subset XY_B$. A point $a_B \in C_B$ is called a double interior point of C_B if there exists a bi-open set U_B such that $a_B \in U_B \subset C_B$. The set of all double interior points of C_B is called the double interior set of C_B and is denoted by C_B°.

Remark 5.1. The double interior C_B° of a bi-set C_B is bi-open and it is the largest bi-open set bi-contained in C_B.

Theorem 5.1. *A bi-subset C_B of a bi-topological space (XY_B, τ_D) is bi-open if and only if $C_B^\circ = C_B$.*

Proposition 5.1. *Let (XY_B, τ_D) be a double topological space, $H_B = (H_1, H_2)$ be a bi-subset of XY_B and $p_B = (p_1, p_2)$ be a bi-point of XY_B. If $p_B \in H_B^\circ$, then $p_1 \in H_1^\circ$ and $p_2 \in H_2^\circ$, i.e. $H_B^\circ \subset (H_1^\circ,\ H_2^\circ)$. The converse may not be true.*

Example 5.1. Let (XY_B, τ_D) be a double topological space, where XY_B, $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$ and $\tau_D = \{XY_B, \phi_B, (\{a\}, \{1\}),$ $(\{b, c\}, \{2\}), (\{c\}, \{3\}), (X, \{1, 2\}), (\{a, c\}, \{1, 3\}), (\{b, c\}, \{2, 3\}),$ $(\{c\}, \phi), (\{a, c\}, \{1\})\}$. The induced topologies are $\tau_1 = \{X, \phi, \{a\},$ $\{c\}, \{a, c\}, \{b, c\}\}$ and $\tau_2 = \{Y, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Let $H_B = (H_1, H_2) = (\{a, c\}, \{1, 2\})$. Then $H_B^\circ = (\{a, c\}, \{1\})$, $H_1^\circ = \{a, c\}$ and $H_2^\circ = \{1, 2\}$. This shows that $H_B^\circ \neq (H_1^\circ,\ H_2^\circ)$. Since $a \in H_1^\circ$ and $2 \in H_2^\circ$ but $(a, 2) \notin\ H_B^\circ$.

Definition 5.2. Let (XY_B, τ_D) be a double topological space on a bi-set XY_B and $C_B \subset XY_B$. A point $a_B \in XY_B$ is called a double limit point of C_B if for each bi-open set U_B bi-containing a_B, we have $(U_B \backslash \{a_B\}) \cap C_B \neq \phi$. The set of all double limit points of C_B is called the double derived set of C_B and is denoted by C'_B.

Theorem 5.2. *A bi-subset B_B of a bi-topological space (XY_B, τ_D) is bi-closed if and only if $B'_B \subset B_B$.*

Proof. Suppose that B_B is bi-closed and $p_B \notin B_B$. Then $p_B \in B_B^c$. Since B_B^c is bi-open, bi-complement of a bi-closed set, and $B_B \cap B^c_B = \phi_B$, then $p_B \notin B'_B$. Thus $B'_B \subset B_B$. Now, assume that $B'_B \subset B_B$. We show that B^c_B is bi-open. Let $p_B \in B^c_B$, then $p_B \notin B'_B$. So, the exists a bi-open set U_B such that $p_B \in U_B$ and $(U_B \backslash \{p_B\}) \cap B_B = \phi$. But $p_B \notin B_B$, hence, $U_B \cap B_B = \phi_B$. So $U_B \subset B^c_B$, and p_B is a double interior point of B^c_B and so B^c_B is bi-open. ■

DEFINITION 5.3. Let (XY_B, τ_D) be a double topological space on a bi-set XY_B and $C_B \subset XY_B$. The double closure of C_B , denoted by $\overline{C_B}$ is defined as the bi-intersection of all bi-closed sets bi-containing C_B, that is , $\overline{C_B} = \cap\{F_B : \ F_B \text{ is bi} - \text{closed and } \ C_B \subset F_B\}$.

REMARK 5.2. The double closure $\overline{C_B}$ of a bi-set C_B is bi-closed and it is the smallest bi-closed set bi-containing C_B.

THEOREM 5.3. *A bi-subset B_B of a double topological space (XY_B, τ_D) is bi-closed if and only if $\overline{B_B} = B_B$.*

THEOREM 5.4. *Let (XY_B, τ_D) be a double topological space on a bi-set XY_B, $a_B \in XY_B$ and $C_B \subset XY_B$. Then $a_B \in \overline{C_B}$ if and only if for each bi-open set U_B bi-containing a_B, we have $U_B \cap C_B \neq \phi$.*

Proof. Let $a_B \in \overline{C_B}$ and suppose that U_B is a bi-open set bi-containing a_B such that $U_B \cap C_B = \phi_B$. Then $C_B \subset U^c_B$. Since U^c_B is a bi-closed set and $a_B \notin U^c_B$, then $a_B \notin \overline{C_B}$, a contradiction. Therefore, $U_B \cap C_B \neq \phi$.

On the other hand, suppose that for each bi-open set U_B bi-containing a_B, we have $U_B \cap C_B \neq \phi$. Suppose that $a_B \notin \overline{C_B}$. Then there exists a bi-closed set F_B such that $C_B \subset F_B$ and $a_B \notin F_B$. Then F^c_B is a bi-open set, $a_B \in F^c_B$ and $C_B \cap F^c_B = \phi_B$, a contradiction. Therefore $a_B \in \overline{C_B}$. ■

THEOREM 5.5. *Let (XY_B, τ_D) be a double topological space on a bi-set XY_B and $C_B, D_B \subset XY_B$. Then*

(i) $C_B \subset \overline{C_B}$

(ii) $\overline{C_B \cup D_B} = \overline{C_B} \cup \overline{D_B}$

(iii) *If* $C_B \subset D_B$ *then* $\overline{C_B} \subset \overline{D_B}$.

Theorem 5.6. *Let* (XY_B, τ_D) *be a double topological space on a bi-set* XY_B *and* $C_B \subset XY_B$. *Then* $\overline{C_B} = C_B \cup C'_B$.

Proposition 5.2. *Let* (XY_B, τ_D) *be a double topological space. Let* $H_B = (H_1, H_2)$ *be a bi-subset of* XY_B *and* $p_B = (p_1, p_2)$ *be a bi-point of* XY_B. *If* $p_B \in \overline{H_B}$, *then* $p_1 \in \overline{H_1}$ *and* $p_2 \in \overline{H_2}$, *i.e.* $\overline{H_B} \subset (\overline{H_1},\ \overline{H_2})$. *The converse may not be true.*

Example 5.2. Let $X = \{1, 2, 3, 4\}$ and $Y = \{x, y, z\}$. Then

$$\begin{gathered}\tau_D = \{XY_B, \phi_B, (\{1\}, \{x, y\}), (\{2\}, \{x\}), (\{3\}, \{y, z\}), (\{4\}, \{x, z\}),\\ (\{1, 2\}, \{x, y\}), (\phi, \{x\}), (\{1, 3\}, Y), (\phi, \{y\}), (\{1, 4\}, Y), (\{2, 3\}, Y)\\ (\{2, 4\}, \{x, z\}), (\phi, \{x, y\}), (\{1, 2, 3\}, Y), (\{2\}, \{x, y\}), (\{1, 2, 4\}, Y),\\ (\{3, 4\}, Y), (\phi, \{z\}), (\{3\}, Y), (\{1, 3, 4\}, Y), (\phi, \{y, z\}), (\{2, 3, 4\}, Y)\\ (\{4\}, Y), (\{2, 3, 4\}, Y), (\{2, 4\}, Y), (\{1\}, Y), (\{2\}, \{x, z\}), (\{2\}, Y)\\ (\{1, 2\}, Y), (\phi, \{x, z\}), (\phi, Y)\}\end{gathered}$$

is a double topology on XY_B. Let $C_B = (\{2, 3\}, \{y\})$, then $\overline{C_B} = (\{1, 2, 3\}, \{y\})$.

Now, the induced topologies are the discrete topologies on X and Y. Therefore, $\overline{\{2, 3\}} = \{2, 3\}$ and $\overline{\{y\}} = \{y\}$. Therefore $\overline{C_B} \neq (\overline{\{2, 3\}}, \overline{\{y\}})$. Note that $(1, y) \in \overline{(\{1, 2, 3\}, \{y\})}$ but $(1, y) \notin (\overline{\{2, 3\}}, \overline{\{y\}})$.

6. Double Neighborhoods and Double Neighborhood Systems

Definition 6.1. Let p_B be a bi-point of a double topological space (XY_B, τ_D). A bi-subset N_B of XY_B is called a double neighborhood of p_B, if there exists a bi-open set G_B bi-containing p_B and is bi-contained in N_B, i.e. $p_B \in G_B \subset N_B$.

In other words, the relation "N_B is a double neighborhood of p_B" is the inverse of the relation "p_B is a double interior point of N_B".

The class of all double neighborhoods of p_B, denoted by $\mathcal{N}_{p_B}$, is called the double neighborhood system of p_B.

Proposition 6.1. *Let (XY_B, τ_D) be a double topological space, where $XY_B = (X, Y)$, and τ_1 and τ_2 be the topologies induced by τ_D on X and Y, respectively. Let $p_B = (p_1, p_2)$ be a bi-point of XY_B, where $p_1 \in X$ and $p_2 \in Y$. If $N_B = (N_1, N_2)$ is a double neighborhood of p_B, then N_1 is a neighborhood of p_1 in (X, τ_1) and N_2 is a neighborhood of p_2 in (Y, τ_2). The converse may not be true.*

Example 6.1. Let $X = \{r, s, t, u\}$ and $Y = \{1, 2, 3, 4\}$. Then $\tau_D = \{XY_B, \phi_B, (\{r\}, \{1, 2\}), (\{r\}, \{3, 4\}), (\{r\}, Y), (\{r\}, \phi), (\{s\}, \{1, 3\}), (\phi, \{1\}), (\{r, s\}, \{1, 2, 3\}), (\phi, \{3\}), (\{r, s\}, \{1, 3, 4\}), (\{r\}, \{1, 3, 4\}), (\{r, s\}, Y), (\{r\}, \{3\}), (\phi, \{1, 3\}), (\{r\}, \{1, 2, 3\}), (\{r, s\}, \{1, 3\}), (\{r\}, \{1\}), (\{r\}, \{1, 3\})\}$ is a double topology on XY_B.

Consider the bi-point $p_B = (s, 1) \in XY_B$, Then $\mathcal{N}_{p_B} = \{XY_B, (\{s\}, \{1, 3\}), (\{r, s\}, \{1, 2, 3\}), (\{r, s\}, \{1, 3, 4\}), (\{r, s\}, Y), (\{r, s\}, \{1, 3\}), (\{s\}, \{1, 2, 3\}), (\{s\}, \{1, 3, 4\}), (\{s\}, Y), (\{s, t\}, \{1, 3\}), (\{s, t\}, \{1, 2, 3\}), (\{s, t\}, \{1, 3, 4\}), (\{s, t\}, Y), (\{s, u\}, \{1, 3\}), (\{s, u\}, \{1, 2, 3\}), (\{s, u\}, \{1, 3, 4\}), (\{s, u\}, Y), (\{r, s, t\}, \{1, 3\}), (\{r, s, t\}, \{1, 3, 4\}), (\{r, s, t\}, \{1, 2, 3\}), (X, \{1, 3\}), (X, \{1, 2, 3\}), (X, \{1, 3, 4\})\}$.

The induced topologies by τ_D on X and Y, respectively are

$\tau_1 = \{X, \phi, \{r\}, \{s\}, \{r, s\}\}$ and $\tau_2 = \{Y, \phi, \{1\}, \{3\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}\}$.

Note that $\{s, t\}$ is a neighborhood of s in (X, τ_1) and $\{1, 2, 4\}$ is a neighborhood of 1 in (Y, τ_2), but $(\{s, t\}, \{1, 2, 4\})$ is not a double neighborhood of $p_B = (s, 1)$ in (XY_B, τ_D).

Theorem 6.1. *Let $\mathcal{N}_{p_B}$ be the double neighborhood system of p_B in the double topological space (XY_B, τ_D). Then:*

(i) *$\mathcal{N}_{p_B} \neq \phi_B$ and p_B belongs to each member of $\mathcal{N}_{p_B}$.*

(ii) T*he bi-intersection of any two members of $\mathcal{N}_{p_B}$ is a member of $\mathcal{N}_{p_B}$.*

(iii) *Every bi-super set of a member of $\mathcal{N}_{p_B}$ belongs to $\mathcal{N}_{p_B}$.*

(iv) *Each member* $N_B \in \mathcal{N}_{p_B}$ *is a bi-super set of a member* $G_B \in \mathcal{N}_{p_B}$*, where* G_B *is a double neighborhood of each of its bi-points.*

CONCLUSION: In this paper, any double topology induces two topologies. These topologies can be used to construct a new double topology finer than the original one. The induced topologies of the new double topology is the same as the induced topologies of the first one (Propositions 4.1 and 4.2). Also, non double discrete double topology can induced two discrete topologies, but non discrete topologies cannot construct a discrete double topology(Remark 4.4). Furthermore, any property for a double topology is inherited by the two induced topologies, but not conversely (Propositions 4.3, 5.1, 5.2 and 6.1). We hope that the results of our paper will be a starting point for a sufficiently general but simple theory of objects that are suitable for modelling various aspects of computation and useful in modern applications of bi-open sets to general topology and mathematical analysis. Also, we may stress once more the importance of double topological spaces for the possible application in double separation axioms, double compact spaces, etc.

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