Hindawi
Journal of Mathematics
Volume 2022, Article ID 9352861, 10 pages
https://doi.org/10.1155/2022/9352861

Review Article

On Q_p-Closed Sets in Topological Spaces

Huan Zhou,[1] O. G. Hammad,[2] and Ahmed Mostafa Khalil[3]

[1]*Aviation Engineering School, Air Force Engineering University, Xi'an 710038, China*
[2]*Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt*
[3]*Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt*

Correspondence should be addressed to Ahmed Mostafa Khalil; a.khalil@azhar.edu.eg

Received 23 December 2021; Accepted 25 January 2022; Published 18 February 2022

Academic Editor: Naeem Jan

In the present paper, we will propose the novel notions (e.g., Q_p-closed set, Q_p-open set, Q_p-continuous mapping, Q_p-open mapping, and Q_p-closed mapping) in topological spaces. Then, we will discuss the basic properties of the above notions in detail. The category of all Q_p-closed (resp. Q_p-open) sets is strictly between the class of all preclosed (resp. preopen) sets and gp-closed (resp. gp-open) sets. Also, the category of all Q_p-continuity (resp. Q_p-open (Q_p-closed) mappings) is strictly among the class of all precontinuity (resp., preopen (preclosed) mappings) and gp-continuity (resp. gp-open (gp-closed) mappings). Furthermore, we will present the notions of Q_p-closure of a set and Q_p-interior of a set and explain some of their fundamental basic properties. Several relations are equivalent between two different topological spaces. The novel two separation axioms (i.e., Q_p-$\mathbb{R}_0$ and Q_p-$\mathbb{R}_1$) based on the notion of Q_p-open set and Q_p-closure are investigated. The space of Q_p-$\mathbb{R}_0$ (resp., Q_p-$\mathbb{R}_1$) is strictly between the spaces of pre-$\mathbb{R}_0$ (resp., pre-$\mathbb{R}_1$) and gp-$\mathbb{R}_o$ (resp., gp-$\mathbb{R}_1$). Finally, some relations and properties of Q_p-$\mathbb{R}_0$ and Q_p-$\mathbb{R}_1$ spaces are explained.

1. Introduction

In the early eighties, the novel notions of preopen and preclosed sets (i.e., as a novel type of generalized of open sets in $(\mathscr{X}, \tilde{\tau})$ (i.e., topological space) or a space $\mathscr{X}$) and preconlinuous mappings are proposed in [1]. Consequently, many researchers turned their study to the generalizations of many different notions in $(\mathscr{X}, \tilde{\tau})$ (for instance, semiopen sets [2], α-open sets [3], and β-open sets [4] or semi-preopen sets [5]). Furthermore, the notion of generalized closed (resp., generalized open) sets (for short, g-closed (resp., g-open) sets) in space $\mathscr{X}$ is presented in [6]. The relationship among g-closed (resp., g-open) sets and generalizing closedness (resp., openness) sets (i.e., generalized preclosed (resp., generalized preopen) set (for short, gp-closed (resp., gp-open) set) [7], α-generalized closed (resp., α-generalized open) set (for short, αg-closed (resp., αg-open) set) [8], pregeneralized closed (resp., pregeneralized open) set (for short, pg-closed (resp., pg-open) set) [7], and generalized α-closed (resp., generalized α-open) set (for short, $g\alpha$-closed (resp., $g\alpha$-open) set) [9]. The basic properties of five generalizing continuous mappings (i.e., precontinuous mapping [1], g-continuous mapping [10], gp-continuous mapping [11], αg-continuous mapping [12], pg-continuous mapping [11], and $g\alpha$-continuous mapping [12]) between $\tilde{\sigma}(\mathscr{Y})$ (i.e., a topology on $\mathscr{Y}$) and $\tilde{\tau}(\mathscr{X})$ (i.e., a topology on $\mathscr{X}$) are presented. Furthermore, the fundamental relations of generalizing open (closed) mappings (i.e., preopen (preclosed) mapping [1, 13], α-open (α-closed) mapping [14], g-open (g-closed) mapping [15], gp-open (gp-closed) mapping [11], αg-open (αg-closed) mapping [12], pg-open (pg-closed) mapping [11], $g\alpha$-open ($g\alpha$-closed) mapping [12]) between $\tilde{\sigma}(\mathscr{Y})$ and $\tilde{\tau}(\mathscr{X})$ are studied. On the contrary, the characterizations between separation axioms classes (i.e., pre-$\mathbb{R}_0$, pre-$\mathbb{R}_1$, gp-$\mathbb{R}_0$ and gp-$\mathbb{R}_1$ spaces) (see, [16, 17]) in $(\mathscr{X}, \tilde{\tau})$ are defined.

Regarding the above discussions, as the motivation of the present paper, we will define novel sets called Q_p-closed sets and Q_p-open sets and investigate several of their fundamental properties. The relation between Q_p-closed set (resp., Q_p-open set) and other sets (for example, preclosed set (resp., preopen set), α-closed set (resp., α-pen set), g-closed set (resp., g-open set), gp-closed set (resp., gp-open set), αg-closed set (resp., αg-open set), pg-closed set (resp., pg-open set), and $g\alpha$-closed set (resp., $g\alpha$-open set)) in space $\mathcal{X}$ is introduced. Then, we define the Q_p-continuous mapping and study the relations between Q_p-continuous mapping and other mappings (for example, precontinuous mapping, g-continuous mapping, gp-continuous mapping, αg-continuous mapping, pg-continuous mapping, and $g\alpha$-continuous mapping) between two different topological spaces. Also, we present the notion of Q_p-open (Q_p-closed) mapping and investigate relations between Q_p-open (Q_p-closed) mapping and other mappings (for example, preopen (preclosed) mapping, α-open (α-closed) mapping, g-open (g-closed) mapping, gp-open (gp-closed) mapping, αg-open (αg-closed) mapping, pg-open (pg-closed) mapping, and $g\alpha$-open ($g\alpha$-closed) mapping) between two different topological spaces. Finally, we propose the novel separation axioms classes (i.e., Q_p-$\mathbb{R}_0$ and Q_p-$\mathbb{R}_1$ spaces) in $(\mathcal{X}, \tilde{\tau})$.

Next, the sections of this paper are arranged as follows. In Section 2, we will present many notions related to topological spaces as indicated from Definitions 1 to 4. In Section 2, we propose the novel notions of Q_p-closed sets and Q_p-open sets and explain the interesting properties of them. In Section 3, we give the notions of Q_p-continuous mappings, Q_p-open mappings, and Q_p-closed mappings. In Section 4, we define Q_p-$\mathbb{R}_0$ and Q_p-$\mathbb{R}_1$ spaces. Section 5 is conclusions.

In the current paper, we will use several expressions (i.e., $\mathscr{C}(\mathfrak{A})$ (the closure of a set $\mathfrak{A}$), $\mathscr{I}(\mathfrak{A})$ (the interior of a set $\mathfrak{A}$), $\tilde{\tau}(\mathcal{X})$ (the all of open sets in $\mathcal{X}$), and $\mathscr{F}_{\mathcal{X}}$ (the all of closed sets in $\mathcal{X}$)).

Next, we will present several notions which are used in this section as indicated below.

Definition 1 (Cf. [1, 3]). Assume $(\mathcal{X}, \tilde{\tau})$ is a topological space. Then,

(1)

(i) $\mathfrak{A}$ is preclosed set if $\mathscr{C}(\mathscr{I}(\mathfrak{A}))\subseteq\mathfrak{A}$
(ii) $\mathfrak{A}$ is preopen set if $\mathfrak{A}\subseteq\mathscr{I}(\mathscr{C}(\mathfrak{A}))$

$\mathbb{C}_p(\mathcal{X})$ (resp., $\mathbb{O}_p(\mathcal{X})$) is the set of all preclosed (resp. preopen) sets.

(2)

(i) $\mathfrak{A}$ is α-closed set if $\mathscr{C}(\mathscr{I}(\mathscr{C}(\mathfrak{A})))\subseteq\mathfrak{A}$
(ii) $\mathfrak{A}$ is α-open set if $\mathfrak{A}\subseteq\mathscr{I}(\mathscr{C}(\mathscr{I}(\mathfrak{A})))$

$\mathbb{C}_\alpha(\mathcal{X})$ (resp., $\mathbb{O}_\alpha(\mathcal{X})$) is the set of all α-closed (resp. α-open) sets.

Definition 2 (Cf. [6–9]). Assume $(\mathcal{X}, \tilde{\tau})$ is a topological space. Then,

(1)

(i) $\mathfrak{A}$ is g-closed set if $\mathscr{C}(\mathfrak{A})\subseteq\mathfrak{A}$ whenever $\mathfrak{A}\subseteq\mathfrak{L}$ and $\mathfrak{L}\in\tilde{\tau}(\mathcal{X})$, where $\mathscr{C}(\mathfrak{A})$ is a closure of $\mathfrak{A}$, i.e.,

$$\mathscr{C}(\mathfrak{A}) = \cap\{\mathfrak{F}|\mathfrak{A}\subseteq\mathfrak{F} \text{ and } \mathfrak{F}\in\mathscr{F}_{\mathcal{X}}\}. \tag{1}$$

$\mathbb{GC}(\mathcal{X})$ be the set of all g-closed sets.
(ii) $\mathfrak{A}$ is g-open set if $\mathcal{X}\backslash\mathfrak{A}\in\mathbb{GC}(\mathcal{X})$ and $\mathbb{GO}(\mathcal{X})$ be the set of all g-open sets.

(2)

(i) $\mathfrak{A}$ is gp-closed set if $\mathscr{C}_p(\mathfrak{A})\subseteq\mathfrak{A}$ whenever $\mathfrak{A}\subseteq\mathfrak{L}$ and $\mathfrak{L}\in\tilde{\tau}(\mathcal{X})$, where $\mathscr{C}_p(\mathfrak{A})$ is a preclosure of $\mathfrak{A}$, i.e.,

$$\mathscr{C}_p(\mathfrak{A}) = \cap\left\{\mathfrak{F}|\mathfrak{A}\subseteq\mathfrak{F} \text{ and } \mathfrak{F}\in\mathbb{C}_p(\mathcal{X})\right\}. \tag{2}$$

$\mathbb{GC}_p(\mathcal{X})$ be the set of all gp-closed sets.
(ii) $\mathfrak{A}$ is gp-open set if $\mathcal{X}\backslash\mathfrak{A}\in\mathbb{GC}_p(\mathcal{X})$ and $\mathbb{GO}_p(\mathcal{X})$ be the set of all gp-open sets.

(3)

(i) $\mathfrak{A}$ is αg-closed set if $\mathscr{C}_\alpha(\mathfrak{A})\subseteq\mathfrak{A}$ whenever $\mathfrak{A}\subseteq\mathfrak{L}$ and $\mathfrak{L}\in\tilde{\tau}(\mathcal{X})$, where $\mathscr{C}_\alpha(\mathfrak{A})$ is a α-closure *of* $\mathfrak{A}$, *i.e.,*

$$\mathscr{C}_\alpha(\mathfrak{A}) = \cap\{\mathfrak{F}|\mathfrak{A}\subseteq\mathfrak{F} \text{ and } \mathfrak{F}\in\mathbb{C}_\alpha(\mathcal{X})\}. \tag{3}$$

$\mathbb{C}_\alpha\mathbb{G}(\mathcal{X})$ be the set of all αg-closed sets.
(ii) $\mathfrak{A}$ is αg-open set if $\mathcal{X}\backslash\mathfrak{A}\in\mathbb{C}_\alpha\mathbb{G}(\mathcal{X})$ and $\mathbb{O}_\alpha\mathbb{G}(\mathcal{X})$ be the set of all αg-open sets.

(4)

(i) $\mathfrak{A}$ is pg-closed set if $\mathscr{C}_p(\mathfrak{A})\subseteq\mathfrak{A}$ whenever $\mathfrak{A}\subseteq\mathfrak{L}$ and $\mathfrak{L}\in\mathbb{O}_p(\mathcal{X})$ and $\mathbb{C}_p\mathbb{G}(\mathcal{X})$ be the set of all pg-closed sets.
(ii) $\mathfrak{A}$ is pg-open set if $\mathcal{X}\backslash\mathfrak{A}\in\mathbb{C}_p\mathbb{G}(\mathcal{X})$ and $\mathbb{O}_p\mathbb{G}(\mathcal{X})$ be the set of all gp-open sets.

(5)

(i) $\mathfrak{A}$ is $g\alpha$-closed set if $\mathscr{C}_\alpha(\mathfrak{A})\subseteq\mathfrak{A}$ whenever $\mathfrak{A}\subseteq\mathfrak{L}$ and $\mathfrak{L}\in\mathbb{O}_\alpha(\mathcal{X})$, and $\mathbb{GC}_\alpha(\mathcal{X})$ be the set of all $g\alpha$-closed sets.
(ii) $\mathfrak{A}$ is $g\alpha$-open set if $\mathcal{X}\backslash\mathfrak{A}\in\mathbb{GC}_\alpha(\mathcal{X})$, and $\mathbb{GO}_\alpha(\mathcal{X})$ be the set of all $g\alpha$-open sets.

Definition 3 (Cf. [1, 10, 13–15]). Let $\psi: \mathcal{X}\longrightarrow\mathcal{Y}$ be mapping and $\tilde{\sigma}(\mathcal{Y})$ be a topology on $\mathcal{Y}$ and $\tilde{\tau}(\mathcal{X})$ is a topology on $\mathcal{X}$. Then,

(1) ψ is precontinuous mapping (resp., α-continuous mapping and g-continuous mapping) if $\mathfrak{L}\in\tilde{\sigma}(\mathcal{Y})$, $\psi^{-1}(\mathfrak{L})\in\mathbb{O}_p(\mathcal{X})$ (*resp.*, $\psi^{-1}(\mathfrak{L})\in\mathbb{O}_\alpha(\mathcal{X})$, $\psi^{-1}(\mathfrak{L})\in\mathbb{GO}(\mathcal{X})$).

(2)

(i) ψ is preopen mapping (resp., α-open mapping and g-open mapping) if $\mathfrak{L}\in\tilde{\tau}(\mathcal{X})$, $\psi(\mathfrak{L})\in Q_p\mathbb{O}(\mathcal{Y})$ (*resp.*, $\psi(\mathfrak{L})\in\mathbb{O}_\alpha(\mathcal{Y})$, $\psi(\mathfrak{L})\in\mathbb{GO}(\mathcal{Y})$).
(ii) ψ is preclosed mapping (resp., α-closed mapping and g-closed mapping) if $\mathfrak{L}\in\tilde{\tau}(\mathcal{X})$, $\psi(\mathfrak{L})\in Q_p\mathbb{C}(\mathcal{Y})$ (*resp.*, $\psi(\mathfrak{L})\in\mathbb{C}_\alpha(\mathcal{Y})$, $\psi(\mathfrak{L})\in\mathbb{GC}(\mathcal{Y})$).

Definition 4 (Cf. [16, 17]). A topological space $(\mathcal{X}, \tilde{\tau})$ is said to be

(1) Pre-$\mathbb{R}_0$ space (resp., gp-$\mathbb{R}_0$ space) if $\forall x \in \mathfrak{L} \in \mathbb{O}_p(\mathcal{X})$ s.t. $\mathscr{C}_p(\{x\}) \subset \mathfrak{L}$ (resp., $\forall x \in \mathfrak{L} \in \mathbb{GO}_p(\mathcal{X})$ s.t., $\mathscr{C}_{gp}(\{x\}) \subset \mathfrak{L}$) and $\mathscr{C}_{gp}(\{x\})$ is a gp-closure of $\{x\}$, defined as

$$\mathscr{C}_{gp}(\{x\}) = \cap\left\{\mathfrak{F} \mid \{x\} \subseteq \mathfrak{F} \text{ and } \mathfrak{F} \in \mathbb{GC}_p(\mathcal{X})\right\}. \tag{4}$$

(2) Pre-$\mathbb{R}_1$ space (resp., gp-$\mathbb{R}_1$ space) if $\forall x, y \in \mathcal{X}$, with $\mathscr{C}_p(\{x\}) \neq \mathscr{C}_p(\{y\})$ (resp., $\mathscr{C}_{gp}(\{x\}) \neq \mathscr{C}_{gp}(\{y\})$), there exist disjoint preopen sets (resp., gp-open sets) $\mathfrak{L}$ and $\mathfrak{M}$ s.t. $\mathscr{C}_p(\{x\}) \subseteq \mathfrak{L}$ (resp., $\mathscr{C}_{gp}(\{x\}) \subseteq \mathfrak{L}$) and $\mathscr{C}_p(\{y\}) \subseteq \mathfrak{M}$ (resp., $\mathscr{C}_{gp}(\{y\}) \subseteq \mathfrak{M}$).

2. ERROR!!Q_p-Closed Sets and Q_p-Open Sets

In the following section, we propose novel sets (i.e., Q_p-closed sets and Q_p-open sets) and discuss several interesting theorems and examples.

Definition 5. We call $\mathfrak{A}$ is Q_p-closed set in $(\mathcal{X}, \tilde{\tau})$ if

$$\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A})) \subseteq \mathfrak{A}, \tag{5}$$

where $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A}))$ is a gp-closure of $\mathscr{I}(\mathfrak{A})$, i.e.,

$$\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A})) = \cap\left\{\mathfrak{F} \mid \mathscr{I}(\mathfrak{A}) \subseteq \mathfrak{F} \text{ and } \mathfrak{F} \in \mathbb{GC}_p(\mathcal{X})\right\}. \tag{6}$$

$Q_p\mathbb{C}(\mathcal{X})$ is the set of all Q_p-closed sets in $\mathcal{X}$.

Lemma 1. *Let $\mathfrak{F} \in \mathbb{GC}_p(\mathcal{X})$ s.t. $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{F})) \subseteq \mathfrak{A} \subseteq \mathfrak{F}$. Then, $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$.*

Proof. As $\mathfrak{F} \in \mathbb{GC}_p(\mathcal{X})$ implies $\mathscr{C}_{gp}(\mathfrak{F}) = \mathfrak{F}$, thus, $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A})) \subseteq \mathscr{C}_{gp}(\mathscr{I}(\mathfrak{F})) \subseteq \mathfrak{A}$. Therefore, $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$. □

The converse of Lemma 1 (i.e., $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{F})) \notin \subseteq \mathfrak{F}$) does not hold by the following example.

Example 1. Assume that $\mathcal{X} = \{1, 2, 3\}$ (i.e., $(\mathcal{X}, \tilde{\tau})$ be topological space) and $\tilde{\tau} = \{\mathcal{X}, \phi, \{1\}\}$. Then,

$$\begin{aligned} \mathscr{F}_{\mathcal{X}} &= \{\mathcal{X}, \phi, \{2, 3\}\}, \\ \mathbb{C}_p(\mathcal{X}) &= \{\mathcal{X}, \phi, \{2\}, \{3\}, \{2, 3\}\}, \\ \mathbb{GC}_p(\mathcal{X}) &= \{\mathcal{X}, \phi, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}, \end{aligned} \tag{7}$$

$$Q_p\mathbb{C}(\mathcal{X}) = 2^{\mathcal{X}}. \tag{8}$$

Let $\mathfrak{A} = \{3\} \in Q_p\mathbb{C}(\mathcal{X})$ and $\mathfrak{F} = \mathcal{X} \in \mathbb{GC}_p(\mathcal{X})$. Then, $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{F})) = \mathscr{C}_{gp}(\mathcal{X}) = \notin \subseteq \mathfrak{F}$.

Theorem 1. *The following two properties are holding in $(\mathcal{X}, \tilde{\tau})$.*

(1) If $\mathfrak{A} \in \mathbb{C}_p(\mathcal{X})$, then $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$

(2) If $\mathfrak{A} \in \mathbb{GC}_p(\mathcal{X})$, then $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$

Proof.

(1) As $\mathscr{C}_{gp}(\mathfrak{A}) \subseteq \mathscr{C}_p(\mathfrak{A})$ and $\mathscr{C}_p(\mathfrak{A}) \subseteq \mathscr{C}(\mathfrak{A})$ imply that $\mathscr{C}_{gp}(\mathfrak{A}) \subseteq \mathscr{C}(\mathfrak{A})$, since $\mathfrak{A}$ is preclosed set (i.e., $\mathscr{C}(\mathscr{I}(\mathfrak{A})) \subseteq \mathfrak{A}$), then $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A})) \subseteq \mathscr{C}(\mathscr{I}(\mathfrak{A})) \subseteq \mathfrak{A}$. Thus, $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$.

(2) Let $\mathfrak{A} \in \mathbb{GC}_p(\mathcal{X})$ (i.e., $\mathscr{C}_{gp}(\mathfrak{A}) = \mathfrak{A}$). Then, $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A})) \subseteq \mathscr{C}_{gp}(\mathfrak{A}) = \mathfrak{A}$. Thus, $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$. □

The converse of Theorem 1 (i.e., $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$ but $\mathfrak{A} \notin \mathbb{C}_p(\mathcal{X})$ and $\mathfrak{A} \notin \mathbb{GC}_p(\mathcal{X})$) does not hold by the following example.

Example 2. (continued from Example 1). As $\{1, 3\} \in Q_p\mathbb{C}(\mathcal{X})$, $\{1, 3\} \notin \mathbb{C}_p(\mathcal{X})$ and $\{1\} \in Q_p\mathbb{C}(\mathcal{X})$ but $\{1\} \notin \mathbb{GC}_p(\mathcal{X})$.

Theorem 2. *Arbitrary intersection of Q_p-closed sets is Q_p-closed set.*

Proof. Suppose that $\{\mathfrak{F}_k \mid k \in \Lambda\}$ be a collection of Q_p-closed sets in $\mathcal{X}$. Then, $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{F}_k)) \subseteq \mathfrak{F}_k$, for every k. As $\cap\mathfrak{F}_k \subseteq \mathfrak{F}_k$, for every k, $\mathscr{C}_{gp}(\cap\mathfrak{F}_k) \subseteq \mathscr{C}_{gp}(\mathfrak{F}_k)$ for every k. Thus, $\mathscr{C}_{gp}(\cap\mathfrak{F}_k) \subseteq \cap\mathscr{C}_{gp}(\mathfrak{F}_k), k \in \Lambda$. Hence, $\mathscr{C}_{gp}(\mathscr{I}(\cap\mathfrak{F}_k)) \subseteq \mathscr{C}_{gp}(\cap\mathscr{I}(\mathfrak{F}_k)) \subseteq \cap\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{F}_k)) \subseteq \cap\mathfrak{F}_k$. Therefore, $\cap\mathfrak{F}_k$ is Q_p-closed set. □

Remark 1. The union of two Q_p-closed sets need not be Q_p-closed set (i.e., $\mathfrak{A}, \mathfrak{B} \in Q_p\mathbb{C}(\mathcal{X})$, but $\mathfrak{A} \cup \mathfrak{B} \notin Q_p\mathbb{C}(\mathcal{X})$) as the next example; let $\mathcal{X} = \{1, 2, 3\}$ (i.e., $(\mathcal{X}, \tilde{\tau})$ be topological space) and $\tilde{\tau} = \{\mathcal{X}, \phi, \{1, 2\}\}$. Then,

$$\begin{aligned} \mathscr{F}_{\mathcal{X}} &= \{\mathcal{X}, \phi, \{3\}\}, \\ \mathbb{C}_p(\mathcal{X}) &= \mathbb{GC}_p(\mathcal{X}) = Q_p\mathbb{C}(\mathcal{X}) \\ &= \{\mathcal{X}, \phi, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}. \end{aligned} \tag{9}$$

Let $\mathfrak{A} = \{1\} \in Q_p\mathbb{C}(\mathcal{X})$ and $\mathfrak{B} = \{2\} \in Q_p\mathbb{C}(\mathcal{X})$. Then, $\mathfrak{A} \cup \mathfrak{B} = \{1, 2\} \notin Q_p\mathbb{C}(\mathcal{X})$.

Corollary 1. *The following two properties are holding in $(\mathcal{X}, \tilde{\tau})$.*

(1) Let $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$ and $\mathfrak{B} \in \mathbb{C}_p(\mathcal{X})$. Then, $\mathfrak{A} \cap \mathfrak{B} \in Q_p\mathbb{C}(\mathcal{X})$.

(2) Let $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X})$ and $\mathfrak{B} \in \mathbb{GC}_p(\mathcal{X})$. Then, $\mathfrak{A} \cap \mathfrak{B} \in Q_p\mathbb{C}(\mathcal{X})$.

Proof. From Theorems 1 and 2, the proof is clear. □

Definition 6. $\mathscr{C}_{Q_p}(\mathfrak{A})$ is called Q_p-closure of $\mathfrak{A}$ in $(\mathcal{X}, \tilde{\tau})$ if

$$\mathscr{C}_{Q_p}(\mathfrak{A}) = \cap\left\{\mathfrak{F} \mid \mathfrak{A} \subseteq \mathfrak{F} \text{ and } \mathfrak{F} \in Q_p\mathbb{C}(\mathcal{X})\right\}. \tag{10}$$

Theorem 3. *The following seven properties are holding in $(\mathcal{X}, \tilde{\tau})$.*

(1) $\mathfrak{A} \in Q_p\mathbb{C}(\mathcal{X}) \Leftrightarrow \mathscr{C}_{Q_p}(\mathfrak{A}) = \mathfrak{A}$

(2) $\mathfrak{A} \subseteq \mathscr{C}_{Q_p}(\mathfrak{A}) \subseteq \mathscr{C}_p(\mathfrak{A})$ and $\mathscr{C}_{Q_p}(\mathfrak{A}) \subseteq \mathscr{C}_{gp}(\mathfrak{A})$

(3) $\mathscr{C}_{Q_p}(\phi)=\phi$ *and* $\mathscr{C}_{Q_p}(\mathscr{X})=\mathscr{X}$

(4) *If* $\mathfrak{A}\subseteq\mathfrak{B}$, *then* $\mathscr{C}_{Q_p}(\mathfrak{A})\subseteq\mathscr{C}_{Q_p}(\mathfrak{B})$

(5) $\mathscr{C}_{Q_p}(\mathscr{C}_{Q_p}(\mathfrak{A}))=\mathscr{C}_{Q_p}(\mathfrak{A})$

(6) $\mathscr{C}_{Q_p}(\mathfrak{A})\cup\mathscr{C}_{Q_p}(\mathfrak{B})\subseteq\mathscr{C}_{Q_p}(\mathfrak{A}\cup\mathfrak{B})$

(7) $\mathscr{C}_{Q_p}(\mathfrak{A}\cap\mathfrak{B})\subseteq\mathscr{C}_{Q_p}(\mathfrak{A})\cap\mathscr{C}_{Q_p}(\mathfrak{B})$

Proof.

(1) Let $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$, and from Definition 6, we have $\mathscr{C}_{Q_p}(\mathfrak{A})=\mathfrak{A}$. Conversely, let $\mathscr{C}_{Q_p}(\mathfrak{A})=\mathfrak{A}$. Then, from Theorem 2, we have $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$which follows from Theorem 1 (1) and (2), respectively, and are obvious.

(2) Let $\mathfrak{A}\subseteq\mathfrak{F}$ such that $\mathfrak{F}\in Q_p\mathbb{C}(\mathscr{X})$; then, from (1) above, we have $\mathscr{C}_{Q_p}(\mathfrak{A})\subseteq\mathscr{C}_{Q_p}(\mathfrak{F})=\mathfrak{F}$. Again $\mathscr{C}_{Q_p}(\mathscr{C}_{Q_p}(\mathfrak{A}))\subseteq\mathscr{C}_{Q_p}(\mathfrak{F})=\mathfrak{F}$. Thus, $\mathscr{C}_{Q_p}(\mathscr{C}_{Q_p}(\mathfrak{A}))\subseteq\cap\{\mathfrak{F}|\mathfrak{A}\subseteq\mathfrak{F}\text{ and }\mathfrak{F}\in Q_p\mathbb{C}(\mathscr{X})\}=\mathscr{C}_{Q_p}(\mathfrak{A})$ and follows from (4). □

The equality of Theorem 3 (6) and (7) (i.e., $\mathscr{C}_{Q_p}(\mathfrak{A})\cup\mathscr{C}_{Q_p}(\mathfrak{B})\neq\mathscr{C}_{Q_p}(\mathfrak{A}\cup\mathfrak{B})$ and $\mathscr{C}_{Q_p}(\mathfrak{A}\cap\mathfrak{B})\neq\mathscr{C}_{Q_p}(\mathfrak{A})\cap\mathscr{C}_{Q_p}(\mathfrak{B})$) does not hold by the following example.

Example 3. (continued from Remark 1). As $\mathfrak{A}=\{1\},\mathfrak{B}=\{2\}$, and $\mathfrak{A}\cup\mathfrak{B}=\{1,2\}$, then $\mathscr{C}_{Q_p}(\{1\})=\{1\}$, $\mathscr{C}_{Q_p}(\{2\})=\{2\}$, and$\mathscr{C}_{Q_p}(\{1,2\})=\mathscr{X}$, and hence, $\mathscr{C}_{Q_p}(\mathfrak{A})\cup\mathscr{C}_{Q_p}(\mathfrak{B})=\{1,2\}\neq\mathscr{X}=\mathscr{C}_{Q_p}(\mathfrak{A}\cup\mathfrak{B})$.

Example 4. Assume that $\mathscr{X}=\{1,2,3\}$ (i.e., $(\mathscr{X},\tilde{\tau})$ be topological space) and $\tilde{\tau}=\{\mathscr{X},\phi,\{2\},t\{3\}n,q\{2,3\}\}$. Then,

$$\mathscr{F}_{\mathscr{X}}=\mathbb{C}_p(\mathscr{X})=\mathbb{GC}_p(\mathscr{X})=Q_p\mathbb{C}(\mathscr{X})=\{\mathscr{X},\phi,\{1\},\{1,2\},\{1,3\}\}. \tag{11}$$

Let $\mathfrak{A}=\{1\}$, $\mathfrak{B}=\{2\}$, and $\mathfrak{A}\cap\mathfrak{B}=\phi$. Then, $\mathscr{C}_{Q_p}(\{1\})=\{1\}$, $\mathscr{C}_{Q_p}(\{2\})=\{1,2\}$, and$\mathscr{C}_{Q_p}(\phi)=\phi$, and hence, $\mathscr{C}_{Q_p}(\mathfrak{A})\cap\mathscr{C}_{Q_p}(\mathfrak{B})=\{1\}\neq\phi=\mathscr{C}_{Q_p}(\mathfrak{A}\cap\mathfrak{B})$.

The relationship among the Q_p-closed sets and other sets (i.e., closed sets, α-closed sets, g-closed sets, $g\alpha$-closed sets, αg-closed sets, and pg-closed sets) is presented by the following theorem.

Theorem 4. *The following six properties is holding in* $(\mathscr{X},\tilde{\tau})$.

(1) *If* $\mathfrak{A}\in\mathscr{F}_{\mathscr{X}}$, *then* $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$

(2) *If* $\mathfrak{A}\in\mathbb{C}_\alpha(\mathscr{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$

(3) *If* $\mathfrak{A}\in\mathbb{GC}(\mathscr{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$

(4) *If* $\mathfrak{A}\in\mathbb{GC}_\alpha(\mathscr{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$

(5) *If* $\mathfrak{A}\in\mathbb{C}_\alpha\mathbb{G}(\mathscr{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$

(6) *If* $\mathfrak{A}\in\mathbb{C}_p\mathbb{G}(\mathscr{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$

Proof.

(1) As $\mathscr{C}_p(\mathfrak{A})\subseteq\mathscr{C}(\mathfrak{A})$ and by Theorem 1 (1), we have $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A}))\subseteq\mathscr{C}_p(\mathscr{I}(\mathfrak{A}))\subseteq\mathscr{C}(\mathscr{I}(\mathfrak{A}))\subseteq\mathfrak{A}$. Thus, $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$.

(2) As $\mathscr{C}_p(\mathfrak{A})\subseteq\mathscr{C}_\alpha(\mathfrak{A})$ and by Theorem 1 (1), we have $\mathscr{C}_{gp}(\mathscr{I}(\mathfrak{A}))\subseteq\mathscr{C}_p(\mathscr{I}(\mathfrak{A}))\subseteq\mathscr{C}(\mathscr{I}(\mathfrak{A}))\subseteq\mathscr{C}(\mathscr{I}(\mathscr{C}(\mathfrak{A})))\subseteq\mathfrak{A}$. Thus, $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$.

(3) As $\mathscr{C}_{gp}(\mathfrak{A})\subseteq\mathscr{C}_g(\mathfrak{A})$ and by Theorem 1 (2), we have $\mathscr{C}_{Q_p}(\mathfrak{A})\subseteq\mathscr{C}_{gp}(\mathfrak{A})\subseteq\mathscr{C}_g(\mathfrak{A})\subseteq\mathfrak{A}$. Thus, $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$.

(4) As $\mathscr{C}_{gp}(\mathfrak{A})\subseteq\mathscr{C}_g(\mathfrak{A})$ and by Theorem 1 (2), we have $\mathscr{C}_{Q_p}(\mathfrak{A})\subseteq\mathscr{C}_{gp}(\mathfrak{A})\subseteq\mathscr{C}_g(\mathfrak{A})\subseteq\mathfrak{A}$. Thus, $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$. □

The converse of Theorem 4 (i.e., $\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$, but $\mathfrak{A}\notin\mathscr{F}_{\mathscr{X}}$, $\mathfrak{A}\notin\mathbb{C}_\alpha(\mathscr{X})$, $\mathfrak{A}\notin\mathbb{GC}(\mathscr{X})$, $\mathfrak{A}\notin\mathbb{GC}_\alpha(\mathscr{X})$, $\mathfrak{A}\notin\mathbb{O}_\alpha\mathbb{G}(\mathscr{X})$, and $\mathfrak{A}\notin\mathbb{C}_p\mathbb{G}(\mathscr{X})$) does not hold by the following example.

Example 5. (continued from Example 1). Clearly,

$$\mathbb{GC}_\alpha(\mathscr{X})=\mathbb{C}_p\mathbb{G}(\mathscr{X})=\mathbb{C}_\alpha\mathbb{G}(\mathscr{X})=\mathbb{C}_\alpha(\mathscr{X})=\mathbb{C}_p(\mathscr{X}). \tag{12}$$

and $\mathbb{GC}(\mathscr{X})=\mathbb{GC}_p(\mathscr{X})$. Thus, $\mathbb{GC}(\mathscr{X})=\mathbb{GC}_p(\mathscr{X})$, but $\{1\}\notin\mathscr{F}_{\mathscr{X}},\{1\}\notin\mathbb{C}_\alpha(\mathscr{X}),\{1\}\notin\mathbb{GC}(\mathscr{X}),\{1\}\notin\mathbb{GC}_\alpha(\mathscr{X})$, $\{1\}\notin\mathbb{C}_\alpha\mathbb{G}(\mathscr{X})$, and $\{1\}\notin\mathbb{C}_p\mathbb{G}(\mathscr{X})$.

Definition 7. $\mathfrak{A}$ is called Q_p-open set if $\mathscr{X}\backslash\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$ and $Q_p\mathbb{O}(\mathscr{X})$ is the set of all Q_p-open sets in $\mathscr{X}$.

Lemma 2. *The following properties are holding in* $(\mathscr{X},\tilde{\tau})$.

(1) $\mathscr{X}\backslash\mathscr{C}_{gp}(\mathscr{X}\backslash\mathfrak{A})=\mathscr{I}_{gp}(\mathfrak{A})$

(2) $\mathscr{X}\backslash\mathscr{I}_{gp}(\mathscr{X}\mathfrak{A})=\mathscr{C}_{gp}(\mathfrak{A})$

Proof. It is clear. □

Theorem 5. *The following properties are holding in* $(\mathscr{X},\tilde{\tau})$:

$$\mathfrak{A}\in Q_p\mathbb{O}(\mathscr{X})\Leftrightarrow\mathfrak{A}\subseteq\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{A})). \tag{13}$$

Proof. Suppose that $\mathfrak{A}\in Q_p\mathbb{O}(\mathscr{X})$. Then, $\mathscr{X}\backslash\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$ and $\mathscr{C}_{gp}(\mathscr{I}(\mathscr{X}\backslash\mathfrak{A}))\subseteq\mathscr{X}\backslash\mathfrak{A}$. From Lemma 2, we have $\mathfrak{A}\subseteq\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{A}))$. Conversely, $\mathfrak{A}\subseteq\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{A}))$. Then, $\mathscr{X}\backslash\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{A}))\subseteq\mathscr{X}\backslash\mathfrak{A}$. Thus, $\mathscr{I}_{gp}(\mathscr{C}(\mathscr{X}\mathfrak{A}))\subseteq\mathscr{X}\backslash\mathfrak{A}$. Thus, $\mathscr{X}\backslash\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$, and hence, $\mathfrak{A}\in Q_p\mathbb{O}(\mathscr{X})$. □

Lemma 3. *Let* $\mathfrak{L}\in\mathbb{GO}_p(\mathscr{X})$ *such that* $\mathfrak{L}\subseteq\mathfrak{A}\subseteq\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{L}))$. *Then,* $\mathfrak{A}\in Q_p\mathbb{O}(\mathscr{X})$.

Proof. As $\mathfrak{L}\in\mathbb{GO}_p(\mathscr{X})$, $\mathscr{X}\backslash\mathfrak{L}\in\mathbb{GC}_p(\mathscr{X})$, and $\mathscr{X}\backslash\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{L}))\subseteq\mathscr{X}\backslash\mathfrak{A}\subseteq\mathscr{X}\backslash\mathfrak{L}$, thus, by Lemma 2, we have $\mathscr{C}_{gp}(\mathscr{I}(\mathscr{X}\backslash\mathfrak{L}))\subseteq\mathscr{X}\backslash\mathfrak{A}\subseteq\mathscr{X}\backslash\mathfrak{L}$. By Lemma 1, we get $\mathscr{X}\backslash\mathfrak{A}\in Q_p\mathbb{C}(\mathscr{X})$, and hence, $\mathfrak{A}\in Q_p\mathbb{O}(\mathscr{X})$. □

The converse of Lemma 3 (i.e., $\mathfrak{L}\subseteq\not\subseteq\mathscr{I}_{gp}(\mathscr{C}(\mathfrak{L}))$) does not hold by the following example.

Example 6. (continued from Example 1). As

$$\mathbb{O}_p(\mathcal{X}) = \{\mathcal{X}, \phi, \{1\}, \{1,2\}, \{1,3\}\},$$
$$\mathbb{GO}_p(\mathcal{X}) = \{\mathcal{X}, \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}, \tag{14}$$

$$Q_p\mathbb{O}(\mathcal{X}) = 2^{\mathcal{X}}, \tag{15}$$

let $\mathfrak{A} = \{1,2\} \in Q_p\mathbb{O}(\mathcal{X})$ and $\mathfrak{L} = \phi \in \mathbb{GO}_p(\mathcal{X})$. Then, $\mathfrak{L} \subseteq A \nsubseteq \phi = \mathcal{I}_{gp}(\mathcal{C}(\mathfrak{L}))$.

Theorem 6. *The following two properties are holding in* $(\mathcal{X}, \tilde{\tau})$.

(1) *If* $\mathfrak{A} \in \mathbb{O}_p(\mathcal{X})$, *then* $\mathfrak{A} \in Q_p\mathbb{O}(\mathcal{X})$
(2) *If* $\mathfrak{A} \in \mathbb{GO}_p(\mathcal{X})$, *then* $\mathfrak{A} \in Q_p\mathbb{O}(\mathcal{X})$

Proof. From Theorem 1 and Lemma 2, the proof is clear. □

The converse of Theorem 6 (i.e., $\mathfrak{A} \in Q_p\mathbb{O}(\mathcal{X})$, but $\mathfrak{A} \notin \mathbb{O}_p(\mathcal{X})$ and $\mathfrak{A} \notin \mathbb{GO}_p(\mathcal{X})$) does not hold by the following example.

Example 7. (continued from Examples 1 and 6). $\{2\} \in Q_p\mathbb{O}(\mathcal{X})$, but $\{2\} \notin \mathbb{O}_p(\mathcal{X})$, and $\{2,3\} \in Q_p\mathbb{O}(\mathcal{X})$, but $\{2,3\} \notin \mathbb{GO}_p(\mathcal{X})$.

Theorem 7. *Arbitrary union of* Q_p*-open sets is* Q_p*-open set.*

Proof. From Theorem 2 and Lemma 2, the proof is clear. □

Remark 2. The intersection of two Q_p-open sets need not be Q_p-open set (i.e., $\mathfrak{A}, \mathfrak{B} \in Q_p\mathbb{O}(\mathcal{X})$, but $\mathfrak{A} \cap \mathfrak{B} \notin Q_p\mathbb{O}(\mathcal{X})$) as given in Remark 1. As

$$\mathbb{O}_p(\mathcal{X}) = \mathbb{GO}_p(\mathcal{X}) = Q_p\mathbb{O}(\mathcal{X}) = \{\mathcal{X}, \phi, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}\}, \tag{16}$$

let $\mathfrak{A} = \{2,3\} \in Q_p\mathbb{O}(\mathcal{X})$ and $\mathfrak{B} = \{1,3\} \in Q_p\mathbb{O}(\mathcal{X})$. Then, $\mathfrak{A} \cap \mathfrak{B} = \{3\} \notin Q_p\mathbb{O}(\mathcal{X})$.

Corollary 2. *The following two properties are holding in* $(\mathcal{X}, \tilde{\tau})$.

(1) *Let* $\mathfrak{A} \in Q_p\mathbb{O}(\mathcal{X})$ *and* $\mathfrak{B} \in \mathbb{O}_p(\mathcal{X})$. *Then* $\mathfrak{A} \cup \mathfrak{B} \in Q_p\mathbb{O}(\mathcal{X})$.
(2) *Let* $\mathfrak{A} \in Q_p\mathbb{O}(\mathcal{X})$ *and* $\mathfrak{B} \in \mathbb{GO}_p(\mathcal{X})$. *Then* $\mathfrak{A} \cup \mathfrak{B} \in Q_p\mathbb{O}(\mathcal{X})$.

Proof. From Theorem 6 and Lemma 7, the proof is clear.
□ □

Definition 8. $\mathcal{I}_{Q_p}(\mathfrak{A})$ is called Q_p-interior of $\mathfrak{A}$ in $(\mathcal{X}, \tilde{\tau})$ if

$$\mathcal{I}_{Q_p}(\mathfrak{A}) = \cup \left\{\mathfrak{L} | \mathfrak{L} \subseteq \mathfrak{A} \text{ and } \mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X})\right\}. \tag{17}$$

Lemma 4. *The following two properties are holding in* $(\mathcal{X}, \tilde{\tau})$:

(1) $\mathcal{X} \backslash \mathcal{C}_{Q_p}(\mathfrak{A}) = \mathcal{I}_{Q_p}(\mathcal{X} \backslash \mathfrak{A})$
(2) $\mathcal{X} \backslash \mathcal{I}_{Q_p}(\mathfrak{A}) = \mathcal{C}_{Q_p}(\mathcal{X} \backslash \mathfrak{A})$

Proof. It is clear. □

Theorem 8. *The following properties are holding in* $(\mathcal{X}, \tilde{\tau})$.

(1) $\mathfrak{A} \in Q_p\mathbb{O}(\mathcal{X}) \Leftrightarrow \mathcal{I}_{Q_p}(\mathfrak{A}) = \mathfrak{A}$
(2) $\mathcal{I}_p(\mathfrak{A}) \subseteq \mathcal{I}_{Q_p}(\mathfrak{A}) \subseteq \mathfrak{A}, \mathcal{I}_{gp}(\mathfrak{A}) \subseteq \mathcal{I}_{Q_p}(\mathfrak{A})$
(3) $\mathcal{I}_{Q_p}(\phi) = \phi$ *and* $\mathcal{I}_{Q_p}(\mathcal{X}) = \mathcal{X}$
(4) *If* $\mathfrak{A} \subseteq \mathfrak{B}$, *then* $\mathcal{I}_{Q_p}(\mathfrak{A}) \subseteq \mathcal{I}_{Q_p}(\mathfrak{B})$
(5) $\mathcal{I}_{Q_p}(\mathcal{I}_{Q_p}(\mathfrak{A})) = \mathcal{I}_{Q_p}(\mathfrak{A})$
(6) $\mathcal{I}_{Q_p}(\mathfrak{A}) \cup \mathcal{I}_{Q_p}(\mathfrak{B}) \subseteq \mathcal{I}_{Q_p}(\mathfrak{A} \cup \mathfrak{B})$
(7) $\mathcal{I}_{Q_p}(\mathfrak{A} \cap \mathfrak{B}) \subseteq \mathcal{I}_{Q_p}(\mathfrak{A}) \cap \mathcal{I}_{Q_p}(\mathfrak{B})$

Proof. It is similar to Theorem 3. □

The equality of Theorem 8 (6) and (7) (i.e., $\mathcal{I}_{Q_p}(\mathfrak{A}) \cup \mathcal{I}_{Q_p}(\mathfrak{B}) \neq \mathcal{I}_{Q_p}(\mathfrak{A} \cup \mathfrak{B})$ and $\mathcal{I}_{Q_p}(\mathfrak{A} \cap \mathfrak{B}) \neq \mathcal{I}_{Q_p}(\mathfrak{A}) \cap \mathcal{I}_{Q_p}(\mathfrak{B})$) does not hold by the following examples.

Example 8 (continued from Remarks 1 and 2). As $\mathfrak{A} = \{2\}, \mathfrak{B} = \{3\}$, and $\mathfrak{A} \cup \mathfrak{B} = \{2,3\}$, then $\mathcal{I}_{Q_p}(\{2\}) = \{2\}, \mathcal{I}_{Q_p}(\{3\}) = \phi$, and $\mathcal{I}_{Q_p}(\{2,3\}) = \{2,3\}$, and hence, $\mathcal{I}_{Q_p}(\mathfrak{A}) \cup \mathcal{I}_{Q_p}(\mathfrak{B}) = \{2\} \neq \{2,3\} = \mathcal{I}_{Q_p}(\mathfrak{A} \cup \mathfrak{B})$.

Example 9 (continued from Remarks 1 and 2). As $\mathfrak{A} = \{1,3\}, \mathfrak{B} = \{2,3\}$, and $\mathfrak{A} \cap \mathfrak{B} = \{3\}$, then $\mathcal{I}_{Q_p}(\{1,3\}) = \{1,3\}$, $\mathcal{I}_{Q_p}(\{2,3\}) = \{2,3\}$, and $\mathcal{I}_{Q_p}(\{3\}) = \phi$, and hence, $\mathcal{I}_{Q_p}(\mathfrak{A}) \cap \mathcal{I}_{Q_p}(\mathfrak{B}) = \{3\} \neq \phi = \mathcal{I}_{Q_p}(\mathfrak{A} \cap \mathfrak{B})$.

Theorem 9. *The following properties is holding in* $(\mathcal{X}, \tilde{\tau})$. *Then,*

$$x \in \mathcal{C}_{Q_p}(\mathfrak{A}) \Leftrightarrow \mathfrak{L} \cap \mathfrak{A} \neq \phi, \quad \forall x \in \mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X}). \tag{18}$$

Proof. The proof is clear. □

Lemma 5. *The following properties are holding in* $(\mathcal{X}, \tilde{\tau})$. *Then,*

(1) $\mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A}))) \in Q_p\mathbb{O}(\mathcal{X})$
(2) $\mathfrak{A} \cup \mathcal{C}_{gp}(\mathcal{I}(\mathcal{C}_{gp}(\mathfrak{A}))) \in Q_p\mathbb{C}(\mathcal{X})$

Proof. $\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))))) = \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A}) \cap t\mathcal{I}_{gp}n(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A}))))) = \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))$.

Then, we have $\mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A}))) = \mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))))) \subseteq \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))))).$

Hence, $\mathfrak{A} \cap \mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A}))) \in Q_p\mathbb{O}(\mathcal{X})$.

In (2), by (1), we have $\mathcal{X}\backslash(\mathfrak{A}\cup\mathcal{C}_{gp}(\mathcal{I}(\mathcal{C}_{gp}(\mathfrak{A})))) = (\mathcal{X}\backslash\mathfrak{A})\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathcal{X}\mathfrak{A}))) \in Q_p\mathbb{O}(\mathcal{X})$. Therefore, $\mathfrak{A}\cup\mathcal{C}_{gp}(\mathcal{I}(\mathcal{C}_{gp}(\mathfrak{A}))) \in Q_p\mathbb{C}(\mathcal{X})$. □

Theorem 10. *The following two properties are holding in* $(\mathcal{X},\tilde{\tau})$. *Then,*

(1) $\mathcal{I}_{Q_p}(\mathfrak{A}) = \mathfrak{A}\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))$

(2) $\mathcal{C}_{Q_p}(\mathfrak{A}) = \mathfrak{A}\cup\mathcal{C}_{gp}(\mathcal{I}(\mathcal{C}_{gp}(\mathfrak{A})))$

Proof.

(1) Suppose $\mathfrak{B} = \mathcal{I}_{Q_p}(\mathfrak{A})$. Since $\mathfrak{B}\in Q_p\mathbb{O}(\mathcal{X})$ and $\mathfrak{B}\subseteq\mathfrak{A}$, then, we have $\mathfrak{B}\subseteq\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{B})))\subseteq\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))$. Thus, $\mathfrak{B}\subseteq\mathfrak{A}\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))$. By Lemma 5, we get $\mathfrak{A}\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A}))) \in Q_p\mathbb{O}(\mathcal{X})$. From Definition 8, we have $\mathfrak{A}\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))\subseteq\mathfrak{B}$. Therefore, $\mathfrak{B} = \mathfrak{A}\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))$ and hence, $\mathcal{I}_{Q_p}(\mathfrak{A}) = \mathfrak{A}\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathfrak{A})))$.

(2) By Lemma 4, we have $\mathcal{C}_{Q_p}(\mathfrak{A}) = \mathcal{X}\backslash\mathcal{I}_{Q_p}(\mathcal{X}\backslash\mathfrak{A}) = \mathcal{X}\backslash((\mathcal{X}\backslash\mathfrak{A})\cap\mathcal{I}_{gp}(\mathcal{C}(\mathcal{I}_{gp}(\mathcal{X}\backslash\mathfrak{A})))) = \mathfrak{A}\cup\mathcal{C}_{gp}(\mathcal{I}(\mathcal{C}_{gp}(\mathfrak{A})))$. □

The relationship among the Q_p-open sets and other sets (i.e., open sets, α-open sets, g-open sets, $g\alpha$-open sets, αg-open sets, and pg-open sets) is presented by the following theorem.

Theorem 11. *The following properties is holding in* $(\mathcal{X},\tilde{\tau})$.

(1) If $\mathfrak{A}\in\tilde{\tau}(\mathcal{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$

(2) If $\mathfrak{A}\in\mathbb{O}_\alpha(\mathcal{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$

(3) If $\mathfrak{A}\in\mathbb{GO}(\mathcal{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$

(4) If $\mathfrak{A}\in\mathbb{GO}_\alpha(\mathcal{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$

(5) If $\mathfrak{A}\in\mathbb{O}_\alpha\mathbb{G}(\mathcal{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$

(6) If $\mathfrak{A}\in\mathbb{O}_p\mathbb{G}(\mathcal{X})$, *then* $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$

Proof. It is similar to Theorem 4. □

The converse of Theorem 11 (i.e., $\mathfrak{A}\in Q_p\mathbb{O}(\mathcal{X})$, but $\mathfrak{A}\notin\tilde{\tau}(\mathcal{X})$, $\mathfrak{A}\notin\mathbb{O}_\alpha(\mathcal{X})$, $\mathfrak{A}\notin\mathbb{GO}(\mathcal{X})$, $\mathfrak{A}\notin\mathbb{GO}_\alpha(\mathcal{X})$, $\mathfrak{A}\notin\mathbb{O}_\alpha\mathbb{G}(\mathcal{X})$, and $\mathfrak{A}\notin\mathbb{O}_p\mathbb{G}(\mathcal{X})$) does not hold by the following example.

Example 10. (continued from Examples 1, 5, and 6). Clearly,

$$\mathbb{GO}_\alpha(\mathcal{X}) = \mathbb{O}_p\mathbb{G}(\mathcal{X}) = \mathbb{O}_\alpha\mathbb{G}(\mathcal{X}) = \mathbb{O}_\alpha(\mathcal{X}) = \mathbb{O}_p(\mathcal{X}). \tag{19}$$

and $\mathbb{GO}(\mathcal{X}) = \mathbb{GO}_p(\mathcal{X})$. Thus, $\{2,3\}\in Q_p\mathbb{O}(\mathcal{X})$, but $\{2,3\}\in\tilde{\tau},(2,3)\in\mathcal{X})(\mathbb{O}_\alpha(\mathcal{X}),\{2,3\}\notin\mathbb{GO}(\mathcal{X})$, $\{2,3\}\notin\mathbb{GO}_\alpha(\mathcal{X}),\{2,3\}\notin\mathbb{O}_\alpha\mathbb{G}(\mathcal{X}))$, and $\{2,3\}\notin\mathbb{O}_p\mathbb{G}(\mathcal{X})$.

3. ERROR!!Q_p-Continuous Mappings, Q_p-Open Mappings, and Q_p-Closed Mappings

Definition 9. A mapping $\psi\colon \mathcal{X}\longrightarrow\mathcal{Y}$ is called Q_p-continuous if

$$\begin{gathered}\mathfrak{L}\in\tilde{\sigma}(\mathcal{Y}),\\ \psi^{-1}(\mathfrak{L})\in Q_p\mathbb{O}(\mathcal{X}),\end{gathered} \tag{20}$$

where $\tilde{\sigma}(\mathcal{Y})$ is a topology on $\mathcal{Y}$ and $\mathcal{X}$ is defined on a topology $\tilde{\tau}$

Theorem 12. *The following two properties are holding in* $(\mathcal{X},\tilde{\tau})$ *and* $(\mathcal{Y},\tilde{\sigma})$.

(1) Every precontinuous mapping is Q_p*-continuous mapping*

(2) Every gp-continuous mapping is Q_p*-continuous mapping*

Proof. From Theorem 6, the proof is clear. □

The converse of Theorem 12 (i.e., ψ is Q_p-continuous mapping, but ψ not precontinuous mapping and ψ is Q_p-continuous mapping but ψ not gp-continuous mapping) does not hold by the following example.

Example 11. Assume that $\mathcal{X}=\{1,2,3\}$ (i.e., $(\mathcal{X},\tilde{\tau})$ be topological space, $\tilde{\tau}=\{\mathcal{X},\phi,\{1\}\}$) and $\mathcal{Y}=\{u,v,w\}$ (i.e., $(\mathcal{Y},\tilde{\sigma})$ be topological space, $\tilde{\sigma}=\{\mathcal{Y},\phi,\{u,v\},\{w\}\}$).

(1) Suppose $\psi\colon \mathcal{X}\longrightarrow\mathcal{Y}$ be a mapping defined by

$$\psi(1)=\psi(2)=u,\text{and}\,\psi(3)=w. \tag{21}$$

Then,

$$\begin{aligned}\mathcal{F}_{\mathcal{X}} &= \{\mathcal{X},\phi,\{2,3\}\},\\ \mathbb{C}_p(\mathcal{X}) &= \{\mathcal{X},\phi,\{2\},\{3\},\{2,3\}\},\\ \mathbb{GC}_p(\mathcal{X}) &= \{\mathcal{X},\phi,\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\}\},\\ \mathbb{O}_p(\mathcal{X}) &= \{\mathcal{X},\phi,\{1\},\{1,2\},\{1,3\}\},\\ \mathbb{GO}_p(\mathcal{X}) &= \{\mathcal{X},\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\}\},\\ Q_p\mathbb{C}(\mathcal{X}) &= Q_p\mathbb{O}(\mathcal{X}) = 2^{\mathcal{X}}.\end{aligned} \tag{22}$$

As $\{w\}\in\tilde{\sigma}(\mathcal{Y}),\psi^{-1}(\{w\})=\{3\}\in Q_p\mathbb{O}(\mathcal{X})$, but $\{3\}\notin\mathbb{O}_p(\mathcal{X})$.

Thus, ψ is Q_p-continuous mapping but ψ not precontinuous mapping.

(2) Suppose $\psi\colon \mathcal{X}\longrightarrow\mathcal{Y}$ be a mapping defined by

$$\psi(1)=u\,\text{and}\,\psi(2)=\psi(3)=w. \tag{23}$$

As $\{w\} \in \tilde{\sigma}(\mathscr{Y}), \psi^{-1}(\{w\}) = \{2,3\} \in Q_p\mathbb{O}(\mathscr{X})$, but $\{2,3\} \notin \mathbb{GO}_p(\mathscr{X})$.

Thus, ψ is Q_p-continuous mapping but ψ not gp-continuous mapping.

Theorem 13. *Assume that ψ: $\mathscr{X} \longrightarrow \mathscr{Y}$ be a mapping. Then, the following six properties are equivalent:*

(1) ψ is Q_p-continuous

(2) For every $x \in \mathscr{X}$ and every open set $\mathfrak{L} \subset \mathscr{Y}$ containing $\psi(x)$, there exists Q_p-open set $\mathfrak{P} \subset \mathscr{X}$ containing x such that $\psi(\mathfrak{P}) \subset \mathfrak{L}$

(3) $\psi^{-1}(\mathfrak{F}) \in Q_p\mathbb{C}(\mathscr{X})$, $\mathfrak{F} \subset \mathscr{Y}$, be a closed set

(4) $\psi(\mathscr{C}_{Q_p}(\mathfrak{A}))\subseteq\mathscr{C}(\psi(\mathfrak{A}))(\mathfrak{A}\subseteq\mathscr{X})$

(5) $\mathscr{C}_{Q_p}(\psi^{-1}(\mathfrak{B}))\subseteq\psi^{-1}(\mathscr{C}(\mathfrak{B}))(\mathfrak{B}\subseteq\mathscr{Y})$

(6) $\psi^{-1}(\mathscr{I}(\mathfrak{B}))\subseteq\mathscr{I}_{Q_p}(\psi^{-1}(\mathfrak{B}))(\mathfrak{B}\subseteq\mathscr{Y})$

Proof.

(1) ⇒ (2) Since $\mathfrak{L} \subset \mathscr{Y}$ containing $\psi(x)$ is the open set, then $\psi^{-1}(\mathfrak{L}) \in Q_p\mathbb{O}(\mathscr{X})$. Put $\mathfrak{P} = \psi^{-1}(\mathfrak{L})$ which contains x; hence, $\psi(\mathfrak{P}) \subset \mathfrak{L}$.

(2) ⇒ (1) Suppose $\mathfrak{L} \subset \mathscr{Y}$ be the open set, and let $x \in \psi^{-1}(\mathfrak{L})$; then, $\psi(x) \in \mathfrak{L}$ and hence, there exists $\mathfrak{P}_x \in Q_p\mathbb{O}(\mathscr{X})$ such that $x \in \mathfrak{P}_x$ and $\psi(\mathfrak{P}_x) \subset \mathfrak{L}$. Thus, $x \in \mathfrak{P}_x \subset \psi^{-1}(\mathfrak{L})$, so $\psi^{-1}(\mathfrak{L}) = \cup_{x\in\psi^{-1}(\mathfrak{L})}\mathfrak{P}_x$, but $\cup_{x\in\psi^{-1}(\mathfrak{L})}\mathfrak{P}_x \in Q_p\mathbb{O}(\mathscr{X})$. Therefore, $\psi^{-1}(\mathfrak{L}) \in Q_p\mathbb{O}(\mathscr{X})$, and thus, ψ is Q_p-continuous.

(1) ⇒ (3) Suppose $\mathfrak{F} \subset \mathscr{Y}$ be closed set. Thus, $\mathscr{Y}\backslash\mathfrak{F}$ is the open set and $\psi^{-1}(\mathscr{Y}\backslash\mathfrak{F}) \in Q_p\mathbb{O}(\mathscr{X})$, i.e., $\mathscr{X}\backslash\psi^{-1}(\mathfrak{F}) \in Q_p\mathbb{O}(\mathscr{X})$. Hence, $\psi^{-1}(\mathfrak{F}) \in Q_p\mathbb{C}(\mathscr{X})$.

(3) ⇒ (4) Suppose $\mathfrak{A}\subseteq\mathscr{X}$ and $\mathfrak{F}$ be a closed set in $\mathscr{Y}$ containing $\psi(\mathfrak{A})$. By (3), we have $\psi^{-1}(\mathfrak{F})$ is Q_p-closed set containing $\mathfrak{A}$. Then, $\mathscr{C}_{Q_p}(\mathfrak{A})\subseteq\mathscr{C}_{Q_p}(\psi^{-1}(\mathfrak{F})) = \psi^{-1}(\mathfrak{F})$, and thus, $\psi(\mathscr{C}_{Q_p}(\mathfrak{A}))\subseteq\mathfrak{F}$. Hence, $\psi(\mathscr{C}_{Q_p}(\mathfrak{A}))\subseteq\mathscr{C}(\psi(\mathfrak{A}))$.

(4) ⇒ (5) Suppose $\mathfrak{B}\subseteq\mathscr{Y}$ and $\mathfrak{A} = \psi^{-1}(\mathfrak{B})$. By assumption, we have $\psi(\mathscr{C}_{Q_p}(\mathfrak{A}))\subseteq\mathscr{C}(\psi(\mathfrak{A}))\subseteq\mathscr{C}(\mathfrak{B})$. Then, $\mathscr{C}_{Q_p}(\mathfrak{A})\subseteq\psi^{-1}(\mathscr{C}(\mathfrak{B}))$. Therefore, $\mathscr{C}_{Q_p}(\psi^{-1}(\mathfrak{B}))\subseteq\psi^{-1}(\mathscr{C}(\mathfrak{B}))$.

(5) ⇒ (6) Suppose $\mathfrak{B}\subseteq\mathscr{Y}$. By assumption, we have $\mathscr{C}_{Q_p}(\psi^{-1}(\mathscr{Y}\backslash\mathfrak{B}))\subseteq\psi^{-1}(\mathscr{C}(\mathscr{Y}\mathfrak{B}))$. Then, $\mathscr{C}_{Q_p}(\mathscr{X}\backslash\psi^{-1}(\mathfrak{B}))\subseteq\psi^{-1}(\mathscr{Y}\backslash\mathscr{I}(\mathfrak{B}))$, and thus, $\mathscr{X}\backslash\mathscr{I}_{Q_p}(\psi^{-1}(\mathfrak{B}))\subseteq\mathscr{X}\backslash\psi^{-1}(\mathscr{I}(\mathfrak{B}))$. By taking complement, we obtain $\psi^{-1}(\mathscr{I}(\mathfrak{B}))\subseteq\mathscr{I}_{Q_p}(\psi^{-1}(\mathfrak{B}))$.

(6) ⇒ (1) Suppose $\mathfrak{L}$ be any open set in $\mathscr{Y}$. Then, $\mathscr{I}(\mathfrak{L}) = \mathfrak{L}$. By assumption, $\psi^{-1}(\mathscr{I}(\mathfrak{L})) \subseteq\mathscr{I}_{Q_p}(\psi^{-1}(\mathfrak{L}))$, and hence, $\psi^{-1}(\mathfrak{L})\subseteq\mathscr{I}_{Q_p}(\psi^{-1}(\mathfrak{L}))$. Thus, $\psi^{-1}(\mathfrak{L}) = \mathscr{I}_{Q_p}(\psi^{-1}(\mathfrak{L}))$, and we have $\psi^{-1}(\mathfrak{L}) \in Q_p\mathbb{O}(\mathscr{X})$. Therefore, ψ is Q_p-continuous. □

Remark 3. Composition of two Q_p-continuous mappings does not need to be Q_p-continuous mapping, as shown by the following example.

Example 12. Assume that $\mathscr{X} = \{1,2,3\}$ (i.e., $(\mathscr{X},\tilde{\tau})$ be topological space, $\tilde{\tau} = \{\mathscr{X},\phi,\{2\},t\{1,2\}\}$), $\mathscr{Y} = \{u,v,w\}$ (i.e., $(\mathscr{Y},\tilde{\sigma})$ be topological space, $\tilde{\sigma} = \{\mathscr{Y},\phi,\{u\}\}$), and $\mathscr{Z} = \{s,r,t\}$ (i.e., $(\mathscr{Z},\tilde{\theta})$ be topological space, $\tilde{\theta} = \{\mathscr{Z},\phi,\{t\}\}$).

(1) Suppose ψ: $\mathscr{X} \longrightarrow \mathscr{Y}$ be a mapping defined by

$$\psi(1) = v\,, \psi(2) = u, \text{and} \psi(3) = w. \tag{24}$$

Then,

$$\begin{aligned}
\mathscr{F}_{\mathscr{X}} &= \{\mathscr{X},\phi,\{3\}\{1,3\}\},\\
\mathbb{C}_p(\mathscr{X}) &= \{\mathscr{X},\phi,\{1\},\{3\},\{1,3\}\},\\
\mathbb{GC}_p(\mathscr{X}) = Q_p\mathbb{C}(\mathscr{X}) &= \{\mathscr{X},\phi,\{1\},\{3\},\{1,3\},\{2,3\}\},\\
\mathbb{O}_p(\mathscr{X}) &= \{\mathscr{X},\phi,\{2\},\{1,2\},\{2,3\}\},\\
\mathbb{GO}_p(\mathscr{X}) = Q_p\mathbb{O}(\mathscr{X}) &= \{\mathscr{X},\phi,\{1\},\{2\},\{1,2\},\{2,3\}\}.
\end{aligned} \tag{25}$$

Thus, ψ is Q_p-continuous mapping.

(2) Suppose φ: $\mathscr{Y} \longrightarrow \mathscr{Z}$ be a mapping defined by

$$\varphi(u) = \varphi(v) = s, \text{ and } \varphi(w) = t. \tag{26}$$

Then,

$$\begin{aligned}
\mathscr{F}_{\mathscr{Y}} &= \{\mathscr{Y},\phi,\{v,w\}\},\\
\mathbb{C}_p(\mathscr{Y}) &= \{\mathscr{Y},\phi,\{v\},\{w\},\{v,w\}\},\\
\mathbb{GC}_p(\mathscr{Y}) &= \{\mathscr{Y},\phi,\{v\},\{w\},\{u,v\},\{u,w\},\{v,w\}\},\\
\mathbb{O}_p(\mathscr{Y}) &= \{\mathscr{Y},\phi,\{u\},\{u,v\},\{u,w\}\},\\
\mathbb{GO}_p(\mathscr{Y}) &= \{\mathscr{Y},\phi,\{u\},\{v\},\{w\},\{u,v\},\{u,w\}\},\\
Q_p\mathbb{C}(\mathscr{Y}) &= Q_p\mathbb{O}(\mathscr{Y}).
\end{aligned} \tag{27}$$

Thus, φ is Q_p-continuous mapping.

From (1) and (2), $\{t\} \in \tilde{\theta}(\mathscr{Z})$, but $(\varphi^\circ\psi)^{-1}(\{t\}) = \psi^{-1}(\varphi^{-1}(\{t\})) = \psi^{-1}(\{w\}) = \{3\} \notin Q_p\mathbb{O}(\mathscr{X})$. Thus, $\varphi^\circ\psi$ is not Q_p-continuous mapping.

Definition 10. A mapping ψ: $\mathscr{X} \longrightarrow \mathscr{Y}$ is called

(1) Q_p-open if

$$\mathfrak{L} \in \tilde{\tau}(\mathscr{X}),\ \psi(\mathfrak{L}) \in Q_p\mathbb{O}(\mathscr{Y}), \tag{28}$$

where $\tilde{\tau}(\mathscr{X})$ is a topology on $\mathscr{X}$ and $\mathscr{Y}$ is defined on a topology $\tilde{\sigma}$.

(2) Q_p-closed if

$$\mathfrak{F} \in \mathscr{F}_{\mathscr{X}},\ \psi(\mathfrak{F}) \in Q_p\mathbb{C}(\mathscr{Y}), \tag{29}$$

where $\mathscr{F}_{\mathscr{X}}$ is the closed sets of $\mathscr{X}$.

Theorem 14. *The following two properties are holding in $(\mathscr{X},\tilde{\tau})$ and $(\mathscr{Y},\tilde{\sigma})$.*

(1) *Every preopen (resp. preclosed) mapping is Q_p-open (resp. Q_p-closed) mapping*

(2) *Every gp-open (resp. gp-closed) mapping is Q_p-open (resp. Q_p-closed) mapping*

Proof. It follows from Theorem 6. □

Example 13. (continued from Example 11).

(1) Suppose $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ be a mapping defined by

$$\psi(1) = \psi(2) = v, \text{and} \psi(3) = u. \tag{30}$$

(i) As $\{2\} \in \tilde{\tau}(\mathscr{X}), \psi(\{2\}) = \{v\} \in Q_p\mathbb{O}(\mathscr{Y})$, but $\{v\} \notin \mathbb{O}_p(\mathscr{Y})$. Thus, ψ is Q_p-open mapping but ψ not preopen mapping.

(ii) As $\{3\} \in \mathscr{F}_{\mathscr{X}}$, $\psi(\{3\}) = \{u\} \in Q_p\mathbb{O}(\mathscr{Y})$, but $\{u\} \notin \mathbb{C}_p(\mathscr{Y})$. Thus, ψ is Q_p-closed mapping but ψ not preclosed mapping.

(2) Suppose $\varphi: \mathscr{X} \longrightarrow \mathscr{Y}$ be a mapping defined by

$$\varphi(1) = v, \varphi(2) = w, \text{and } \varphi(w) = u. \tag{31}$$

(i) As $\{1,2\} \in \tilde{\tau}(\mathscr{X})$, $\varphi(\{1,2\}) = \{v,w\} \in Q_p\mathbb{O}(\mathscr{Y})$, but $\{v,w\} \notin \mathbb{GO}_p(\mathscr{Y})$. Thus, φ is Q_p-open mapping but φ not gp-open mapping.

(ii) As $\{3\} \in \mathscr{F}_{\mathscr{X}}$, $\{u\} \notin \mathbb{GC}_p(\mathscr{Y})$. Thus, φ is Q_p-closed mapping but φ not gp-closed mapping.

Theorem 15. *Assume that $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ is a mapping. Then, the following two properties are equivalent:*

(1) *ψ is Q_p-open.=*

(2) *For every $x \in \mathscr{X}$ and $\mathfrak{U}$ is a neighborhood of x, there exists Q_p-open set $\mathfrak{B} \subset \mathscr{Y}$ containing $\psi(x)$ such that $\mathfrak{B} \subset \psi(\mathfrak{U})$*

Proof. The proof is clear. □

Theorem 16. *Assume that $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ is Q_p-open (resp. Q_p-closed) mapping and $\mathfrak{B} \subset \mathscr{Y}$. If $\mathfrak{A} \subset \mathscr{X}$ is a closed (resp. open) set containing $\psi^{-1}(\mathfrak{B})$, then there exists Q_p-open (resp. Q_p-closed) set $\mathfrak{H} \subset \mathscr{Y}$ containing $\mathfrak{B}$ such that $\psi^{-1}(\mathfrak{H}) \subset \mathfrak{A}$.*

Proof. Obvious. □

Corollary 3. *For every set $\mathfrak{B} \subseteq \mathscr{Y}$, if $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ is Q_p-open, then $\psi^{-1}(\mathscr{C}_{Q_p}(\mathfrak{B})) \subseteq \mathscr{C}(\psi^{-1}(\mathfrak{B}))$.*

Proof. Obvious. □

Theorem 17. *For any subset $\mathfrak{A}$ of $\mathscr{X}$, a mapping $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ is Q_p-open $\Leftrightarrow \psi(\mathscr{I}(\mathfrak{A})) \subseteq \mathscr{I}_{Q_p}(\psi(\mathfrak{A}))$.*

Proof. Assume that $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ is Q_p-open mapping and $\mathfrak{A} \subseteq \mathscr{X}$. Then, $\mathscr{I}(\mathfrak{A}) \in \tilde{\tau}(\mathscr{X})$ and $\psi(\mathscr{I}(\mathfrak{A}))$ is Q_p-open set contained in $\psi(\mathfrak{A})$. Hence, we have $\psi(\mathscr{I}(\mathfrak{A})) \subseteq \mathscr{I}_{Q_p}(\psi(\mathfrak{A}))$. Conversely, for every $\mathfrak{A}$ of $\mathscr{X}$, $\psi(\mathscr{I}(\mathfrak{A})) \subseteq \mathscr{I}_{Q_p}(\psi(\mathfrak{A}))$ and $\mathfrak{L} \in \tilde{\tau}(\mathscr{X})$. Then, $\mathscr{I}(\mathfrak{L}) = \mathfrak{L}$, $\psi(\mathfrak{L}) \subseteq \mathscr{I}_{Q_p}(\psi(\mathfrak{L}))$. Therefore, $\psi(\mathfrak{L}) = \mathscr{I}_{Q_p}(\psi(\mathfrak{L}))$, and we have $\psi(\mathfrak{L})$ is Q_p-open mapping. □

Theorem 18. *Assume that $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ is a bijective mapping. Then, the following three properties are equivalent:*

(1) *ψ^{-1} is Q_p-continuous*

(2) *ψ is Q_p-open*

(3) *ψ is Q_p-closed*

Proof.

(1) ⇒ (2) Let $\mathfrak{U} \in \tilde{\tau}(\mathscr{X})$. Then, $\mathscr{X} \backslash \mathfrak{U} \in \mathscr{F}_{\mathscr{X}}$. Since ψ^{-1} is Q_p-continuous, $(\psi^{-1})^{-1}(\mathscr{X} \backslash \mathfrak{U}) = \psi(\mathscr{X} \backslash \mathfrak{U}) = \mathscr{Y} \backslash \psi(\mathfrak{U}) \in Q_p\mathbb{C}(\mathscr{Y})$. So, $\psi(\mathfrak{U}) \in Q_p\mathbb{O}(\mathscr{Y})$. Hence, ψ is Q_p-open mapping.

(2) ⇒ (3) Let $\mathfrak{F} \in \mathscr{F}_{\mathscr{X}}$. Then, $\mathscr{X} \backslash \mathfrak{F} \in \tilde{\tau}(\mathscr{X})$. Since ψ is Q_p-open, $\psi(\mathscr{X} \backslash \mathfrak{F}) = \mathscr{Y} \backslash \psi(\mathfrak{F}) \in Q_p\mathbb{O}(\mathscr{Y})$. So, $\psi(\mathfrak{U}) \in Q_p\mathbb{C}(\mathscr{Y})$. Hence, ψ is Q_p-closed mapping.

(3) ⇒ (1) Let $\mathfrak{F} \in \mathscr{F}_{\mathscr{X}}$. Since ψ is Q_p-closed, $(\psi^{-1})^{-1}(\mathfrak{F}) = \psi(\mathfrak{F}) \in Q_p\mathbb{C}(\mathscr{Y})$. Hence, ψ^{-1} is Q_p-continuous mapping. □

Remark 4. Composition of two Q_p-open (Q_p-closed) mappings do not need to be Q_p-open (Q_p-closed) as shown by the following example.

Example 14. Assume that $\mathscr{X} = \{1,2,3\}$ (i.e., $(\mathscr{X}, \tilde{\tau})$ be topological space, $\tilde{\tau} = \{\mathscr{X}, \phi, \{1,2\}, \{3\}\}$), $\mathscr{Y} = \{u,v,w\}$ (i.e., $(\mathscr{Y}, \tilde{\sigma})$ be topological space, $\tilde{\sigma} = \{\mathscr{Y}, \phi, \{u\}\}$), and $\mathscr{Z} = \{r,s,t\}$ (i.e., $(\mathscr{Z}, \tilde{\theta})$ be topological space, $\tilde{\theta} = \{\mathscr{Z}, \phi, \{s\}, t\{r,s\}\}$).

(1) Consider $\mathscr{F}_{\mathscr{Y}}, \mathbb{C}_p(\mathscr{Y}), \mathbb{O}_p(\mathscr{Y}), \mathbb{GC}_p(\mathscr{Y}), \mathbb{GO}_p(\mathscr{Y})$, $Q_p\mathbb{C}(\mathscr{Y})$, and $Q_p\mathbb{O}(\mathscr{Y})$ are computing in Example 12. Suppose $\psi: \mathscr{X} \longrightarrow \mathscr{Y}$ be a mapping defined by

$$\psi(1) = u, \psi(2) = v, \text{and} \psi(3) = w. \tag{32}$$

Thus, ψ is Q_p-open (Q_p-closed) mapping.

(2) Suppose $\varphi: \mathscr{Y} \longrightarrow \mathscr{Z}$ be a mapping defined by

$$\varphi(u) = s, \varphi(v) = r, \text{and } \varphi(w) = t. \tag{33}$$

Then,

$$\begin{aligned}
\mathscr{F}_{\mathscr{Z}} &= \{\mathscr{Z}, \phi, \{t\}, \{r,t\}\}, \\
\mathbb{C}_p(\mathscr{Z}) &= \{\mathscr{Z}, \phi, \{t\}, \{r\}, \{r,t\}\}, \\
\mathbb{GC}_p(\mathscr{Z}) = Q_p\mathbb{C}(\mathscr{Z}) &= \{\mathscr{Z}, \phi, \{r\}, \{t\}, \{s,t\}, \{r,t\}\}, \\
\mathbb{O}_p(\mathscr{Z}) &= \{\mathscr{Z}, \phi, \{s\}, \{s,t\}, \{r,s\}\}, \\
\mathbb{GO}_p(\mathscr{Z}) = Q_p\mathbb{O}(\mathscr{Z}) &= \{\mathscr{Z}, \phi, \{r\}, \{s\}, \{r,s\}, \{s,t\}\}.
\end{aligned} \tag{34}$$

Thus, φ is Q_p-open (Q_p-closed) mapping.

From (1) and (2), as $\{3\}$ is open in $\mathcal{X}$, $\varphi(\psi(\{3\})) = \varphi(\{w\}) = \{t\} \notin Q_p\mathbb{O}(\mathcal{Z})$. Therefore, $\varphi^\circ\psi$ is not Q_p-open mapping. Also, as $\{1,2\}$ is closed in $\mathcal{X}$, $\varphi(\psi(\{1,2\})) = \varphi(\{u,v\}) = \{r,s\} \notin Q_p\mathbb{C}(\mathcal{Z})$. Therefore, $\varphi^\circ\psi$ is not Q_p-closed mapping.

4. ERROR!!Q_p-$\mathbb{R}_0$ and Q_p-$\mathbb{R}_1$ Spaces

Definition 11.

(1) $\mathbb{K}_{Q_p}(\mathfrak{A})$ is called Q_p-kernel of $\mathfrak{A}$ (i.e., $\mathfrak{A}$ be subset of a space $\mathcal{X}$) if

$$\mathbb{K}_{Q_p}(\mathfrak{A}) = \cap\left\{\mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X}) | \mathfrak{A} \subset \mathfrak{L}\right\}. \tag{35}$$

(2) $\mathbb{K}_{Q_p}(\{x\})$ is called Q_p-kernel of x (i.e., x be a point of a space $\mathcal{X}$) if

$$\mathbb{K}_{Q_p}(\{x\}) = \cap\left\{\mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X}) | x \in \mathfrak{L}\right\}. \tag{36}$$

Lemma 6. *The following properties are holding in $(\mathcal{X},\widetilde{\tau})$. Then,*

(1) $y \in \mathbb{K}_{Q_p}(\{x\}) \Leftrightarrow x \in \mathscr{C}_{Q_p}(\{y\})$ $(x \in \mathcal{X})$

(2) $\mathbb{K}_{Q_p}(\mathfrak{A}) = \cap\left\{x \in \mathcal{X} | \mathscr{C}_{Q_p}(\{x\}) \cap \mathfrak{A} \neq \phi\right\}$

Proof. It is similar to Lemmas 3.1 and 3.2 of [16]. □

Lemma 7. *For any elements x and y in $(\mathcal{X},\widetilde{\tau})$, then the following two properties are equivalent:*

(1) $\mathbb{K}_{Q_p}(\{x\}) \neq \mathbb{K}_{Q_p}(\{y\})$

(2) $\mathscr{C}_{Q_p}(\{x\}) \neq \mathscr{C}_{Q_p}(\{y\})$

Proof. It is similar to Lemma 3.6 of [16]. □

Definition 12. We call Q_p-$\mathbb{R}_0$ space in $(\mathcal{X},\widetilde{\tau})$ if $\forall x \in \mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X})$ such that $\mathscr{C}_{Q_p}(\{x\}) \subset \mathfrak{L}$.

Theorem 19. *The following two properties are holding in $(\mathcal{X},\widetilde{\tau})$:*

(1) Every pre-$\mathbb{R}_0$ space is Q_p-$\mathbb{R}_0$ space

(2) Every gp-$\mathbb{R}_0$ space is Q_p-$\mathbb{R}_0$ space

Proof. It follows from Theorem 6. □

Theorem 20. *Let $x,y \in \mathcal{X}$. Then, Q_p-$\mathbb{R}_0$ space in $(\mathcal{X},\widetilde{\tau}) \Leftrightarrow \mathscr{C}_{Q_p}(\{x\}) \neq \mathscr{C}_{Q_p}(\{y\})$ implies $\mathscr{C}_{Q_p}(\{x\}) \cap \mathscr{C}_{Q_p}(\{y\}) = \phi$.*

Proof. From Definition 12, the proof is clear. □

Theorem 21. *Let $x,y \in \mathcal{X}$. Then, Q_p-$\mathbb{R}_0$ space in $(\mathcal{X},\widetilde{\tau}) \Leftrightarrow \mathbb{K}_{Q_p}(\{x\}) \neq \mathbb{K}_{Q_p}(\{y\})$ implies $\mathbb{K}_{Q_p}(\{x\}) \cap \mathbb{K}_{Q_p}(\{y\}) = \phi$.*

Proof. From Lemmas 6 (1), Lemma 7, Definition 12, and Theorem 20, the proof is clear. □

Theorem 22. *The following five properties are equivalent in $(\mathcal{X},\widetilde{\tau})$:*

(1) $(\mathcal{X},\widetilde{\tau})$ is an Q_p-$\mathbb{R}_0$ space

(2) For any $\mathfrak{A} \neq \phi$ and $\mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X})$ such that $\mathfrak{A} \cap \mathfrak{L} \neq \phi$, there exists $\mathfrak{F} \in Q_p\mathbb{C}(\mathcal{X})$ such that $\mathfrak{A} \cap \mathfrak{F} \neq \phi$ and $\mathfrak{F} \subset \mathfrak{L}$

(3) For any $\mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X}), \mathfrak{L} = \cup\left\{\mathfrak{F} \in Q_p\mathbb{C}(\mathcal{X}) | \mathfrak{F} \subset \mathfrak{L}\right\}$

(4) For any $\mathfrak{F} \in Q_p\mathbb{O}(\mathcal{X}), \mathfrak{F} = \cap\left\{\mathfrak{L} \in Q_p\mathbb{O}(\mathcal{X}) | \mathfrak{F} \subset \mathfrak{L}\right\}$

(5) For any $x \in \mathcal{X}$, $\mathscr{C}_{Q_p}(\{x\}) \subset \mathbb{K}_{Q_p}(\{x\})$

Proof. It is similar to Theorem 3.8 of [16]. □

Corollary 4. *The following two properties are equivalent in $(\mathcal{X},\widetilde{\tau})$:*

(1) $(\mathcal{X},\widetilde{\tau})$ is an Q_p-$\mathbb{R}_0$ space

(2) $\mathscr{C}_{Q_p}(\{x\}) = \mathbb{K}_{Q_p}(\{x\})(\forall x \in \mathcal{X})$

Proof. From Definition 12 and Theorem 22, the proof is clear. □

Theorem 23. *The following two properties are equivalent in $(\mathcal{X},\widetilde{\tau})$:*

(1) $(\mathcal{X},\widetilde{\tau})$ is an Q_p-$\mathbb{R}_0$ space

(2) $x \in \mathscr{C}_{Q_p}(\{y\}) \Leftrightarrow y \in \mathscr{C}_{Q_p}(\{x\})(\forall x, y \in \mathcal{X})$

Proof. The proof is clear. □

Theorem 24. *The following four properties are equivalent in $(\mathcal{X},\widetilde{\tau})$:*

(1) $(\mathcal{X},\widetilde{\tau})$ is an Q_p-$\mathbb{R}_0$ space

(2) If $\mathfrak{F} \in Q_p\mathbb{C}(\mathcal{X})$, then $\mathfrak{F} = \mathbb{K}_{Q_p}(\mathfrak{F})$

(3) If $\mathfrak{F} \in Q_p\mathbb{C}(\mathcal{X})$ and $x \in \mathfrak{F}$, then $\mathbb{K}_{Q_p}(\{x\}) \subset \mathfrak{F}$

(4) If $x \in \mathcal{X}$, then $\mathbb{K}_{Q_p}(\{x\}) \subset \mathscr{C}_{Q_p}(\{x\})$

Proof. From Lemma 6, Theorem 23, and Definition 12, the proof is clear. □

Definition 13. We call $(\mathcal{X},\widetilde{\tau})$ is Q_p-$\mathbb{R}_1$ space if

(i) $\forall x, y \in \mathcal{X}$ with $\mathscr{C}_{Q_p}(\{x\}) \neq \mathscr{C}_{Q_p}(\{y\})$

(ii) There exist disjoint Q_p-open sets $\mathfrak{L}$ and $\mathfrak{M}$ s.t. $\mathscr{C}_{Q_p}(\{x\}) \subseteq \mathfrak{L}$ and $\mathscr{C}_{Q_p}(\{y\}) \subseteq \mathfrak{M}$

Theorem 25. *Let $(\mathcal{X},\widetilde{\tau})$ be a topological space. Then,*

(1) Every pre-$\mathbb{R}_1$ space is Q_p-$\mathbb{R}_1$ space

(2) Every gp-$\mathbb{R}_1$ space is Q_p-$\mathbb{R}_1$ space

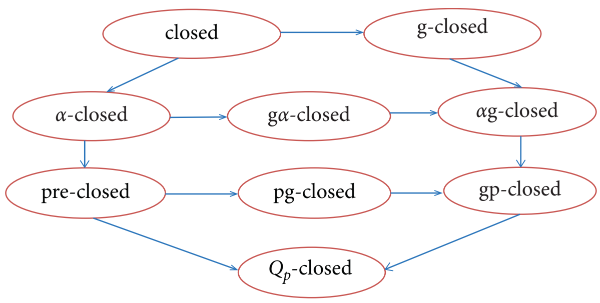

FIGURE 1: The relationship among the Q_p-closed sets and other sets.

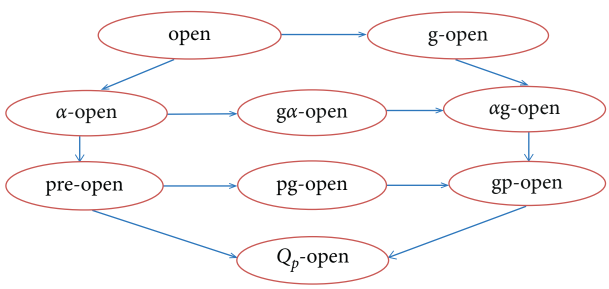

FIGURE 2: The relationship among the Q_p-open sets and other sets.

Proof. It is obvious. □

Theorem 26. *If* $(\mathcal{X}, \tilde{\tau})$ *is an* Q_p-$\mathbb{R}_1$ *space, then* $(\mathcal{X}, \tilde{\tau})$ *is* Q_p-$\mathbb{R}_0$.

Proof. It is from Definitions 12 and 13. □

5. Conclusion

We proposed novel notions (i.e., Q_p-closed set, Q_p-open set, Q_p-continuous mapping, Q_p-open mapping, and Q_p-closed mapping) and explained the basic interesting relations and properties of above notions. The relationship among the Q_p-closed sets (resp., Q_p-open sets) and other sets are given in Figure 1 (resp., Figure 2). Finally, a novel two separation axioms (i.e., Q_p-$\mathbb{R}_0$ and Q_p-$\mathbb{R}_1$) based on the notion of Q_p-open set and Q_p-closure are discussed. In the future, we will add new works (i.e., weakly Q_p-$\mathbb{R}_0$ space) and also extend several results from [18, 19] to Q_p-closed sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Natural Science Foundation of Shaanxi Province, China (2020JQ-481 and 2019JQ-014).

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