Afr. Mat. (2014) 25:757–777
DOI 10.1007/s13370-013-0149-3

On Rice's matrix polynomials

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Received: 3 August 2012 / Accepted: 16 March 2013 / Published online: 9 April 2013

Abstract The main aim of this paper is to define and study of a new matrix polynomials, say, the Rice's matrix polynomials. The convergence, radius of regularity, integral form, generating matrix functions and matrix recurrence relations satisfied by these Rice's matrix polynomials are derived. Furthermore, we study the operation of differential operators of Rice's matrix polynomials and their applications are presented. The matrix differential equation are obtained by them is presented. Finally, the study of the composition of Rice's matrix polynomials is investigated.

Keywords Hypergeometric matrix function · Rice's matrix polynomials · Matrix differential equations · Integral representation · Recurrence relations · Differential operator

Mathematics Subject Classification (2000) 15A60 · 33C05 · 33C20 · 33C60 · 33C70 · 34A05

1 Introduction

Special matrix functions seen on statistics, Lie group theory and number theory are well known in [5,17,23]. Recently, an extension to the matrix framework of the classical families orthogonal polynomials of Hermite, Humbert, Jacobi, Gegenbauer, Laguerre, Bessel, Chebyshev and pseudo Legendre matrix polynomials were introduced and studied in a number of previous papers, see for example, [1,6,7,10,11,14,19,22,30,32] for matrix in $\mathbb{C}^{N\times N}$. Furthermore, one can see more papers concerning orthogonal matrix polynomials in [4,9,13]. Hermite matrix polynomials have been introduced and studied in [2,3,20,21,25] for matrices

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in $\mathbb{C}^{N\times N}$ whose eigenvalues are all situated in the right open half-plane. In [7,33], the authors introduced and studied Jacobi matrix polynomials. The hypergeometric matrix function has been introduced as a matrix power series and an integral representation and the hypergeometric matrix differential equation in [15,18,27,28,31] and the explicit closed form general solution of it has been given in [16]. In [6], the authors introduced the Chebyshev matrix polynomials and gave some results with Chebyshev matrix polynomials. In [13], the authors studied a new system of matrix polynomials, namely the Gegenbauer matrix polynomials in [26,29]. The reason of interest for this family of Rice's matrix polynomials is due to their intrinsic mathematical importance.

The primary goal of this paper is to consider a new system of matrix polynomials, namely the Rice's matrix polynomials. The structure of the paper is as follows: In Sect. 2 a definition of Rice's matrix polynomials is given and the convergence properties, radius of convergence and an integral form are given. Some matrix differential recurrence relations, generating matrix functions and matrix recurrence relations are established, the effect of differential operator on these polynomials is investigated and the matrix differential equation satisfied by them is presented in Sect. 3. Finally, we define the composite Rice's matrix polynomials and the convergence properties are investigated in Sect. 4.

Throughout this paper for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. If A is a matrix in $\mathbb{C}^{N\times N}$, its two-norm denoted by $||A||_2$ is defined by [12]

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector y in $\mathbb{C}^N$, $||y||_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y. We say that A in $\mathbb{C}^{N\times N}$ is a positive stable matrix [14], if $Re(z) > 0$ for all $\lambda \in \sigma(z)$, and denotes

$$M(A) = \max\{Re(z) : z \in \sigma(A)\}; \quad m(A) = \min\{Re(z) : z \in \sigma(A)\}. \tag{1.1}$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [8], it follows that

$$f(A)g(A) = g(A)f(A). \tag{1.2}$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and also if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{1.3}$$

Let P and Q be two positive stable matrices in $\mathbb{C}^{N\times N}$. The gamma matrix function $\Gamma(P)$ and the beta matrix function $B(P, Q)$ have been defined in [14], as follows

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I} dt; \quad t^{P-I} = \exp((P-I)\ln t), \tag{1.4}$$

and

$$B(P, Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} dt, \tag{1.5}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$. Furthermore, if A is a matrix in $\mathbb{C}^{N\times N}$ such that

$$A + nI \text{ is invertible for every non-negative integer } n, \tag{1.6}$$

then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula [15]

$$(A)_n = A(A+I)(A+2I), \ldots, (A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1; \quad (A)_0 = I. \tag{1.7}$$

From the relation (1.3) of [3], one obtains

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!}; \quad 0 \leq k \leq n. \tag{1.8}$$

Jódar and Cortés have proved in [14,15] and $n \geq 1$ is an integer, then

$$\Gamma(A) = \lim_{n \to \infty} (n-1)![(A)_n]^{-1} n^A; \, n^A = e^{A \ln n}, \tag{1.9}$$

where $(A)_n$ is defined by (1.7). The hypergeometric matrix function ${}_2F_1(A, B; C; z)$ has been given in the form

$${}_2F_1(A, B; C; z) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k [(C)_k]^{-1}}{n!} z^k, \tag{1.10}$$

for matrices A, B and C in $\mathbb{C}^{N \times N}$ such that $C + nI$ is invertible for all integer $n \geq 0$ and for $|z| < 1$. It has been seen by Jódar and Cortés [15] that the series is absolutely converges for $|z| = 1$ when

$$m(C) > M(A) + M(B),$$

where $M(A)$, $M(B)$ and $m(C)$ are defined by (1.1).

We will exploit the following relation due to [15]

$$(1-z)^{-A} = {}_1F_0(A; -; z) = \sum_{n=0}^{\infty} \frac{1}{n!} (A)_n z^n; \quad |z| < 1. \tag{1.11}$$

In the following, we introduce to define and study of a new matrix polynomial which represents of the Rice's matrix polynomials as given by the relation and the convergence properties, radius of convergence and an integral form are given.

2 On the Rice's matrix polynomials

The Rice's matrix polynomials $H_n(A, B, z)$ is defined by means of the relation

$$\begin{aligned} H_n(A, B, z) &= {}_3F_2(-nI, (n+1)I, A; I, B; z) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} (-nI)_k ((1+n)I)_k (A)_k [(I)_k]^{-1} [(B)_k]^{-1}, \end{aligned} \tag{2.1}$$

for matrices A and B in $\mathbb{C}^{N \times N}$ and commutative matrices in $\mathbb{C}^{N \times N}$ such that $B + kI$ is invertible for all integer $k \geq 0$.

For the sake of brevity we shall denote the expressions $H_n(A, B, z)$, $\frac{z^k}{k!}(-nI)_k((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$ and $\frac{1}{k!}(-nI)_k((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$ by H_n, $U_k(z)$ and U_k respectively.

Now, we are going to study of the convergence properties of Rice's matrix polynomials. Note that, If k is large enough so then for $k > \|B\|$, then we will mention to the following relation which existed in Jódar and Cortés [14] in the form

$$\|(B + kI)^{-1}\| \leq \frac{1}{k - \|B\|}; \quad k > \|B\|. \tag{2.2}$$

Let us denote

$$\begin{aligned} \alpha_1(k) &= \|I^{-1}\|\|(2I)^{-1}\|, \ldots, \|(kI)^{-1}\|; \quad k > 1, \\ \alpha_2(k) &= \|(B)^{-1}\|\|(B + I)^{-1}\|, \ldots, \|(B + (k-1)I)^{-1}\|; \quad k > 1, \end{aligned} \tag{2.3}$$

and note that

$$\|(A)_k\| \leq (\|A\|)_k, \tag{2.4}$$

and taking into account the Pochhammer symbol or shifted factorial defined by

$$(a)_k = a(a+1)(a+2), \ldots, (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}; \quad k \geq 1; \quad (a)_0 = 1; a \neq 0.$$

By (2.2), (2.3) and (2.4) for $k > \|B\|$, we have

$$\begin{aligned} &\left\| \frac{z^k}{k!}(-nI)_k((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1} \right\| \\ &\quad \leq \frac{|z|^k}{k!}\|(-nI)_k\|\|((1+n)I)_k\|\|(A)_k\|\alpha_1(k)\alpha_2(k) \\ &\quad \leq \frac{|z|^k}{k!}(\|-nI\|)_k(\|(1+n)I\|)_k(\|A\|)_k\alpha_1(k)\alpha_2(k). \end{aligned} \tag{2.5}$$

Now, we will investigate the convergence of the following series

$$\sum_{k=0}^{\infty} \frac{|z|^k}{k!}(\|-nI\|)_k(\|(1+n)I\|)_k(\|A\|)_k\alpha_1(k)\alpha_2(k).$$

By using the ratio test and the relation (2.3), it follows

$$\begin{aligned} &\lim_{k\longrightarrow\infty}\left|\frac{(\|A\|)_{k+1}(\|-nI\|)_{k+1}(\|(1+n)I\|)_{k+1}\alpha_1(k+1)\alpha_2(k+1)k!}{(\|A\|)_k(\|-nI\|)_k(\|(1+n)I\|)_k\alpha_1(k)\alpha_2(k)(k+1)!}\frac{z^{k+1}}{z^k}\right| \\ &\leq \lim_{k\to\infty}\left|\frac{(\|A\|+k)(\|-nI\|+k)(\|(1+n)I\|+k)\|(I+kI)^{-1}\|\|(B+kI)^{-1}\|}{(k+1)}\frac{z^{k+1}}{z^k}\right| \\ &\leq \lim_{k\to\infty}\left|\frac{(\|A\|+k)(\|-nI\|+k)(\|(1+n)I\|+k)}{(k-\|I\|)(k-\|B\|)(k+1)}\right||z| = |z|, \end{aligned} \tag{2.6}$$

where

$$\|U_k\| \leq \frac{(\|A\|)_k(\|-nI\|)_k(\|(1+n)I\|)_k\alpha_1(k)\alpha_2(k)}{k!}.$$

The last limit shows that, thus, the power series (2.1) is absolutely convergent for $|z| < 1$ and divergent for $|z| > 1$. The Rice's matrix polynomials is absolutely convergent for $|z| = 1$ when $m(B) + m(I) > M(-nI) + M((1+n)I) + M(A)$ where $M(-nI)$, $M((1+n)I)$, $M(A)$, $m(I)$ and $m(B)$ as defined in (1.1).

Now, we begin the study of this function by calculating it radius of convergence R. For this purpose, we recall relation of [24], by (1.9), then

$$\begin{aligned}
\frac{1}{R} &= \limsup_{k\to\infty} (\|U_k\|)^{\frac{1}{k}} \\
&= \limsup_{k\to\infty} \left(\left\| \frac{(-nI)_k((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}}{k!} \right\|\right)^{\frac{1}{k}} \\
&= \limsup_{k\to\infty} \left[\left\| \frac{k^{-A}(A)_k}{(k-1)!}(k-1)!k^A \right.\right. \\
&\quad \frac{k^{nI}(-nI)_k}{(k-1)!}(k-1)!k^{-nI}\frac{k^{-(1+n)I}((1+n)I)_k}{(k-1)!}(k-1)!k^{(1+n)I} \\
&\quad \left.\left. \frac{k^{-I}}{(k-1)!}(k-1)![(I)_k]^{-1}k^I \frac{k^{-B}}{(k-1)!}(k-1)![(B)_k]^{-1}k^B \frac{1}{k!} \right\|\right]^{\frac{1}{k}} \\
&= \limsup_{k\to\infty} \left[\left\| \left(\Gamma^{-1}(A)\Gamma^{-1}(-nI)\Gamma^{-1}((1+n)I)\Gamma(I)\Gamma(B)\right) k^A k^{-B}\frac{(k-1)!}{k!} \right\|\right]^{\frac{1}{k}} \\
&\le \limsup_{k\to\infty} \left[\left\| k^A k^{-B}\frac{1}{k} \right\|\right]^{\frac{1}{k}} \le \limsup_{k\to\infty} \left[\frac{\|k^A\|\|k^{-B}\|}{k}\right]^{\frac{1}{k}}.
\end{aligned} \tag{2.7}$$

Using relation (8) of [15] for any square complex matrix A of size N, it follows that in the form

$$\| e^{tA} \| \le e^{tM(A)} \sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} t)^j}{j!}; \ t \ge 0, \tag{2.8}$$

and considering that $k^A = e^{A \ln k}$ one gets

$$\| k^A \| \le k^{M(A)} \sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} \ln k)^j}{j!}. \tag{2.9}$$

Substitute from (2.8) and (2.9) into (2.7) one gets

$$\frac{1}{R} \le \limsup_{k\to\infty} \left\{ k^{M(A)} \sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} \ln k)^j}{j!} k^{-m(B)} \sum_{j=0}^{N-1} \frac{\left(\| B \| N^{\frac{1}{2}} \ln k\right)^j}{j!} \frac{1}{k} \right\}^{\frac{1}{k}}. \tag{2.10}$$

Since

$$\sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} \ln k)^j}{j!} \le (N \ln k)^{N-1} \sum_{j=0}^{N-1} \frac{(\| A \|)^j}{j!} = (N \ln k)^{N-1} e^{\|A\|},$$

then

$$\frac{1}{R} \le \limsup_{k\to\infty} \left\{k^{M(A)} k^{-m(B)} (N \ln k)^{N-1} e^{\|A\|} (N \ln k)^{N-1} e^{\|B\|}\right\}^{\frac{1}{k}} = 1,$$

i.e. the radius of convergence of the Rice's matrix polynomials $H_n(A, B, z)$ is one and it is regular in a circle of radius $r = 1$.

2.1 Integral form of the Rice's matrix polynomials

To get an integral form for the Rice's matrix polynomials of complex variable. Suppose that $-nI$, $(n+1)I$, A and I, B are commuting matrices in $\mathbb{C}^{N\times N}$, such that

$$AB = BA, \tag{2.11}$$

and

$$A,\, B \text{ and } B - A \text{ are positive stable matrices.} \tag{2.12}$$

By (1.7) and (2.12) one gets

$$\begin{aligned}(A)_k[(B)_k]^{-1} &= \Gamma(A+kI)\Gamma^{-1}(A)[\Gamma(B+kI)\Gamma^{-1}(B)]^{-1} \\ &= \Gamma^{-1}(A)\Gamma^{-1}(B-A)\Gamma(B-A)\Gamma(A+kI)\Gamma^{-1}(B+kI)\Gamma(B).\end{aligned} \tag{2.13}$$

By Lemma 2 of [14] and (2.12), we see that

$$\begin{aligned}\int_0^1 t^{A+(k-1)I}(1-t)^{B-A-I}dt &= B(A+kI, B-A) \\ &= \Gamma(B-A)\Gamma(A+kI)\Gamma^{-1}(B+kI),\end{aligned} \tag{2.14}$$

and by (2.13) and (2.14) one get

$$(A)_k[(B)_k]^{-1} = \Gamma^{-1}(A)\Gamma^{-1}(B-A)\left(\int_0^1 t^{A+(k-1)I}(1-t)^{B-A-I}dt\right)\Gamma(B), \tag{2.15}$$

where $Re(b-a) > 0$ for all $b-a \in \sigma(B-A)$ and $Re(a+k) > 0$ for all $a+k \in \sigma(A+kI)$.

Hence, by (1.10) and (2.15), one can write

$$\begin{aligned}H_n(A,B,z) &= \sum_{k=0}^{\infty}\frac{(-nI)_k((n+1)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}}{k!}z^k \\ &= \sum_{k=0}^{\infty}\frac{(-nI)_k((n+1)I)_k[(I)_k]^{-1}\Gamma^{-1}(A)\Gamma^{-1}(B-A)\Gamma(B)}{k!} \\ &\quad \left(\int_0^1 t^{A-I}(1-t)^{B-A-I}(tz)^k dt\right) \\ &= \left(\int_0^1 {}_2F_1(-nI,(n+1)I;I;tz)t^{A-I}(1-t)^{B-A-I}dt\right) \\ &\quad \Gamma^{-1}(A)\Gamma^{-1}(B-A)\Gamma(B).\end{aligned} \tag{2.16}$$

This is the integral form of Rice's matrix polynomials.

In the following, we derive several matrix differential recurrence relations, the pure matrix recurrence relations and Rice's matrix differential equations from this Rice's matrix polynomials.

3 The matrix contiguous function relations

Some matrix recurrence relation is carried out on the Rice's matrix polynomials. In this connection the following contiguous functions relations follow, directly by increasing or decreasing one in original relation $A(A+I)_k = (A+kI)(A)_k$ together with the definitions of the matrix contiguous functions relations follow, yield the formulas

$$\begin{aligned} H_n(A+, B, z) &= {}_3F_2(-nI, (n+1)I, A+; I, B; z) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!}(-nI)_k((n+1)I)_k(A+I)_k[(I)_k]^{-1}[(B)_k]^{-1} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!}(-nI)_k((n+1)I)_k(A+kI)A^{-1}(A)_k[(I)_k]^{-1}[(B)_k]^{-1} \\ &= \sum_{k=0}^{\infty}(A+kI)A^{-1}U_k(z). \end{aligned} \tag{3.1}$$

Similarly, we have

$$H_n(A-, B, z) = \sum_{k=0}^{\infty}(A-I)(A+(k-1)I)^{-1}U_k(z), \tag{3.2}$$

$$H_n(A, B+, z) = \sum_{k=0}^{\infty} B(B+kI)^{-1}U_k(z), \tag{3.3}$$

and

$$H_n(A, B-, z) = \sum_{k=0}^{\infty}(B+(k-1)I)(B-I)^{-1}U_k(z). \tag{3.4}$$

Using the differential operator $\theta = z\frac{d}{dz}$. Since $\theta z^k = kz^k$, we see that

$$(\theta I + A)H_n = \sum_{k=0}^{\infty}(A+kI)U_k(z). \tag{3.5}$$

Hence, with the aid of (3.1)–(3.5), yield

$$(\theta I + A)H_n = A\, H_n(A+, B, z). \tag{3.6}$$

Similarly, it follows that

$$(\theta I + B - I)H_n = (B-I)\, H_n(A, B-, z). \tag{3.7}$$

The result is the set of simple relations of four:

$$\begin{aligned} (A-B+I)\, H_n &= A\, H_n(A+, B, z) - (B-I)\, H_n(A, B-, z), \\ (A+nI)\, H_n + (A-nI)\, H_{n-1}(A, B, z) &= A\, H_n(A+, B, z) + A\, H_{n-1}(A+, B, z), \\ B[2A-I-(A+B-I)z]\, H_n &= AB(1-z)\, H_n(A+, B, z) + B(A-I)\, H_n(A-, B, z) \\ &\quad + (B+nI)(nI-B+I)z\, H_n(A, B+, z), \\ B\, H_n + B\, H_{n-1}(A, B, z) &= (B+nI)\, H_n(A, B+, z) + (B-nI)\, H_{n-1}(A, B+, z). \end{aligned} \tag{3.8}$$

The $\Xi(\theta)$ differential operator has been defined by Sayyed [24] in the form

$$\Xi(\theta) = 1 + \sum_{k=1}^{N} \theta^k; \quad \theta^k = \theta\, \theta^{k-1}. \tag{3.9}$$

From (3.7), we obtain

$$\theta\, H_n = (B - I)\, [H_n(A, B-, z) - H_n(A, B, z)]\,, \tag{3.10}$$

and

$$\begin{aligned} \theta^2\, H_n &= (B - I)(B - 2I)\, H_n(A, B - 2I, z) \\ &\quad - \left[(B - I)(B - 2I) + (B - I)^2\right] H_n(A, B - I, z) \\ &\quad + (B - I)^2\, H_n(A, B, z). \end{aligned} \tag{3.11}$$

Thus by mathematical induction, we have the following general form

$$\begin{aligned} \Xi(\theta) H_n(A, B, z) &= \left(1 + \sum_{k=1}^{N} \theta^k\right) H_n(A, B, z) \\ &= H_n(A, B, z) + \sum_{k=1}^{N} \Bigg\{ \prod_{j=1}^{k} (B - jI)\, H_n(A, B - jI, z) \\ &\quad - \left[\prod_{j=1}^{k} (B - jI) + \prod_{j=1}^{k-1} (B - jI) \sum_{k=1}^{N-1} (B - jI)\right] H_n(A, B - (j-1)I, z) \\ &\quad + \Bigg[\prod_{j=1}^{k-1} (B - jI) \sum_{j=1}^{k-1} (B - jI) + \prod_{j=1}^{k-2} (B - jI) \left(\sum_{j=1}^{k-2} (B - jI)\right)^2 \\ &\quad + \sum_{j=1}^{k-3} (B - jI)(B - (j+1)I) + \sum_{j=1}^{k-4} (B - jI)(B - (j+1)I) \ldots \Bigg] \\ &\quad H_n(A, B - (j-2)I, z) + \cdots + (-1)^k (B - I)^k H_n(A, B, z) \Bigg\}, \end{aligned} \tag{3.12}$$

where N is a finite positive integer.

Consider the differential operator $\theta = z\frac{d}{dz}$, $\theta z^k = k z^k$, yields that

$$\begin{aligned} &\theta\, I(\theta\, I + I - I)(\theta\, I + B - I) H_n \\ &= \sum_{k=1}^{\infty} \frac{k\, z^k}{k!} (kI + I - I)(kI + B - I)(-nI)_k ((n+1)I)_k \\ &(A)_k [(I)_k]^{-1} [(B)_k]^{-1} \\ &= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} (-nI)_k ((n+1)I)_k (A)_k [(I)_{k-1}]^{-1} [(B)_{k-1}]^{-1}. \end{aligned} \tag{3.13}$$

Now, we replace k by $k+1$ and we have

$$\begin{aligned}\theta\, I(\theta\, I)(\theta\, I+B-I)H_n &= \sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}(-nI)_{k+1}((n+1)I)_{k+1}(A)_{k+1}[(I)_k]^{-1}[(B)_k]^{-1}\\ &= \sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}(-nI+kI)((n+k+1)I)(A+kI)(-nI)_k((n+1)I)_k\\ &\quad (A)_k[(I)_k]^{-1}[(B)_k]^{-1} \qquad (3.14)\\ &= z(\theta\, I-nI)(\theta\, I+(n+1)I)(\theta\, I+A)H_n.\end{aligned}$$

Thus, we have shown that $H_n(A, B, z)$ is a solution of the following matrix differential equation

$$[\theta\, I(\theta\, I)(\theta I + B - I) - z(\theta\, I - n\, I)(\theta\, I + (n+1)I)(\theta I + A)]\, H_n = 0, \qquad (3.15)$$

an equation which may also be written

$$\begin{aligned}&(1-z)z^2\, H_n''' + (B+2I-(A+4I)zI)z\, H_n'' + [B-(2A+(n+2)(1-n)I)z]\\ &\quad H_n' + n(n+1)A\, H_n = 0.\end{aligned}$$

These results are summarized below.

Theorem 3.1 *For each natural number $n > 0$, the Rice's matrix polynomials $H_n(A, B, z)$ satisfies the matrix differential equation*

$$\begin{aligned}&(1-z)z^2\, H_n''' + (B+2I-(A+4I)zI)z\, H_n'' + [B-(2A+(n+2)(1-n)I)z]\\ &\quad H_n' + n(n+1)A\, H_n = 0. \qquad (3.16)\end{aligned}$$

Since the series $H_n(A, B, z)$ is absolutely convergent we can differentiate it term by term. We can find some properties related with differentiation of the Rice's matrix polynomials of complex variables with respect to z in the following forms

$$\begin{aligned}&\frac{d}{dz}H_n\\ &= \sum_{k=1}^{\infty}\frac{(-nI)_k((n+1)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}}{(k-1)!}z^{k-1}\\ &= \sum_{k=0}^{\infty}\frac{(-nI)_{k+1}((n+1)I)_{k+1}(A)_{k+1}[(I)_{k+1}]^{-1}[(B)_{k+1}]^{-1}}{k!}z^{k}\\ &= \sum_{k=0}^{\infty}\frac{(-nI)((n+1)I)A(I)^{-1}(B)^{-1}(-nI+I)_k((n+1)I+I)_k(A+I)_k[(I+I)_k]^{-1}[(B+I)_k]^{-1}}{k!}z^{k}\\ &= (-nI)((n+1)I)A(I)^{-1}(B)^{-1}\, H_{n+1}(A+I, B+I, z), \qquad (3.17)\end{aligned}$$

and in general

$$\frac{d^k}{dz^k}H_n(A, B, z) = (-nI)_k((n+1)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}\, H_{n+k}(A+kI, B+kI, z), \qquad (3.18)$$

for $k = 1, 2, 3, \ldots,$

$$\frac{d^n}{dz^n} H_n(A, B, z) = (-nI)_n((n+1)I)_n(A)_n[(I)_n]^{-1}[(B)_n]^{-1}\, H_{2n}(A+nI, B+nI, z). \tag{3.19}$$

In the next theorem, we obtain another representation for the Rice's matrix polynomials.

Theorem 3.2 *Let A and B be matrices in* $\mathbb{C}^{N\times N}$. *The Rice's matrix polynomials defined by* (2.1) *have the following properties*

$$H_n(A, B, z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!(n-k)!} 2^{2k} \left(\frac{1}{2}I\right)_k (I)_{n+k}(A)_k[(I)_{2k}]^{-1}[(B)_k]^{-1}. \tag{3.20}$$

Proof By (2.1), we can write

$$H_n(A, B, z) = \sum_{k=0}^{\infty} \frac{(-nI)_k((1+n)I)_k z^k}{k!}(A)_k[(I)_k]^{-1}[(B)_k]^{-1}; \quad 0 \le k \le n. \tag{3.21}$$

From the relation (1.8), we get

$$\begin{aligned} H_n(A, B, z) &= \sum_{k=0}^{\infty} \frac{(-1)^k n! z^k}{k!(n-k)!}((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!(n-k)!}(I)_n((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}. \end{aligned}$$

Clearly

$$(I)_n = I(I+I)(I+2I), \ldots, (I+(n-1)I) = (1, 2, 3, \ldots, n)I = n!I.$$

According to (1.7), we find that

$$(I)_{n+k} = (I)_n((1+n)I)_k, \tag{3.22}$$

and using (3.22), we get

$$H_n(A, B, z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!(n-k)!}(I)_{n+k}(A)_k[(I)_k]^{-1}[(B)_k]^{-1}.$$

From (1.7), we see that

$$(I)_{2k} = 2^{2k}\left(\frac{1}{2}I\right)_k (I)_k, \tag{3.23}$$

this gives another representation for Rice's matrix polynomials in the form

$$H_n(A, B, z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!(n-k)!} 2^{2k} \left(\frac{1}{2}I\right)_k (I)_{n+k}(A)_k[(I)_{2k}]^{-1}[(B)_k]^{-1},$$

and the proof of Theorem 3.2 is completed. □

We now give representations a generating matrix function in a series of the Rice's matrix polynomials follow readily from (3.20).

Theorem 3.3 *Let A and B be matrices in* $\mathbb{C}^{N\times N}$. *Then a generating matrix function for Rice's matrix polynomials has the following representation*

$$\sum_{n=0}^{\infty} H_n(A, B, z)t^n = (1-t)^{-I}\,{}_2F_1\left(\frac{1}{2}I, A; B; -\frac{4zt}{(1-t)^2}\right), \tag{3.24}$$

where the hypergeometric matrix function ${}_2F_1(\ldots,\ldots;\ldots;\ldots)$ *is given as*

$${}_2F_1\left(\frac{1}{2}I, A; B; -\frac{4zt}{(1-t)^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{4zt}{(1-t)^2}\right)^k \left(\frac{1}{2}I\right)_k (A)_k[(B)_k]^{-1},$$

such that $B + kI$ *is invertible for all integer* $k \geq 0$ *and for* $|\frac{4zt}{(1-t)^2}| < 1$.

Proof We consider the series

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(A, B, z)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(4z)^k t^n}{k!(n-k)!}\left(\frac{1}{2}I\right)_k (I)_{n+k}(A)_k[(I)_{2k}]^{-1}[(B)_k]^{-1}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(4z)^k t^{n+k}}{k!n!}\left(\frac{1}{2}I\right)_k (I)_{n+2k}(A)_k[(I)_{2k}]^{-1}[(B)_k]^{-1}.\end{aligned}$$

From (1.7) and (1.11), we can write

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(A, B, z)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(4z)^k t^{n+k}}{k!n!}((2k+1)I)_n\left(\frac{1}{2}I\right)_k (A)_k[(B)_k]^{-1}\\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(4z)^k t^k}{k!}\left(\frac{1}{2}I\right)_k (A)_k[(B)_k]^{-1}\sum_{n=0}^{\infty}\frac{t^n}{n!}((2k+1)I)_n\\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(4z)^k t^k}{k!}\left(\frac{1}{2}I\right)_k (A)_k[(B)_k]^{-1}(1-t)^{-(2k+1)I}\\ &= (1-t)^{-I}\,{}_2F_1\left(\frac{1}{2}I, A; B; -\frac{4zt}{(1-t)^2}\right).\end{aligned}$$

We obtain an generating matrix function for the Rice's matrix polynomials in the form

$$\sum_{n=0}^{\infty} H_n(A, B, z)t^n = (1-t)^{-I}\,{}_2F_1\left(\frac{1}{2}I, A; B; -\frac{4zt}{(1-t)^2}\right).$$

Hence the equation (3.24) is established and the proof of Theorem 3.3 is completed. □

Now, we can use the expansion of Rice's matrix polynomials together with their properties to prove the following result.

Theorem 3.4 *Let* A *and* B *be matrices in* $\mathbb{C}^{N\times N}$. *For non-negative integral* n *and expansion of Rice's matrix polynomials*

$$z^n I = (B)_n\left(\frac{1}{2}I\right)_n]^{-1}[(A)_n]^{-1}\sum_{k=0}^{n} \frac{(-1)^k n!}{2^{2n}(n-k)!}((1+2k)I)(I)_{2n}[(I)_{n+k+1}]^{-1}H_k(A, B, z). \tag{3.25}$$

Proof We put that

$$v = -\frac{4t}{(1-t)^2}.$$

Then

$$t = 1 - \frac{2}{1+\sqrt{1-\nu}} = -\frac{\nu}{(1+\sqrt{1-\nu})^2},$$

and (3.24) becomes

$${}_2F_1\left(\frac{1}{2}I, A; B; z\nu\right) = \left(\frac{2}{1+\sqrt{1-\nu}}\right)^I \sum_{k=0}^{\infty} \frac{(-1)^k H_k(A,B,z)\nu^k}{(1+\sqrt{1-\nu})^{2k}}$$

or

$${}_2F_1\left(\frac{1}{2}I, A; B; z\nu\right) = \sum_{k=0}^{\infty} \frac{(-1)^k H_k(A,B,z)\nu^k}{2^{2k}} \left(\frac{2}{1+\sqrt{1-\nu}}\right)^{(2k+1)I},$$

we found that

$$\left(\frac{2}{1+\sqrt{1-\nu}}\right)^{(2\alpha-1)I} = {}_2F_1\left(\alpha I, \left(\alpha - \frac{1}{2}\right) I; 2\alpha I; \nu\right). \tag{3.26}$$

The use of (3.26) with $\alpha = k+1$ leads to

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}I, A; B; z\nu\right) &= \sum_{k=0}^{\infty} {}_2F_1\left((k+1)I, \left(k+\frac{1}{2}\right) I; (2k+2)I; \nu\right) \frac{(-1)^k \nu^k}{2^{2k}} H_k(A,B,z) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \nu^{n+k}}{n! 2^{2k}} ((1+k)I)_n \left(\left(k+\frac{1}{2}\right) I\right)_n [((2k+2)I)_n]^{-1} \\ &\quad H_k(A,B,z), \end{aligned}$$

and using (3.22) and (3.23), we get

$$\begin{aligned} ((2k+1)I)_{2n} &= 2^{2n} \left(\left(k+\frac{1}{2}\right) I\right)_n ((k+1)I)_n, \\ (I)_{2k+1} &= ((2k+1)I)(I)_{2k}, \\ (I)_{2k+n+1} &= (I)_{2k+1}((2k+2)I)_n = ((2k+1)I)(I)_{2k}((2k+2)I)_n, \end{aligned}$$

and hence

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}I, A; B; z\nu\right) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \nu^{n+k}}{2^{2n} n! 2^{2k}} ((1+2k)I)_{2n} [(2k+2)_n]^{-1} H_k(A,B,z) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \nu^{n+k}}{2^{2n} n! 2^{2k}} ((1+2k)I)(I)_{2k}((1+2k)I)_{2n}[(I)_{n+2k+1}]^{-1} \\ &\quad H_k(A,B,z) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \nu^{n+k}}{2^{2n+2k} n!} ((1+2k)I)(I)_{2n+2k}[(I)_{n+2k+1}]^{-1} H_k(A,B,z). \end{aligned}$$

Therefore

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{2}I, A; B; z\nu\right) &= \sum_{n=0}^{\infty} \frac{\nu^n z^n}{n!}\left(\frac{1}{2}I\right)_n (A)_n[(B)_n]^{-1} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \nu^{n+k}}{2^{2n+2k} n!}((1+2k)I)(I)_{2n+2k}[(I)_{n+2k+1}]^{-1} H_k(A, B, z) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k \nu^{n}}{2^{2n}(n-k)!}((1+2k)I)(I)_{2n}[(I)_{n+k+1}]^{-1} H_k(A, B, z),
\end{aligned}
\tag{3.27}
$$

this yield

$$
\frac{z^n}{n!}\left(\frac{1}{2}I\right)_n (A)_n[(B)_n]^{-1} = \sum_{k=0}^{n} \frac{(-1)^k}{2^{2n}(n-k)!}((1+2k)I)(I)_{2n}[(I)_{n+k+1}]^{-1} H_k(A, B, z).
$$

Expanding the left-hand side of (3.27) into powers of ν and identifying the coefficients of ν^n on both sides gives (3.25). Therefore, the expression (3.25) is established and the proof of Theorem 3.4 is completed. □

In the next corollary, we obtain another representation an expansion of the Rice's matrix polynomials as follows.

Corollary 3.1 *For non-negative integral n and another representation expansion of Rice's matrix polynomials the following holds*

$$
z^n I = (B)_n[(A)_n]^{-1} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!}((1+2k)I)(I)_n[(I)_{n+k+1}]^{-1} H_k(A, B, z). \tag{3.28}
$$

Proof Using the Theorem 3.4 and (3.23), we get directly the equation (3.28). The proof of Corollary 3.1 is completed. □

In the following theorem, we obtain the properties of Rice's matrix polynomials as follows.

Theorem 3.5 *Let A and B be matrices in* $\mathbb{C}^{N\times N}$ *and* $B + kI$ *is invertible for every integer* $k \geq 0$*. The Rice's matrix polynomials satisfy the following differential recurrence relations*

$$
zH_n'(A, B, z) + zH_{n-1}'(A, B, z) = n\left[H_n(A, B, z) - H_{n-1}(A, B, z)\right]; \qquad n \geq 1, \tag{3.29}
$$

$$
zH_n'(A, B, z) - nH_n(A, B, z) = -\sum_{k=0}^{n-1}\left[H_k(A, B, z) + 2zH_k'(A, B, z)\right]; \qquad n \geq 1, \tag{3.30}
$$

and

$$
zH_n'(A, B, z) - nH_n(A, B, z) = \sum_{k=0}^{n-1}(-1)^{n-k}(1+2k)H_k(A, B, z); \qquad n \geq 1. \tag{3.31}
$$

Proof In order to derive (3.29)–(3.31), we put

$$
\begin{aligned}
W = W(I, A, B, z, t) &= \sum_{n=0}^{\infty} H_n(A, B, z)t^n = (1-t)^{-I}\, {}_2F_1\left(\frac{1}{2}I, A; B; -\frac{4zt}{(1-t)^2}\right) \\
&= (1-t)^{-I}\, \Psi\left(-\frac{4zt}{(1-t)^2}\right),
\end{aligned}
\tag{3.32}
$$

where

$$\Psi = \Psi\left(-\frac{4zt}{(1-t)^2}\right) = {}_2F_1\left(\frac{1}{2}I, A; B; -\frac{4zt}{(1-t)^2}\right).$$

By differentiating (3.32) with respect to z and t yields respectively

$$\begin{aligned}\frac{\partial W}{\partial z} &= -4t(1-t)^{-3I}\Psi', \\ \frac{\partial W}{\partial t} &= (1-t)^{-2I}\Psi - 4z(1+t)(1-t)^{-4I}\Psi'.\end{aligned} \tag{3.33}$$

Therefore W satisfies the matrix partial differential equation

$$z(1+t)\frac{\partial W}{\partial z} - t(1-t)\frac{\partial W}{\partial t} = -tW. \tag{3.34}$$

Equation (3.34) can be rewritten in the forms

$$z\frac{\partial W}{\partial z} - t\frac{\partial W}{\partial t} = -tW - t^2\frac{\partial W}{\partial t} - zt\frac{\partial W}{\partial z}, \tag{3.35}$$

$$z\frac{\partial W}{\partial z} - t\frac{\partial W}{\partial t} = -\frac{t}{1-t}W - \frac{2zt}{1-t}\frac{\partial W}{\partial z}, \tag{3.36}$$

and

$$z\frac{\partial W}{\partial z} - t\frac{\partial W}{\partial t} = -\frac{t}{1+t}W - \frac{2t^2}{1+t}\frac{\partial W}{\partial t}. \tag{3.37}$$

Since

$$W = \sum_{n=0}^{\infty} H_n(A, B, z)t^n,$$

equation (3.35) yields that

$$\begin{aligned}&\sum_{n=0}^{\infty}[zH_n'(A, B, z) - nH_n(A, B, z)]t^n \\ &\quad = -\sum_{n=0}^{\infty} H_n(A, B, z)t^{n+1} - \sum_{n=0}^{\infty} nH_n(A, B, z)t^{n+1} - \sum_{n=0}^{\infty} zH_n'(A, B, z)t^{n+1} \\ &\quad = -\sum_{n=0}^{\infty} nH_{n-1}(A, B, z)t^n - \sum_{n=0}^{\infty} zH_{n-1}'(A, B, z)t^n,\end{aligned}$$

this leads to (3.29). Equation (3.36) implies that

$$\begin{aligned}&\sum_{n=0}^{\infty}[zH_n'(A, B, z) - nH_n(A, B, z)]t^n \\ &\quad = -\sum_{n=0}^{\infty} t^{n+1}\sum_{k=0}^{\infty} H_k(A, B, z)t^k - 2z\sum_{n=0}^{\infty} t^{n+1}\sum_{k=0}^{\infty} H_k'(A, B, z)t^k \\ &\quad = -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_k(A, B, z)t^{n+k+1} - 2z\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_n'(A, B, z)t^{n+k+1}\end{aligned}$$

$$= -\sum_{n=0}^{\infty}\sum_{k=0}^{n} H_k(A, B, z)t^{n+1} - 2z\sum_{n=0}^{\infty}\sum_{k=0}^{n} H'_k(A, B, z)t^{n+1}$$
$$= -\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} H_k(A, B, z)t^{n} - 2z\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} H'_k(A, B, z)t^{n},$$

which leads to (3.30). From (3.37), we obtain

$$\sum_{n=0}^{\infty}[zH'_n(A, B, z) - nH_n(A, B, z)]t^n$$
$$= -\sum_{n=0}^{\infty}(-1)^n t^{n+1}\sum_{k=0}^{\infty} H_k(A, B, z)t^k - 2\sum_{n=0}^{\infty}(-1)^n t^{n+1}\sum_{k=0}^{\infty} kH_k(A, B, z)t^k$$
$$= -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^n H_k(A, B, z)t^{n+k+1} - 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} k(-1)^n H_k(A, B, z)t^{n+k+1}$$
$$= -\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k} H_k(A, B, z)t^{n+1} - 2\sum_{n=0}^{\infty}\sum_{k=0}^{n} k(-1)^{n-k} H_k(A, B, z)t^{n+1}$$
$$= -\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k-1} H_k(A, B, z)t^{n} - 2\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} k(-1)^{n-k-1} H_k(A, B, z)t^{n}$$
$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k} H_k(A, B, z)t^{n} + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} k(-1)^{n-k} H_k(A, B, z)t^{n}$$
$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k}(1 + 2k) H_k(A, B, z)t^{n},$$

this gives (3.31). Formulas (3.29)–(3.31) are called the recurrence formulas for Rice's matrix polynomials. Thus the proof of Theorem 3.5 is completed. □

Now, we can state and prove the following theorem:

Theorem 3.6 *The Rice's matrix polynomials $H_n(A, B, z)$ defined in* (2.1)*, satisfy the pure matrix recurrence relations*

$$\begin{aligned} &n(2n-3)(B+(n-1)I)H_n - (2n-1)\,[(n-2)(B-(n-1)I) \\ &\quad +2(n-1)(2n-3)I - 2(2n-3)(A+(n-1)I)z]\,H_{n-1} + (2n-3) \\ &\left[2(n-1)^2 I - n(B-(n-1)I) + 2(2n-1)(A-(n-1)I)z\right] H_{n-2} \\ &\quad +(n-2)(2n-1)(B-(n-1)I)H_{n-3} = 0. \end{aligned} \tag{3.38}$$

Proof To simplify the exposition, we let

$$H_n(A, B, z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}(-nI)_k((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1} = \sum_{k=0}^{\infty}U_k(z). \tag{3.39}$$

We list a sequence of the $H_k(A, B, z)$ and $zH_k(A, B, z)$ ($k = n, n-1, \ldots$) in which for clarity, we exhibit each in both its explicit form and also in a form involving $U_k(z)$ as defined above. Thus

$$H_{n-1}(A,B,z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}((-n+1)I)_k(nI)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$$
$$=\sum_{k=0}^{\infty}(n-k)I[(n+k)I]^{-1}U_k(z), \tag{3.40}$$
$$zH_{n-1}(A,B,z)=\sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}((-n+1)I)_k(nI)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$$
$$=\sum_{k=1}^{\infty}\frac{z^k}{(k-1)!}((-n+1)I)_{k-1}(nI)_{k-1}(A)_{k-1}[(I)_{k-1}]^{-1}[(B)_{k-1}]^{-1}$$
$$=\sum_{k=0}^{\infty}-k^2(B+(k-1)I)[(n+k)I]^{-1}[(n+k-1)I]^{-1}[A+(k-1)I]^{-1}U_k(z), \tag{3.41}$$
$$H_{n-2}(A,B,z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}((-n+2)I)_k((n-1)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$$
$$=\sum_{k=0}^{\infty}(n-k)I(n-k-1)I[(n+k)I]^{-1}[(n+k-1)I]^{-1}U_k(z), \tag{3.42}$$
$$zH_{n-2}(A,B,z)=\sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}((-n+2)I)_k((n-1)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$$
$$=\sum_{k=1}^{\infty}\frac{z^k}{(k-1)!}((-n+2)I)_{k-1}((n-1)I)_{k-1}(A)_{k-1}[(I)_{k-1}]^{-1}[(B)_{k-1}]^{-1}$$
$$=\sum_{k=0}^{\infty}-k^2(B+(k-1)I)(n-k)(n-k-2)I[(n+k)I]^{-1}[(n+k-1)I]^{-1}$$
$$[(n+k-2)I]^{-1}[A+(k-1)I]^{-1}U_k(z), \tag{3.43}$$

and

$$H_{n-3}(A,B,z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}((-n+3)I)_k((n-2)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}$$
$$=\sum_{k=0}^{\infty}(n-k)I(n-k-1)I(n-k-2)I[(n+k)I]^{-1}[(n+k-1)I]^{-1}$$
$$[(n+k-2)I]^{-1}U_k(z). \tag{3.44}$$

If the coefficients of $U_k(z)$ in the above series are written with a least common denominator, and the numerators are polynomials of degree at most four in k, then a linear combination of the six series would have as numerator of its coefficient of $U_k(z)$ a polynomial of degree four in k. Such a linear combination would leave five undetermined constants with which to make the five coefficients in the fourth degree polynomial vanish. This suggests that for $n \geq 3$ there exists a linear recurrence relation of the form

$$H_n(A,B,z)+(C_1+C_2\,z)H_{n-1}(A,B,z)+(C_3+C_4\,z)H_{n-2}(A,B,z)$$
$$+C_5H_{n-3}(A,B,z)=0 \tag{3.45}$$

in which C_1, C_2, C_3, C_4 and C_5 are rational functions in n and are independent of z. To determine the coefficients in (3.45) we replace the several $H_k(A, B, z)$ and $zH_k(A, B, z)$ by their respective forms where the factor $U_k(z)$ occurs explicitly under the summation sign. Equating coefficients of $U_k(z)$, we have for $k \geq 0$

$$\begin{aligned} I &+ C_1(n-k)I[(n+k)I]^{-1} + C_2\left[-k^2(B+(k-1)I)[(n+k)I]^{-1}[(n+k-1)I]^{-1}\right.\\ &\left.[A+(k-1)I]^{-1}\right]\\ &+C_3\left[(n-k)I(n-k-1)I[(n+k)I]^{-1}[(n+k-1)I]^{-1}\right]\\ &+C_4\left[-k^2(B+(k-1)I)(n-k)I(n-k-2)I[(n+k)I]^{-1}[(n+k-1)I]^{-1}\right.\\ &\quad\left.[(n+k-2)I]^{-1}[A+(k-1)I]^{-1}\right]\\ &+C_5\left[(n-k)I(n-k-1)I(n-k-2)I[(n+k)I]^{-1}[(n+k-1)I]^{-1}\right.\\ &\quad\left.[(n+k-2)I]^{-1}\right]=0. \end{aligned} \tag{3.46}$$

Clearing the above expression of fractions gives the following identity in k:

$$\begin{aligned} &(n+k)I(n+k-1)I(n+k-2)I(A+(k-1)I)\\ &\quad+C_1(n-k)I(n+k-1)I(n+k-2)I(A+(k-1)I)\\ &\quad+C_2\left[-k^2(B+(k-1)I)(n+k-2)I\right]\\ &\quad+C_3\left[(n-k)I(n-k-1)I(n+k-2)I(A+(k-1)I)\right]\\ &\quad+C_4\left[-k^2(B+(k-1)I)(n-k)I(n-k-2)I\right]\\ &\quad+C_5\left[(n-k)I(n-k-1)I(n-k-2)I(A+(k-1)I)\right]=0. \end{aligned} \tag{3.47}$$

From this identity we can readily determine the coefficients in (3.45). Substituting the values thus obtained and clearing the result of fractions, we get, for $n \geq 3$, the four-term recurrence relation:

$$\begin{aligned} &nI(2n-3)I(B+(n-1)I)H_n-(2n-1)I\left[(n-2)I(B-(n-1)I)\right.\\ &\quad\left.+2(n-1)I(2n-3)I-2(2n-3)I(A+(n-1)I)z\right]H_{n-1}+(2n-3)I\\ &\left[2(n-1)^2I-n(B-(n-1)I)+2(2n-1)I(A-(n-1)I)z\right]H_{n-2}\\ &\quad+(n-2)I(2n-1)I(B-(n-1)I)H_{n-3}=0. \end{aligned} \tag{3.48}$$

In the following, we introduce to define of composite Rice's matrix polynomials and the radius of convergence is obtained. □

4 Composite Rice's matrix polynomials

Let us introduce the following notation [27]

$$\begin{aligned} \underline{k} &= (k_1, k_2, \ldots, k_i),\\ (\underline{k}) &= k_1 + k_2 + \cdots + k_i,\\ (\underline{k})! &= k_1!k_2!\ldots k_i!,\\ \underline{z}^{\underline{k}} &= z_1^{k_1} z_2^{k_2}\ldots z_i^{k_i},\\ \underline{A} &= (A_1, A_2, \ldots, A_i),\\ \underline{B} &= (B_1, B_2, \ldots, B_i),\\ (\underline{A})_{\underline{k}} &= (A_1)_{k_1}(A_2)_{k_2}\ldots(A_i)_{k_i},\\ (\underline{B})_{\underline{k}} &= (B_1)_{k_1}(B_2)_{k_2}\ldots(B_i)_{k_i}, \end{aligned}$$

and

$$\underline{H} = (H_1, H_2, \ldots, H_i).$$

Suppose that

$$H_l(A_l, B_l, z_l) = \sum_{k_l \geq 0} \frac{z_l^{k_l}}{(k_l)!}(-n_l I)_{k_l}((n_l+1)I)_{k_l}(A_l)_{k_l}[(I_l)_{k_l}]^{-1}[(B_l)_{k_l}]^{-1},\ l = 1, 2, \ldots, i, \tag{4.1}$$

are i Rice's matrix polynomials with square complex matrices $A_1, A_2, \ldots, A_i$ and B_1, B_2 ,..., B_i of the same order N.

Construct the Rice's matrix polynomials $\underline{H}_{\underline{n}}(\underline{A}, \underline{B}, \underline{z})$ defined by

$$\begin{aligned} \underline{H}_{\underline{n}}(\underline{A}, \underline{B}, \underline{z}) &= {}_3F_2(-\underline{n}I, \underline{(n+1)}I, \underline{A}; \underline{I}, \underline{B}; \underline{z}) \\ &= \sum_{\underline{k} \geq 0} \frac{\underline{z}^{\underline{k}}}{(\underline{k})!}(-\underline{n}I)_{\underline{k}}(\underline{(1+n)}I)_{\underline{k}}(\underline{A})_{\underline{k}}[(\underline{I})_{\underline{k}}]^{-1}[(\underline{B})_{\underline{k}}]^{-1}. \end{aligned} \tag{4.2}$$

This function, will be called the composite Rice's matrix polynomials of several complex variables $z_1, z_2, \ldots, z_i$.

We begin the study of this function by calculating its radius of convergence R. For this purpose, we recall relation (1.3.10) of [24] and keeping in mind that $\sigma_{\underline{k}} \geq 1$. Hence

$$\begin{aligned}
\frac{1}{R} &= \limsup_{(\underline{k}) \to \infty} \left(\frac{\| U_{\underline{k}} \|}{\sigma_{\underline{k}}} \right)^{\frac{1}{(\underline{k})}} \\
&= \limsup_{(\underline{n}) \to \infty} \left(\frac{\| (-\underline{n}I)_{\underline{k}}(\underline{(1+n)}I)_{\underline{k}}(\underline{A})_{\underline{k}}[(\underline{I})_{\underline{k}}]^{-1}[(\underline{B})_{\underline{k}}]^{-1} \|}{(\underline{k})!} \right)^{\frac{1}{(\underline{k})}} \left(\frac{1}{\sigma_{\underline{k}}} \right)^{\frac{1}{(\underline{k})}} \\
&\leq \limsup_{(\underline{k}) \to \infty} \left(\frac{\| ((-nI)_1)_{k_1}, \ldots, ((-nI)_i)_{k_i}(((n+1)I)_1)_{k_1}, \ldots, (((n+1)I)_i)_{k_i}(A_1)_{k_1}, \ldots, (A_i)_{k_i} \|}{k_1! k_2!, \ldots, k_i!} \right)^{\frac{1}{(\underline{k})}} \\
&\quad \left([(I_1)_{k_1}]^{-1}, \ldots, [(I_i)_{k_i}]^{-1}[(B_1)_{k_1}]^{-1}, \ldots, [(B_i)_{k_i}]^{-1} \right)^{\frac{1}{(\underline{k})}} \\
&\leq \limsup_{(\underline{k}) \to \infty} \Bigg(\Bigg\| \left(\frac{k_1^{-(-nI)_1}((-nI)_1)_{k_1}}{(k_1-1)!} \right)(k_1-1)! k_1^{(-nI)_1}, \ldots, \left(\frac{k_i^{-(-nI)_i}((-nI)_i)_{k_i}}{(k_i-1)!} \right)(k_i-1)! k_i^{(-nI)_i} \\
&\quad \left(\frac{k_1^{-((n+1)I)_1}(((n+1)I)_1)_{k_1}}{(k_1-1)!} \right)(k_1-1)! k_1^{((n+1)I)_1}, \ldots, \left(\frac{k_i^{-((n+1)I)_i}(((n+1)I)_i)_{k_i}}{(k_i-1)!} \right)(k_i-1)! k_i^{((n+1)I)_i} \\
&\quad \left(\frac{k_1^{-A_1}(A_1)_{k_1}}{(k_1-1)!} \right)(k_1-1)! k_1^{A_1}, \ldots, \left(\frac{k_i^{-A_i}(A_i)_{k_i}}{(k_i-1)!} \right)(k_i-1)! k_i^{A_i} \\
&\quad \frac{k_1^{I_1}}{(k_1-1)!}(k_1-1)![(I_1)_{k_1}]^{-1} k_1^{-I_1}, \ldots, \frac{k_i^{I_i}}{(k_i-1)!}(k_i-1)![(I_i)_{k_i}]^{-1} k_i^{-I_i} \\
&\quad \frac{k_1^{B_1}}{(k_1-1)!}(k_1-1)![(B_1)_{k_1}]^{-1} k_1^{-B_1}, \ldots, \frac{k_i^{B_i}}{(k_i-1)!}(k_i-1)![(B_i)_{k_i}]^{-1} k_i^{-B_i} \Bigg\| \\
&\quad \frac{1}{k_1! k_2!, \ldots, k_i!} \Bigg)^{\frac{1}{(\underline{k})}} \\
&\leq \limsup_{(\underline{k}) \to \infty} \left(\| \Gamma^{-1}((-nI)_1) \| \cdots \| \Gamma^{-1}((-nI)_i) \| \| \| \Gamma^{-1}(((n+1)I)_1) \| \cdots \| \Gamma^{-1}(((n+1)I)_i) \| \right) \\
&\quad \left(\| \Gamma^{-1}(A_1) \| \cdots \| \Gamma^{-1}(A_i) \| \| \| \Gamma(I_1) \| \cdots \| \Gamma(I_i) \| \| \| \Gamma(B_1) \| \cdots \| \Gamma(B_i) \| \right)^{\frac{1}{(\underline{k})}}
\end{aligned}$$

$$\limsup_{(\underline{k})\to\infty}\left(\frac{\| k_1^{(-nI)_1} \| \cdots \| k_i^{(-nI)_i} \|\| k_1^{((n+1)I)_1} \| \cdots \| k_i^{((n+1)I)_i} \|\| k_1^{A_1} \| \cdots \| k_i^{A_i} \|}{k_1 k_2, \ldots, k_i}\right)^{\frac{1}{(\underline{k})}}$$
$$\left(\| k_1^{-I_1} \| \cdots \| k_i^{-I_i} \|\| k_1^{-B_1} \| \cdots \| k_i^{-B_i} \|\right)^{\frac{1}{(\underline{k})}}, \tag{4.3}$$

where

$$\sigma_{\underline{k}} = \begin{cases} \left(\frac{k_1+\cdots+k_i}{k_1}\right)^{\frac{k_1}{2}} \left(\frac{k_1+\cdots+k_i}{k_2}\right)^{\frac{k_2}{2}}, \ldots, \left(\frac{k_1+\cdots+k_i}{k_i}\right)^{\frac{k_i}{2}}, & \underline{k} \neq 0 \\ 1, & \underline{k} = 0. \end{cases}$$

For positive numbers μ_l and positive integer k, we can write

$$k_l = \mu_l k, \quad l = 1, 2, \ldots, i.$$

Substitute from (2.8) and (2.9) into (4.3) one gets

$$\frac{1}{R} \le \limsup_{k(\mu_1+\cdots+\mu_i)\to\infty}$$
$$\times \left\{ (\mu_1 k)^{M((-nI)_1)} \sum_{j=0}^{N-1} \frac{(\| (-nI)_1 \| N^{\frac{1}{2}} \ln \mu_1 k)^j}{j!}, \ldots, (\mu_i k)^{M((-nI)_i)} \sum_{j=0}^{N-1} \frac{(\| (-nI)_i \| N^{\frac{1}{2}} \ln \mu_i k)^j}{j!} \right.$$
$$(\mu_1 k)^{M(((n+1)I)_1)} \sum_{j=0}^{N-1} \frac{(\| ((n+1)I)_1 \| N^{\frac{1}{2}} \ln \mu_1 k)^j}{j!}, \ldots, (\mu_i k)^{M(((n+1)I)_i)} \sum_{j=0}^{N-1} \frac{(\| ((n+1)I)_i \| N^{\frac{1}{2}} \ln \mu_i k)^j}{j!}$$
$$(\mu_1 k)^{M(A_1)} \sum_{j=0}^{N-1} \frac{(\| A_1 \| N^{\frac{1}{2}} \ln \mu_1 k)^j}{j!}, \ldots, (\mu_i k)^{M(A_i)} \sum_{j=0}^{N-1} \frac{(\| A_i \| N^{\frac{1}{2}} \ln \mu_i k)^j}{j!}$$
$$(\mu_1 k)^{-m(I_1)} \sum_{j=0}^{N-1} \frac{(\| I_1 \| N^{\frac{1}{2}} \ln \mu_1 k)^j}{j!}, \ldots, (\mu_i k)^{-m(I_i)} \sum_{j=0}^{N-1} \frac{(\| I_i \| N^{\frac{1}{2}} \ln \mu_i k)^j}{j!}$$
$$(\mu_1 k)^{-m(B_1)} \sum_{j=0}^{N-1} \frac{(\| B_1 \| N^{\frac{1}{2}} \ln \mu_1 k)^j}{j!}, \ldots, (\mu_i k)^{-m(B_i)} \sum_{j=0}^{N-1} \frac{(\| B_i \| N^{\frac{1}{2}} \ln \mu_i k)^j}{j!}$$
$$\left. \frac{1}{(\mu_1 k)!, \ldots, (\mu_i k)!} \right\}^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}. \tag{4.4}$$

Since

$$\sum_{j=0}^{N-1} \frac{(\| A_i \| N^{\frac{1}{2}} \ln \mu_i k)^j}{j!} \le (N \ln \mu_i k)^{N-1} \sum_{j=0}^{N-1} \frac{(\| A_i \|)^j}{j!} = (N \ln \mu_i k)^{N-1} e^{\| A_i \|},$$

then

$$\frac{1}{R} \le \limsup_{k(\mu_1+\cdots+\mu_i)\to\infty} \{k^{M((-nI)_1)+\cdots+M((-nI)_i)} k^{M(((n+1)I)_1)+\cdots+M(((n+1)I)_i)} k^{M(A_1)+\cdots+M(A_i)}\}^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}$$
$$\limsup_{k(\mu_1+\cdots+\mu_i)\to\infty} \{k^{-m(I_1)-\cdots-m(I_i)} k^{-m(B_1)-\cdots-m(B_i)}\}^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}$$
$$\limsup_{k(\mu_1+\cdots+\mu_i)\to\infty} \left((N \ln \mu_1 k)^{N-1} e^{\|(-nI)_1\|} \ldots (N \ln \mu_i k)^{N-1} e^{\|(-nI)_i\|}\right)^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}$$
$$\limsup_{k(\mu_1+\cdots+\mu_i)\to\infty} \left((N \ln \mu_1 k)^{N-1} e^{\|((n+1)I)_1\|} \ldots (N \ln \mu_i k)^{N-1} e^{\|((n+1)I)_i\|}\right)^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}$$
$$\limsup_{k(\mu_1+\cdots+\mu_i)\to\infty} \left((N \ln \mu_1 k)^{N-1} e^{\|A_1\|} \ldots (N \ln \mu_i k)^{N-1} e^{\|A_i\|}\right)^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}$$

$$\limsup_{k(\mu_1+\cdots+\mu_i)\to\infty}\left((N\ln\mu_1 k)^{N-1}e^{\|I_1\|}\dots(N\ln\mu_i k)^{N-1}e^{\|I_i\|}\right)^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}$$
$$\limsup_{k(\mu_1+\cdots+\mu_i)\to\infty}\left((N\ln\mu_1 k)^{N-1}e^{\|B_1\|}\dots(N\ln\mu_i k)^{N-1}e^{\|B_i\|}\right)^{\frac{1}{k(\mu_1+\cdots+\mu_i)}}=1,$$

i.e. the radius of convergence of the composite Rice's matrix polynomials is one and it is regular in the sphere $\bar{S}_R$; $R=1$ (c.f. [24]). There are many way of investigating of Rice's matrix polynomials and give some properties of the results. Starting from the modified forms of the definition of Rice's and composite Rice's matrix polynomials is one of these direct methods and clearly some directions to develop more researches and studies in that area. The results of this paper are original, variant, significant and so it is interesting and capable to develop its study in the future.

5 Open problem

One can use the same class of new integral representation, operational methods, families of multilinear and multilateral generating functions and orthogonality property for the multivariable extension of the generalized Rice's matrix polynomials. Hence, new results and further applications can be obtained. Further results and applications will be discussed in a forthcoming paper.

Acknowledgments (1) The Author expresses his sincere appreciation to Dr. Mahmoud Tawfik Mohamed, (Department of Mathematics and Science, Faculty of Education(New Valley), Assiut University, New Valley, EL-Kharga 72111, Egypt) for his kind interest, encouragements, help, suggestions, comments and the investigations for this series of papers. (2) The author would like to thank the referees for their valuable comments and suggestions which have led to the better presentation of the paper.

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