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Stability analysis for systems of nonlinear Hill's equations

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Abstract

Systems of nonlinear differential equations with periodic coefficients, which include Hill's and Mathieu's equations as examples in the linear limit, are important from a practical point of view. Nonlinear Hill's equations model a variety of dynamical systems of interest to physics and engineering, in which perturbations enter as periodic modulations of their linear frequencies. As is well known, the stability properties of some fundamental periodic solutions of these systems is often an essential problem. The main purpose of this paper is to concentrate on one such class of nonlinear Hill's equations and study the stability properties of some of their simplest periodic solutions analytically as well as numerically. To accomplish this task, we first use an extension of the generalized averaging method to approximate these solutions and then apply the technique of multiple scaling to perform the stability analysis. A three-particle system with free–free boundary conditions is studied as an example. The accuracy of our results is tested, within the limits of first-order perturbation theory, and is found to be well confirmed by numerical experiments. The stability analysis of these simple periodic solutions, though local in itself, can yield considerable information about more global properties of the dynamics, since it is in the vicinity of such solutions that the largest regions of regular or chaotic motion are usually observed, depending on whether the periodic solution is, respectively, stable or unstable.

Keywords: Hill's equation; Periodic solution; Stable; Unstable; Nonlinear

1. Introduction

Hill's equation [1] is any linear, second-order ordinary differential equation of the form

$$\ddot{x} + [\omega^2 - \lambda f(t)]x = 0\,, \tag{1.1}$$

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where ω^2 and λ are real constants and $f(t)$ is a bounded real-valued periodic function of period π. Eq. (1.1) with $f(t) = \cos 2t$ is called Mathieu's equation. If nonlinear terms in x and/or $\dot{x}$ are present in (1.1), it is called a nonlinear Hill's equation.

In recent years, there have been great advances in our understanding of low-dimensional nonlinear dynamical systems. Thus, an outstanding challenge for current and future research is the analysis of higher-dimensional dynamical systems (or systems with many degrees of freedom). Such systems are known to exhibit regular and chaotic behavior at various levels, but their solutions are, of course, more difficult to analyze globally. An important task in elucidating the properties of such dynamical systems is finding some of their fundamental stable and unstable periodic solutions, and studying the motion in their vicinity.

Recently, Mahmoud [2] studied from this point of view two coupled nonlinear Hill's equations using perturbative techniques. In the present paper, our aim is to extend this analysis to the periodic solutions of n coupled nonlinear differential equations of general Hill's type:

$$\ddot{x}_i + \omega_i^2 x_i + \varepsilon\phi_i(\Omega_i t) f_i(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \varepsilon g_i(v_i t) r_i(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 0 , \tag{1.2}$$

$i=1,2,\ldots,n$, $\omega_i > 0$, where $\boldsymbol{x} \equiv (x_1, x_2, \ldots, x_n)$, $\dot{\boldsymbol{x}} \equiv (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$ are the phase space coordinates, f_i and r_i are nonlinear functions of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}, \phi_i, g_i$ are periodic functions of t with frequencies $v_i \cong \omega_i$, ϕ_i has period π, ε is a small positive parameter, and dots represent as usual differentiation with respect to t.

Systems of nonlinear Hill's equations model a variety of dynamical problems of interest to physics and engineering. In particular, they arise in the study of electrohydrodynamic stability of a fluid layer between two cylindrical interfaces [3,4], two-(or higher-) mode responses of simply supported rectangular laminated plates [5], the forced response of a beam with three-mode interaction [6], robots, shells, arches and elastic pendulums with many degrees of freedom [7,8] and the problem of beam stability in particle accelerators [2,9–11]. Also, the study of autonomous systems with many degrees of freedom can sometimes be reduced to equations of type (1.2), as e.g. the Hamiltonian system:

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}(x^2 + y^2 + z^2) + x^2(y^2 + z^2)$$

which possesses the exact periodic solution $x = 0$, $y = A\cos(t + \delta_1)$, $z = B\cos(t + \delta_2)$, where A, B, δ_1, δ_2 are constants. Variations about this solution yield a system of equations of type (1.2) (after proper rescaling of the variables).

The stability analysis of certain fundamental periodic solutions of (1.2) with frequency $\omega = p/m$ (p and m are positive integers, $T = 2\pi/\omega$) is interesting in its own right. It can also yield, however, considerable information about more global properties of the motion, since when these solutions are stable (unstable) they are surrounded by some of the largest regular (chaotic) regions of the system in phase space.

Special cases of this class of systems have been the subject of many investigations [2,11–18]. In previous studies of coupled Hill's equations [3,4,19], researchers have used some transformations into new coordinates which decouple the set of Hill's equations. In more complicated cases, however, where such decoupling is not possible, different techniques must be implemented.

In Section 2, the stability of certain fundamental periodic solutions of Eq. (1.2) is discussed. In Section 3, we demonstrate the techniques described in Section 2 on a system of coupled nonlinear Hill's oscillators of the form

$$\ddot{x}_1 + (\omega_1^2 + \varepsilon\cos 2t)x_1 - \beta x_1^3 = \gamma(x_2 - x_1)\,,$$

$$\ddot{x}_2 + (\omega_2^2 + \varepsilon\cos 2t)x_2 - \beta x_2^3 = \gamma(x_3 - x_2) + \gamma_1(x_1 - x_2)\,,$$

$$\ddot{x}_3 + (\omega_3^2 + \varepsilon\cos 2t)x_3 - \beta x_3^3 = \gamma(x_2 - x_3) \tag{1.3}$$

with $\beta = O(\varepsilon)$ and $\gamma = O(\varepsilon)$, positive, small constants and ε a small positive parameter. This is a system of three linearly interacting particles with free–free boundary conditions derived from the Hamiltonian

$$H = \sum_{i=1}^{3}\left\{\frac{\dot{x}_i^2}{2} + \frac{1}{2}(\omega_i^2 + \varepsilon\cos 2t)x_i^2 - \frac{\beta x_i^4}{4}\right\} + \frac{1}{2}\gamma(x_1 - x_2)^2 + \frac{1}{2}\gamma(x_2 - x_3)^2\,.$$

A typical 2π-periodic solution of this system is first obtained to $O(\varepsilon)$, using the method of generalized averaging, in the case $\omega_i = 1$ $(i = 1,2,3)$, for $\varepsilon = 0$. Then, the stability regions for this solution are obtained analytically as well as numerically in the ω_i^2, ε-planes and good agreement is found for small values of ε. Our analytical treatment is based on the method of multiple time scales, while in our numerical study we compute the time evolution of phase space orbits starting near our 2π-periodic solution.

Our concluding remarks are presented in Section 4. Currently, we are generalizing this analysis to systems of strongly nonlinear Hill's equations and will report on our results in a future publication [20].

2. Stability of periodic solutions

In this section, we study the stability of periodic solutions with period 2π for a system of nonlinear Hill's equations of the form

$$\ddot{x}_i + \omega_i^2 x_i + \varepsilon\phi_i(\Omega_i t)f_i(\boldsymbol{x},\dot{\boldsymbol{x}}) + \varepsilon g_i(v_i t)r_i(\boldsymbol{x},\dot{\boldsymbol{x}}) = 0, \quad i = 1,2,3,\ldots,n\,. \tag{2.1}$$

At $\varepsilon = 0$, the particular solutions of these equations, $x_i(t)$, are periodic with periods $T_i = 2\pi/\omega_i$, where $\omega_i = n_i$, n_i being positive integers, with $n_n = 1$.

To investigate these periodic solutions for $\varepsilon \neq 0$, we first apply an extension of the generalized averaging method [21] to Eq. (2.1) to get approximate expressions for the solution $\hat{\boldsymbol{x}}(t) = (\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t))$, and the perturbed frequencies ω_i^2, valid to first order in ε. Then, to solve the problem of stability, small variations $u_i(t)$ from this periodic state are considered substituting $\hat{x}_i(t) + u_i(t)$ in place of $x_i(t)$ in Eq. (2.1). Upon neglecting powers of $u_i(t)$ beyond the first, a linear variational equation is thus obtained:

$$\ddot{u}_i + \omega_i^2 u_i + \varepsilon\phi_i(\Omega_i t)f_{il}(\hat{\boldsymbol{x}} + \boldsymbol{u}, \dot{\hat{\boldsymbol{x}}} + \dot{\boldsymbol{u}}) + \varepsilon g_i(v_i t)r_{il}(\hat{\boldsymbol{x}} + \boldsymbol{u}, \dot{\hat{\boldsymbol{x}}} + \dot{\boldsymbol{u}}) = 0\,, \tag{2.2}$$

where $\boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ and f_{il} and r_{il} are the linear parts of f_i and g_i in $u_1, u_2, \ldots, u_n$.

The behavior of $u_i(t)$ with time determines the stability of the corresponding component $\hat{x}_i(t)$: if all components $u_i(t)$ in Eq. (2.2) either vanish or are oscillatory and bounded as t tends to infinity, $\hat{x}_i(t)$ is defined as stable; if $u_i(t)$ grows exponentially as $t \to \infty$, $\hat{x}_i(t)$ is defined as unstable [12,22,23]. The periodic solution $\boldsymbol{x}(t)$ is stable if all the periodic components $\hat{x}_i(t)$ are stable, otherwise the periodic solution is unstable.

Let us consider solutions of (2.2) with ω_i^2 which can be expanded in powers of ε:

$$\omega_i^2 = n_i^2 + \varepsilon a_{i1} + \varepsilon^2 a_{i2} + \cdots . \tag{2.3}$$

Using the standard "multiple-scaling" techniques of perturbation theory [2,8] we also write the solution of (2.2) as a series expansion in ε:

$$u_i(t, \varepsilon) = u_{i0}(t, \tilde{t}) + \varepsilon u_{i1}(t, \tilde{t}) + \cdots , \tag{2.4}$$

where

$$\varepsilon t \equiv \tilde{t} , \tag{2.5}$$

is the "slow" time variable, for ε, sufficiently small.

Substituting then (2.4) into (2.2), using (2.3) and equating like powers of ε yields an infinite hierarchy of equations at order ε^j as follows:

$$\frac{\partial^2 u_{i_0}}{\partial t^2} + n_i^2 u_{i0} = 0, \quad j = 0 , \tag{2.6}$$

$$\frac{\partial^2 u_{i_1}}{\partial t^2} + n_i^2 u_{i1} = d_i(a_{i1}, t, \tilde{t}, u_{i0}, \hat{\boldsymbol{x}}), \quad j = 1 , \tag{2.7}$$

etc. for the u_{ij}, $j = 2, 3, \ldots$.

The general solutions of Eqs. (2.6) are clearly

$$u_{i0}(t, \tilde{t}) = A_{i0}(\tilde{t}) \cos n_i t + B_{i0}(\tilde{t}) \sin n_i t . \tag{2.8}$$

Solving (2.7) for u_{i1} and eliminating terms that produce secular terms in Eq. (2.7) gives a system of first-order differential equations for $A_{i0}(\tilde{t})$ and $B_{i0}(\tilde{t})$. Studying the solutions of this system (with regard to their growth or decay as $\tilde{t} \to \infty$) as a function of the system parameters, yields the boundaries of the stability regions a_{i1} in the ω_i^2, ε-planes to first order in ε. In the next section, we apply this approach to an illustrative example of physical interest.

3. An example of three coupled oscillators

To illustrate the technique of Section 2, we apply it here to the following system of three linearly coupled oscillators with free–free boundary conditions of the form

$$\ddot{x}_1 + (\omega_1^2 + \varepsilon \cos 2t) x_1 - \varepsilon \beta x_1^3 = \varepsilon \gamma (x_2 - x_1) ,$$

$$\ddot{x}_2 + (\omega_2^2 + \varepsilon\cos 2t)x_2 - \varepsilon\beta x_2^3 = \varepsilon\gamma(x_3 - x_2) + \varepsilon\gamma(x_1 - x_2)\,,$$

$$\ddot{x}_3 + (\omega_3^2 + \varepsilon\cos 2t)x_3 - \varepsilon\beta x_3^3 = \varepsilon\gamma(x_2 - x_3), \quad 0 < \varepsilon \ll 1\,. \tag{3.1}$$

Using an extension of the generalized averaging method [11,21,24], we first obtain approximate analytical solutions of Eqs. (3.1) for the case $\omega_i \cong 1$, $(i = 1,2,3)$, which turn out to be (see the appendix):

$$\begin{aligned}\hat{x}_1(t) = {} & A_1\cos(t + \Psi_1) + \varepsilon\{\tfrac{1}{16}\beta A_1^3[3\cos(t + \Psi_1) - \tfrac{1}{2}\cos(3t + 3\Psi_1)] \\ & + \tfrac{1}{16}A_1[\cos(3t + \Psi_1) - 2\cos(t - \Psi_1)] - \tfrac{1}{32}\gamma A_1\cos(t + \Psi_1) \\ & + \tfrac{1}{4}\gamma A_2\cos(t + \Psi_2)\} + O(\varepsilon^2)\,,\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\hat{x}_2(t) = {} & A_2\cos(t + \Psi_2) + \varepsilon\{\tfrac{1}{16}\beta A_2^3[3\cos(t + \Psi_2) - \tfrac{1}{2}\cos(3t + 3\Psi_2) \\ & + \tfrac{1}{16}A_2[\cos(3t + \Psi_2) - 2\cos(t - \Psi_2)] - \tfrac{1}{32}\gamma A_2\cos(t + \Psi_2) \\ & + \tfrac{1}{4}\gamma A_3\cos(t + \Psi_3) + \tfrac{1}{4}\gamma A_1\cos(3t + \Psi_1 + 2\Psi_2) \\ & - \tfrac{1}{4}\gamma A_2\cos(3t + 3\Psi_2)\} + O(\varepsilon^2)\,,\end{aligned} \tag{3.2b}$$

$$\begin{aligned}\hat{x}_3(t) = {} & A_3\cos(t + \Psi_3) + \varepsilon\{\tfrac{1}{16}\beta A_3^3[3\cos(t - \Psi_3) - \tfrac{1}{2}\cos(3t + 3\Psi_3)] \\ & + \tfrac{1}{16}A_3[\cos(3t + \Psi_3) - 2\cos(t - \Psi_3)] - \tfrac{1}{32}\gamma A_3\cos(t + \Psi_3) \\ & + \tfrac{1}{4}\gamma A_2\cos(t + \Psi_2)\} + O(\varepsilon^2)\,,\end{aligned} \tag{3.2c}$$

where A_i and Ψ_i are constants determined by the initial conditions.

In accordance with Eqs. (2.2), one then obtains from (3.1), the variational equations

$$\ddot{u}_1 + \omega_1^2 u_1 - 3\varepsilon\beta\hat{x}_1^2 u_1 + \varepsilon u_1\cos 2t + \varepsilon\gamma(u_1 - u_2) = 0\,, \tag{3.3a}$$

$$\ddot{u}_2 + \omega_2^2 u_2 - 3\varepsilon\beta\hat{x}_2^2 u_2 + \varepsilon u_2\cos 2t + \varepsilon\gamma(u_2 - u_3) + \varepsilon\gamma(u_2 - u_1) = 0\,, \tag{3.3b}$$

$$\ddot{u}_3 + \omega_3^2 u_3 - 3\varepsilon\beta\hat{x}_3^2 u_3 + \varepsilon u_3\cos 2t + \varepsilon\gamma(u_3 - u_2) = 0\,, \tag{3.3c}$$

where the $\hat{x}_i(t)$ are given by (3.2a,b,c), for $i=1,2,3$, respectively. We consider solutions of Eqs. (3.1) which oscillate with frequencies given in the form (2.3). We shall study here only the case $n_i=1$, $i=1,2,3$ (other values of n_i can be similarly treated). Writing thus the solution of (3.3) also as a series expansion in ε, in the form (2.4) with (2.5), we find that Eqs. (2.6) and (2.7) yield:

$$\frac{\partial^2 u_{i0}}{\partial t^2} + u_{i0} = 0, \quad i = 1,2,3 \tag{3.4}$$

$$\begin{aligned}\frac{\partial^2 u_{11}}{\partial t^2} + u_{11} = {} & -2\frac{\partial^2 u_{10}}{\partial t\partial\tilde{t}} - a_{11}u_{10} - u_{10}\cos 2t \\ & -\gamma(u_{10} - u_{20}) + 3\beta u_{10}A_1^2\cos^2(t + \Psi_1)\,,\end{aligned} \tag{3.5}$$

$$\frac{\partial^2 u_{21}}{\partial t^2} + u_{21} = -2\frac{\partial^2 u_{20}}{\partial t \partial \tilde{t}} - a_{21}u_{20} - u_{20}\cos 2t$$

$$-\gamma(u_{20} - u_{30}) - \gamma(u_{20} - u_{10}) + 3\beta u_{20}A_2^2\cos^2(t + \Psi_2)\,, \qquad (3.6)$$

$$\frac{\partial^2 u_{31}}{\partial t^2} + u_{31} = -2\frac{\partial^2 u_{30}}{\partial t \partial \tilde{t}} - a_{31}u_{30} - u_{30}\cos 2t$$

$$-\gamma(u_{30} - u_{20}) + 3\beta u_{30}A_3^2\cos^2(t + \Psi_3)\,. \qquad (3.7)$$

The general solutions of Eqs. (3.4) can be expressed in the form

$$u_{i0}(t,\tilde{t}) = A_{i0}(\tilde{t})\cos t + B_{i0}(\tilde{t})\sin t, \quad i = 1,2,3\,, \qquad (3.8)$$

where $A_{i0}(\tilde{t})$ and $B_{i0}(\tilde{t})$ are as yet undetermined functions of the slow variable $\tilde{t}$. Inserting now (3.8) into the right-hand side of Eqs. (3.5)–(3.7) and requiring (for uniformly valid solutions in time) that secular terms proportional to $\sin t$ and $\cos t$ vanish, leads to the following system for the $A_{i0}(\tilde{t})$ and $B_{i0}(\tilde{t})$:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\begin{bmatrix} A_{10}(\tilde{t}) \\ B_{10}(\tilde{t}) \\ A_{20}(\tilde{t}) \\ B_{20}(\tilde{t}) \\ A_{30}(\tilde{t}) \\ B_{30}(\tilde{t}) \end{bmatrix} = \begin{bmatrix} -m_1 & m_2 & 0 & \frac{\gamma}{2} & 0 & 0 \\ m_3 & m_1 & -\frac{\gamma}{2} & 0 & 0 & 0 \\ 0 & \frac{\gamma}{2} & -m_4 & m_5 & 0 & \frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & m_7 & m_4 & -\frac{\gamma}{2} & 0 \\ 0 & 0 & 0 & \frac{\gamma}{2} & -m_6 & m_8 \\ 0 & 0 & -\frac{\gamma}{2} & 0 & m_9 & m_6 \end{bmatrix} \begin{bmatrix} A_{10} \\ B_{10} \\ A_{20} \\ B_{20} \\ A_{30} \\ B_{30} \end{bmatrix}, \qquad (3.9a)$$

where

$$m_1 = \tfrac{3}{8}\beta A_1^2 \sin 2\Psi_1\,,$$

$$m_2 = -\tfrac{1}{2}a_{11} - \tfrac{3}{8}\beta A_1^2\cos 2\Psi_1 + \tfrac{3}{4}\beta A_1^2 + \tfrac{1}{4} - \tfrac{\gamma}{2}\,,$$

$$m_3 = \tfrac{1}{2}a_{11} - \tfrac{3}{8}\beta A_1^2\cos 2\Psi_1 - \tfrac{3}{4}\beta A_1^2 + \tfrac{1}{4} + \tfrac{\gamma}{2}\,,$$

$$m_4 = \tfrac{3}{8}\beta A_2^2 \sin 2\Psi_2\,,$$

$$m_5 = -\tfrac{1}{2}a_{21} - \tfrac{3}{8}\beta A_2^2\cos 2\Psi_2 + \tfrac{3}{4}\beta A_2^2 + \tfrac{1}{4} - \gamma\,,$$

$$m_6 = \tfrac{3}{8}\beta A_3^2 \sin 2\Psi_3\,,$$

$$m_7 = \tfrac{1}{2}a_{21} - \tfrac{3}{8}\beta A_2^2\cos 2\Psi_2 - \tfrac{3}{4}\beta A_2^2 + \tfrac{1}{4} + \gamma\,,$$

$$m_8 = -\tfrac{1}{2}a_{31} - \tfrac{3}{8}\beta A_3^2\cos 2\Psi_3 + \tfrac{3}{4}\beta A_3^2 + \tfrac{1}{4} - \tfrac{\gamma}{2}\,,$$

$$m_9 = \tfrac{1}{2}a_{31} - \tfrac{3}{8}\beta A_3^2\cos 2\Psi_3 - \tfrac{3}{4}\beta A_3^2 + \tfrac{1}{4} + \tfrac{\gamma}{2}\,. \qquad (3.9b)$$

Looking for exponential solutions $\sim e^{\lambda\tilde{t}}$, we obtain the eigenvalues of the above matrix in (3.9a) from the characteristic equation:

$$\lambda^6 + L_1\lambda^4 + L_2\lambda^2 + L_3 = 0\,, \tag{3.10a}$$

where

$$L_1 = \gamma^2 - m_9m_8 - m_5m_7 - m_2m_3\,,$$

$$L_2 = m_5m_7m_8m_9 - m_5m_8\frac{\gamma^2}{4} - m_7m_9\frac{\gamma^2}{4} + \frac{\gamma^4}{8} - \frac{\gamma^2}{4}m_8^2 + m_2m_3m_8m_9 - \frac{\gamma^2}{4}m_2m_3 + m_2m_3m_5m_7 - \frac{\gamma^2}{4}m_2m_3 - \frac{\gamma^2}{4}m_2m_5 - \frac{\gamma^2}{4}m_3m_7 - \frac{\gamma^2}{4}m_8m_9\,,$$

$$L_3 = -m_2m_3m_5m_7m_8m_9 + \frac{\gamma^2}{4}m_2m_3m_5m_8 + \frac{\gamma^2}{4}m_2m_3m_7m_9 - \frac{\gamma^4}{16}m_2m_3 + \frac{\gamma^2}{4}m_2m_8m_9m_5 - \frac{\gamma^4}{16}m_2m_9 + \frac{\gamma^2}{4}m_3m_7m_8m_9 - \frac{\gamma^4}{16}m_3m_8 + \frac{\gamma^4}{16}m_8m_9\,.$$

Clearly, bifurcations from oscillatory to exponentially growing solutions occur at $\lambda^2=0$, whence

$$L_3 = 0\,. \tag{3.11}$$

Eq. (3.11) yields the following conditions for the boundaries of the stability regions in the planes $(\omega_i^2, \varepsilon)$:

$$\text{left:}\quad a_{i1} = -\tfrac{3}{4}\beta A_1^2\cos 2\Psi_1 + \tfrac{3}{2}\beta A_1^2 + \tfrac{1}{2} - \delta\,,$$

$$\text{right:}\quad a_{i1} = \tfrac{3}{4}\beta A_1^2\cos 2\Psi_1 + \tfrac{3}{2}\beta A_1^2 - \tfrac{1}{2} - \delta\,, \tag{3.12}$$

whence Eq. (2.3), for our choice $n_i^2 = 1$, gives to first order in ε

$$\text{right:}\quad \omega_i^2 = 1 + \varepsilon(\tfrac{3}{4}\beta A_i^2\cos 2\Psi_i + \tfrac{3}{2}\beta A_i^2 - \tfrac{1}{2} - \delta) + O(\varepsilon^2)\,,$$

$$\text{left:}\quad \omega_i^2 = 1 - \varepsilon(\tfrac{3}{4}\beta A_i^2\cos 2\Psi_i - \tfrac{3}{2}\beta A_i^2 - \tfrac{1}{2} + \delta) + O(\varepsilon^2), \tag{3.13}$$

with $i=1,2,3$, $\delta=\gamma$ for $i=1,3$ and $\delta=2\gamma$ for $i=2$. Expressions (3.13) for $\beta=0.2$, $\gamma=0.1$ are plotted as dashed curves in Fig. 1a for $\Psi_1 = 0$ and $A_1 = 0.3$, in Fig. 1b for $\Psi_2 = 0$ and $A_2 = 0.4$ and in Fig. 1c for $\Psi_3 = 0$ and $A_3 = 0.5$.

These stability regions have also been computed numerically (see solid curves in Figs. 1a–c) and good agreement has been found, comparing with the analytical results for small values of $\varepsilon \leqslant 0.1$. The numerical investigation of the stability of our solutions in the six-dimensional phase space has been carried out by computing the phase space distance

$$\mathrm{d}(t) = [(x_{11} - x_{12})^2 + (\dot{x}_{11} - \dot{x}_{12})^2 + (x_{21} - x_{22})^2 + (\dot{x}_{21} - \dot{x}_{22})^2 + (x_{31} - x_{32})^2 + (\dot{x}_{31} - \dot{x}_{32})^2]^{1/2} \tag{3.14}$$

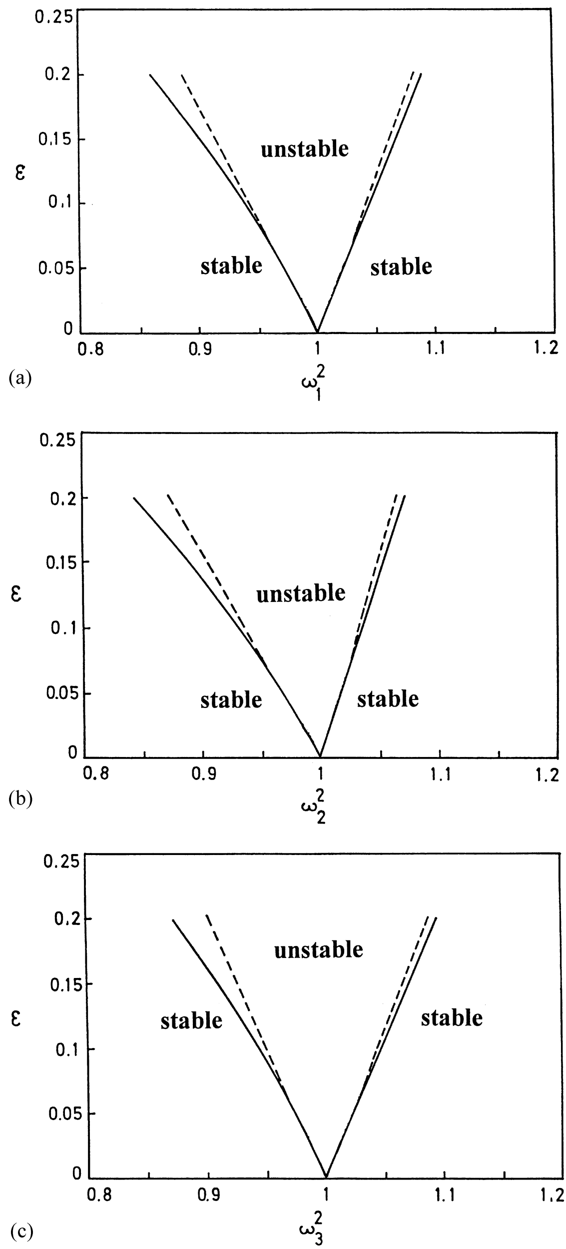
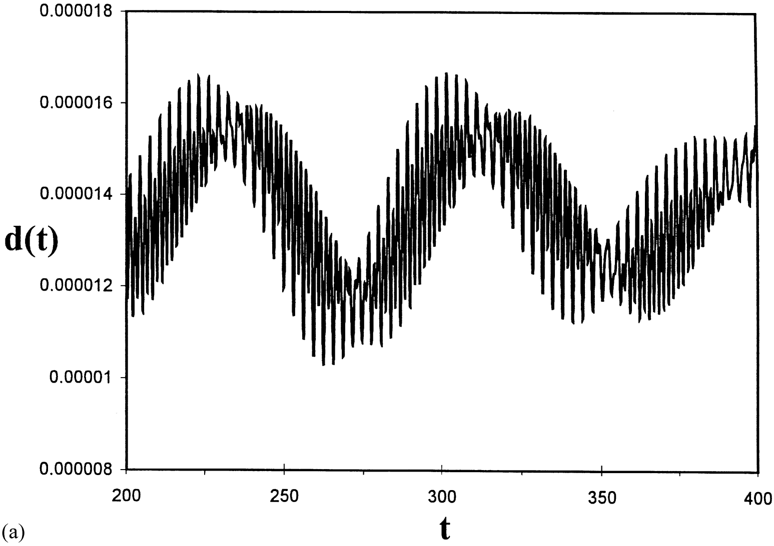
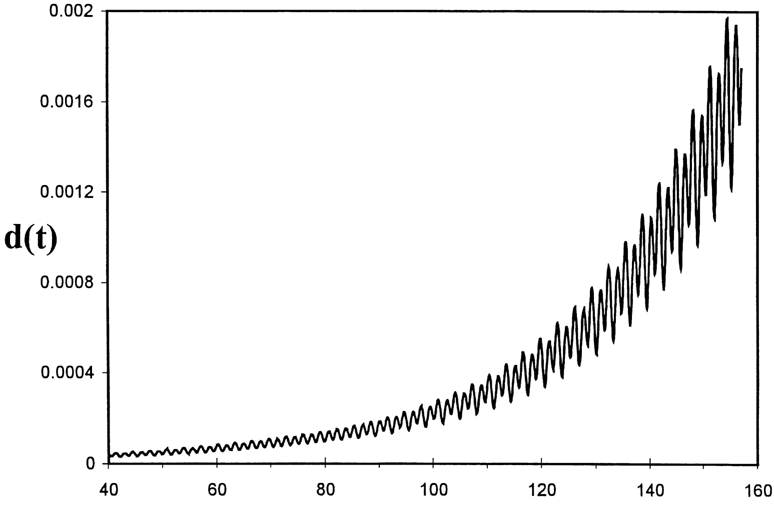


Fig. 1. Stability regions for the 2π-periodic solution of Eq. (3.1) for $\gamma = 0.1$, $\beta = 0.2$ and (a) $\Psi_1 = 0$ and $A_1 = 0.3$ $(\omega_1^2, \varepsilon)$ plane from Eq. (3.13), (b) $\Psi_2 = 0$ and $A_2 = 0.4$ $(\omega_2^2, \varepsilon)$ plane from Eq. (3.13), (c) $\Psi_3 = 0$ and $A_3 = 0.5$ $(\omega_3^2, \varepsilon)$ plane from Eq. (3.13).



(a)



(b)

Fig. 2. Stability results for the 2π-periodic solution of Eq. (3.1) (a) $\varepsilon = 0.1$, $\gamma = 0.1$, $\beta = 0.2$, $t_0 = 0$, $x_1(0) = A_1 = 0.3$, $x_2(0) = A_2 = 0.4$, $x_3(0) = A_3 = 0.5$, $\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$, $\omega_1^2 = 1.1$, $\omega_2^2 = 1.09$ and $\omega_3^2 = 0.9$ (stable); (b) Same as in (a) with $\varepsilon = 0.15$, $\omega_1^2 = 0.98$, $\omega_2^2 = 1.005$ and $\omega_3^2 = 0.97$ (unstable).

between two initially nearby trajectories

$$(x_{1j}, \dot{x}_{1j}, x_{2j}, \dot{x}_{2j}, x_{3j}, \dot{x}_{3j}), \quad j = 1, 2.$$

In the case of Fig. 2a, we have chosen the values

$$\omega_1^2 = 1.1, \quad \omega_2^2 = 1.09, \quad \omega_3^2 = 0.9, \quad \varepsilon = 0.1 \tag{3.15a}$$

which lie inside the stability regions of Fig. 1. Plotting $d(t)$ versus t for a special choice of parameter values, we observe that our 2π-periodic solution is stable, since $d(t)$ is seen to oscillate and remain bounded for the full time interval of integration (see Fig. 2a). On the other hand, if we choose

$$\omega_1^2 = 0.98, \quad \omega_2^2 = 1.005, \quad \omega_3^2 = 0.97, \quad \varepsilon = 0.15 \tag{3.15b}$$

within the instability domain in Fig. 1 we observe that our 2π-periodic solution is unstable, since $d(t)$ is seen to grow on the average exponentially as t increases (see Fig. 2b).

4. Concluding remarks

In this paper, we have investigated analytically as well as numerically the stability of certain simple periodic solutions of systems of coupled nonlinear Hill's equations. These systems can model a variety of higher-dimensional dynamical systems of interest to physics and engineering. They are known, in many cases, to exhibit regular and chaotic behavior, but their general solutions are quite difficult to study analytically.

The stability analysis of simple periodic solutions is important on the other hand, because it is usually around such solutions that the largest regions of regular motion occur, when the solution is stable, while large scale chaotic behavior is observed, when it is unstable. A three-particle Hamiltonian system with free–free boundary conditions was studied as an example. Our analytical results are tested numerically and very good agreement is found for small values of ε.

The results of this paper can be viewed as a generalization of recent work reported in Ref. [2]. Currently we are extending this analysis to systems of strongly nonlinear Hill's equations and results are expected to appear in a future publication [20].

Appendix

In this appendix, we outline the derivation of the solution of Eq. (3.1), which is used to calculate its stability regions. Consider Eq. (3.1):

$$\ddot{x}_1 + (\omega_1^2 + \varepsilon \cos 2t)x_1 - \varepsilon\beta x_1^3 = \varepsilon\gamma(x_2 - x_1)\,, \tag{A.1a}$$

$$\ddot{x}_2 + (\omega_2^2 + \varepsilon \cos 2t)x_2 - \varepsilon\beta x_2^3 = \varepsilon\gamma(x_3 - x_2) + \varepsilon\gamma(x_1 - x_2)\,, \tag{A.1b}$$

$$\ddot{x}_3 + (\omega_3^2 + \varepsilon \cos 2t)x_3 - \varepsilon\beta x_3^3 = \varepsilon\gamma(x_2 - x_3)\,. \tag{A.1c}$$

When $\varepsilon = 0$, the general solutions of (A.1) are:

$$x_i = a_i \cos(\omega_i t + \Psi_i), \quad \dot{x}_i = -a_i\omega_i \sin(\omega_i t + \Psi_i), \qquad i = 1,2,3\,, \tag{A.2}$$

where a_i and Ψ_i are constants determined by the initial conditions. However, for $\varepsilon \neq 0$ "small", we let a_i and Ψ_i be unknown functions of time t in (A.2) and proceed to determine them by an extension of the method of generalized averaging [11,21,24].

We differentiate x_1 and equate with $\dot{x}_1$ in (A.2) with $i=1$ and then differentiate $\dot{x}_1$ and substitute in (A.1a) using (A.2). Doing the same for x_2 and x_3 we get a system of equations, which we solve for $\dot{a}_i(t)$ and $\dot{\psi}_i(t)$, $i=1,2,3$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \end{bmatrix} = \frac{-\varepsilon}{8}\begin{bmatrix} M_1 a_1^3 \\ M_2 a_2^3 \\ M_3 a_3^3 \\ M_4 a_1^2 \\ M_5 a_2^2 \\ M_6 a_3^2 \end{bmatrix} + \frac{\varepsilon}{4}\begin{bmatrix} N_1 a_1 \\ (N_2+T_1)a_2 \\ N_3 a_3 \\ N_4 \\ N_5 + T_2 \\ N_6 \end{bmatrix} + \frac{\varepsilon}{2}\begin{bmatrix} S_1 a_2 \\ S_2 a_3 + T_3 a_1 \\ S_3 a_2 \\ \frac{a_2}{a_1} S_4 \\ S_5 \frac{a_3}{a_2} + T_4 \frac{a_1}{a_2} \\ S_6 \frac{a_2}{a_3} \end{bmatrix}, \tag{A.3a}$$

where

$$M_i = \frac{\beta}{\omega_i}[\sin 4\phi_i + 2\sin 2\phi_i], \quad M_{i+3} = \frac{\beta}{\omega_i}[3 + \cos 4\phi_i + 4\sin 2\phi_i],$$

$$N_i = \frac{1}{\omega_i}[\sin(2\phi_i \pm 2t)] + \frac{2\gamma}{\omega_i}\sin 2\phi_i,$$

$$N_{i+3} = \frac{1}{\omega_i}[2\cos 2t + \cos(2\phi_i \pm 2t)] + \frac{\gamma}{\omega_i}[1 + \cos 2\phi_i],$$

$$T_1 = \frac{-2\gamma}{\omega_2}\sin 2\phi_2, \quad T_2 = \frac{2\gamma}{\omega_2}(1 + \cos 2\phi_2),$$

$$T_3 = \frac{-\gamma}{\omega_2}\sin(\phi_2 \pm \phi_1), \quad T_4 = \frac{-2\gamma}{\omega_2}\cos(\phi_1 \pm \phi_2),$$

$$S_j = \frac{-\gamma}{\omega_j}\sin(\phi_j \pm \phi_{j+1}), \quad S_3 = -S_2\frac{\omega_2}{\omega_3},$$

$$S_{j+3} = \frac{-\gamma}{\omega_j}\cos(\phi_j \pm \phi_{j+1}), \quad S_6 = S_5\frac{\omega_2}{\omega_3},$$

and

$$\phi_i = \omega_i t + \psi_i(t), \quad i = 1,2,3, \ j = 1,2. \tag{A.3b}$$

The nonlinear system (A.3a) with (A.3b) can now be solved to find $a_i(t)$ and $\psi_i(t)$ as follows:

Casting the original system (A.3) in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon f(y,t), \quad y = [a_1, a_2, a_3, \psi_1, \psi_2, \psi_3]^t \tag{A.4}$$

($[\dots]^t$ denoting transpose), we split the space D of functions f into subspaces $\bar{D}$ and $\tilde{D}$, where $\bar{D}$ contains the constant and periodic functions with the smallest frequencies and $\tilde{D}$ the rest of the functions.

To system (A.4) we thus associate the following reduced system

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \varepsilon \bar{F}(v,t)\,, \tag{A.5}$$

by using the transformation

$$y = v + \varepsilon \tilde{G}(v,t)\,. \tag{A.6}$$

Writing

$$f(y,t) = \bar{f}(y,t) + \tilde{f}(y,t)\,, \tag{A.7}$$

we insert (A.6) into (A.4) using (A.5) and (A.7) and get

$$\bar{F} + \varepsilon \bar{F}\frac{\partial \tilde{G}}{\partial v} + \frac{\partial \tilde{G}}{\partial t} = \bar{f}(v+\varepsilon \tilde{G},t) + \tilde{f}(v+\varepsilon \tilde{G},t)\,. \tag{A.8}$$

Separating the terms in $\bar{D}$ and $\tilde{D}$ we obtain

$$\bar{F} = \bar{f}(v+\varepsilon \tilde{G},t)\,, \tag{A.9}$$

$$\frac{\partial \tilde{G}}{\partial t} = \tilde{f}(v+\varepsilon \tilde{G},t) - \varepsilon \bar{f}\frac{\partial \tilde{G}}{\partial v}\,. \tag{A.10}$$

Expanding $\bar{F}$ and $\tilde{G}$ in power series of ε, we have

$$\bar{F}(v,t) = \bar{F}_1 + \varepsilon \bar{F}_2 + \cdots\,, \tag{A.11a}$$

$$\tilde{G}(v,t) = \tilde{G}_1 + \varepsilon \tilde{G}_2 + \cdots\,. \tag{A.11b}$$

Substituting (A.11) into (A.9), (A.10) and expanding in Taylor series around the point (v,t) we obtain, upon equating like powers of ε:

$$\bar{H}(v,t) = \bar{f}(v,t), \ldots\,, \tag{A.12a}$$

$$\tilde{G}_1 = \int \tilde{f}(v,t)\,\mathrm{d}t, \ldots\,. \tag{A.12b}$$

The solution of system (A.4) is also written as an expansion in powers of ε:

$$y = v + \varepsilon \tilde{G}_1(v,t) + \cdots\,, \tag{A.13}$$

while the solution of (A.3) is obtained from (A.13) as

$$a_i(t) = A_i + \varepsilon \tilde{G}_{1A_i} + \cdots\,, \tag{A.14a}$$

$$\psi_i(t) = \Psi_i + \varepsilon \tilde{G}_{1\Psi_i} + \cdots\,, \tag{A.14b}$$

where $\tilde{G}_1 = [G_{1A_i,G_1\psi_i}]^t$, $i=1,2,3$, and A_i and Ψ_i are the initial values of $a_i(t)$ and $\psi_i(t)$, respectively. By substituting from (A.14) into (A.2) we get the approximate solution of Eq. (A.1).

Choosing now $\omega_i \cong 1$, the terms of higher frequency $\tilde{f}$ are

$$\tilde{f}(v,t) = -\frac{1}{8}\begin{bmatrix} \tilde{M}_1 A_1^3 \\ \tilde{M}_2 A_2^3 \\ \tilde{M}_3 A_3^3 \\ \tilde{M}_4 A_1^2 \\ \tilde{M}_5 A_2^2 \\ \tilde{M}_6 A_3^2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \tilde{N}_1 A_1 \\ (\tilde{N}_2 + \tilde{T}_1) A_2 \\ \tilde{N}_3 A_3 \\ \tilde{N}_4 \\ \tilde{N}_5 \\ \tilde{N}_6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \tilde{S}_1 A_1 \\ \tilde{S}_2 A_3 + \tilde{T}_3 A_1 \\ \tilde{S}_3 A_2 \\ \tilde{S}_4 \frac{A_2}{A_1} \\ \tilde{S}_5 \frac{A_3}{A_2} + \tilde{T}_4 \frac{A_1}{A_2} \\ \tilde{S}_6 \frac{A_2}{A_3} \end{bmatrix}, \quad \text{(A.15a)}$$

where

$$\tilde{M}_1 = \beta[\sin 4\Phi_i + 2\sin 2\Phi_i], \quad \tilde{M}_{i+3} = \beta[\cos 4\Phi_i + 4\cos 2\Phi_i]\,,$$

$$\tilde{N}_1 = \sin(2\Phi_i + 2t) + 2\gamma \sin 2\Phi_i, \quad \tilde{N}_{i+3} = 2\cos 2t + \cos(2\Phi_i + 2t) + \gamma\cos 2\Phi_i\,,$$

$$\tilde{T}_1 = -2\gamma \sin 2\Phi_2, \quad \tilde{T}_2 = 2\gamma\cos 2\Phi_2, \quad \tilde{T}_3 = -\gamma\sin(\Phi_2 + \Phi_1)\,,$$

$$\tilde{T}_4 = -\gamma\cos(\Phi_1 + \Phi_2), \quad \tilde{S}_j = -\gamma\sin(\Phi_j + \Phi_{j+1})\,,$$

$$\tilde{S}_3 = -\tilde{S}_2, \quad \tilde{S}_{j+3} = -\gamma\cos(\Phi_j + \Phi_{j+1})\,,$$

$$\tilde{S}_6 = \tilde{S}_5, \quad \Phi_i = t + \Psi_i, \quad i = 1,2,3, \; j = 1,2\,. \quad \text{(A.15b)}$$

Inserting finally (A.15) into (A.12b) we get $\tilde{G}_1$ and using (A.14) we obtain $a_i(t)$ and $\psi_i(t)$. By inserting $a_i(t)$, $\psi_i(t)$ into (A.2) we get the approximate solution $\hat{x}_i$, which are given in Eq. (3.2) in Section 3.

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