ON AUTONOMOUS AND NONAUTONOMOUS MODIFIED HYPERCHAOTIC COMPLEX LÜ SYSTEMS

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In this paper autonomous and nonautonomous modified hyperchaotic complex Lü systems are proposed. Our systems have been generated by using state feedback and complex periodic forcing. The basic properties of these systems are studied. Parameters range for hyperchaotic attractors to exist are calculated. These systems have very rich dynamics in a wide range of parameters. The analytical results are tested numerically and excellent agreement is found. A circuit diagram is designed for one of these hyperchaotic complex systems and simulated using Matlab/Simulink to verify the hyperchaotic behavior.

Keywords: Hyperchaotic attractors; chaotic; complex; fixed points; stability.

1. Introduction

Over the last 30 years, hyperchaotic systems involving both real and complex variables have been proposed and studied in the literature. These systems appeared in many important fields of physics, engineering and computer sciences, such as laser physics, control, flow dynamics and liquid mixing, electronic circuits, secure communications and information sciences ([Barbara & Silvano, 2002; Cang *et al.*, 2010; Chen *et al.*, 2006; Dadras & Momeni, 2010; Grassi & Mascolo, 1999; Hsieh *et al.*, 1999; Li *et al.*, 2004; Li *et al.*, 2005a; Li *et al.*, 2005b; Mahmoud *et al.*, 2007a, 2008a, 2009a; Mahmoud *et al.*, 2008b; Matsumoto *et al.*, 1986; Rössler, 1979; Wang *et al.*, 2006; Wang, 2010]; and references therein).

In 1982, Gibbon and McGuinnes studied and stated the real and complex Lorenz equations in rotating fluids and laser [Gibbon & McGuinnes, 1982]. While, in 1983, Fowler *et al.* introduced complex Lorenz equations and their relevance to physical systems [Fowler *et al.*, 1983]. Zeghlache and Mandel [1985] proposed complex nonlinear equations for detuned lasers. The complex character of the state variables and parameters follows from purely physical considerations [Rauh *et al.*, 1996; Mahmoud *et al.*, 2009b]. Complex state variables (or quantities) are found in equations of problems in laser physics and thermal convection of liquid flows where the electric field and atomic polarization amplitudes are complex quantities, see [Mahmoud *et al.*, 2007b; Mahmoud & Bountis, 2004;

Rauh *et al.*, 1996]. The real and imaginary parts of these variables can display chaotic and hyperchaotic dynamics. Mahmoud *et al.* [2007b] introduced complex Chen and Lü systems and studied their dynamics. The complex Lorenz, Chen and Lü systems do not exhibit hyperchaotic dynamics. Therefore, one wishes to propose complex systems that display hyperchaotic behaviors. Special cases of complex systems have been studied in the recent literature. A system with more than one positive Lyapunov exponent is called a hyperchaotic system. The dynamics of hyperchaotic systems are complicated and rich in the sense that they exhibit chaotic and hyperchaotic behaviors as well as periodic and quasi-periodic solutions for wide and narrow ranges of system parameters. Chaotic behavior and chaos control for a class of complex partial differential equations have been studied in our work [Mahmoud *et al.*, 2001].

In 2006, Wang *et al.* proposed a new modified Lü system as:

$$
\begin{aligned}
\dot{x} &= a(y - x + yz), \\
\dot{y} &= -xz + by + u, \\
\dot{z} &= xy - cz, \\
\dot{u} &= -kx,
\end{aligned}
\tag{1}
$$

where $(x, y, z, u) \in \mathrm{R}^4$, a, b, c, k are constant parameters. This system satisfies the two basic requirements to create hyperchaotic behavior [Rössler, 1979] which are: (i) System (1) is a 4-dimensional (4D) quadratic autonomous system, (ii) the number of terms in the coupled equations giving rise to instability should be at least two, of which at least one term should be a nonlinear function. An electronic circuit is designed for verification of different hyperchaotic attractors of system (1).

In the present paper, based on system (1) we propose a modified hyperchaotic complex Lü system as:

$$
\begin{aligned}
\dot{x} &= a(y - x + yz), \\
\dot{y} &= -xz + by + u, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{u} &= -\frac{k}{2}(\bar{x} + x),
\end{aligned}
\tag{2}
$$

where $x, y \in \mathrm{C}^2$ are complex variables, $z, u \in \mathrm{R}^2$ are real variables, dots represent derivatives with respect to time, overbar denotes complex conjugate

variables, a, b, c and k real (or complex) positive constant parameters. This system is a 6D real first order autonomous ordinary differential equations and has two positive Lyapunov exponents as we calculated in Sec. 2. The basic properties of (2) are studied analytically as well as numerically. The hyperchaotic behavior of our system (2) is verified by physically electronic circuit. As it is cleared from system (2), the control variable u is added to the second equation, so, we can add to it the different equations of (2) to get different forms of modified hyperchaotic complex Lü systems. To create hyperchaotic behavior, we can use another technique which is the complex periodic control signal instead of using state feedback control. There are some options to add the control signal to system variables. For example, we propose the nonautonomous modified hyperchaotic complex Lü systems as follows:

$$
\begin{aligned}
\dot{x} &= a(y - x + yz), \\
\dot{y} &= -xz + by + k\exp(i\omega t), \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz,
\end{aligned}
\tag{3}
$$

where ω is a positive parameter and k is a control parameter. Our system (3) is a 5D complex nonautonomous system. The second technique is not used in [Wang *et al.*, 2006] to generate hyperchaotic attractors. These proposed modified hyperchaotic complex Lü systems may have appeared in several applications in physics, engineering and information systems. The dynamics of these systems is rich and complicated.

This paper is organized as follows: in the next section, symmetry, invariance, fixed points of (2) and stability analysis of the trivial fixed point are discussed. In the third section, the complex behavior of (2) is studied. Numerically the range of parameter values of the system at which hyperchaotic attractors exist is calculated based on the Lyapunov exponents. The signs of Lyapunov exponents provide a good classification of the dynamics of our system. In Sec. 4, we study different forms of autonomous modified hyperchaotic complex Lü systems by adding state feedback controller. In Sec. 5, we study the dynamics of nonautonomous system (3). Section 6 contains the electronic circuit which is designed for our new system (2). A block diagram of this system using Matlab/Simulink is constructed. Numerical and simulation results are compared and good agreement is found between

them. Other circuit diagrams can be similarly designed for other complex systems in Secs. 4 and 5. The last section contains our concluding remarks.

2. Dynamical Properties of System (2)

This section deals with the basic dynamical properties of system (2). The real version of (2) with $x = u_1 + iu_2$, $y = u_3 + iu_4$, $i = \sqrt{-1}$, $z = u_5$ and $u = u_6$ is:

$$
\begin{aligned}
\dot{u}_1 &= a(u_3 - u_1 + u_3 u_5), \\
\dot{u}_2 &= a(u_4 - u_2 + u_4 u_5), \\
\dot{u}_3 &= -u_1 u_5 + bu_3 + u_6, \\
\dot{u}_4 &= -u_2 u_5 + bu_4, \\
\dot{u}_5 &= u_1 u_3 + u_2 u_4 - cu_5, \\
\dot{u}_6 &= -ku_1.
\end{aligned}
\tag{4}
$$

If $(u_1, u_2, u_3, u_4, u_5, u_6)$ is a solution of (4), then $(u_1, -u_2, u_3, -u_4, u_5, u_6)$ is also a solution. This means that the real system (4) is symmetrical about the coordinates u_2 and u_4.

From (4) one gets the divergence as:

$$
\sum_{i=1}^{6} \frac{\partial \dot{u}_i}{\partial u_i} = -(2a - 2b + c).
$$

If

$$
2a - 2b + c > 0,
$$

then (4) is dissipative.

The fixed points of system (4) and their stability can be studied as follows. Solving the equations:

$$
\begin{aligned}
a(u_3 - u_1 + u_3 u_5) &= 0, \\
a(u_4 - u_2 + u_4 u_5) &= 0, \\
-u_1 u_5 + bu_3 + u_6 &= 0, \\
-u_2 u_5 + bu_4 &= 0, \\
u_1 u_3 + u_2 u_4 - cu_5 &= 0, \\
u_1 &= 0,
\end{aligned}
\tag{5}
$$

to get five isolated fixed points:

$$
E_0 = (0, 0, 0, 0, 0, 0),
$$

$$
E_1 = \left(0, \sqrt{cb}, 0, \frac{\sqrt{cb}}{2b}(-1 + \sqrt{1 + 4b}), \frac{-1 + \sqrt{1 + 4b}}{2}, 0\right) \quad \text{and}
$$

$$
E_2 = \left(0, -\sqrt{cb}, 0, -\frac{\sqrt{cb}}{2b}(-1 + \sqrt{1 + 4b}), \frac{-1 + \sqrt{1 + 4b}}{2}, 0\right),
$$

$$
E_3 = \left(0, \sqrt{cb}, 0, \frac{\sqrt{cb}}{2b}(-1 - \sqrt{1 + 4b}), \frac{-1 - \sqrt{1 + 4b}}{2}, 0\right) \quad \text{and}
$$

$$
E_4 = \left(0, -\sqrt{cb}, 0, \frac{\sqrt{cb}}{2b}(1 + \sqrt{1 + 4b}), \frac{-1 - \sqrt{1 + 4b}}{2}, 0\right).
$$

One of the necessary conditions to generate hyperchaotic behavior is that the trivial fixed point E_0 is unstable. The Jacobian matrix of system (2) at E_0 is:

$$
J_{E_0} = \begin{pmatrix}
-a & 0 & a & 0 & 0 & 0 \\
0 & -a & 0 & a & 0 & 0 \\
0 & 0 & b & 0 & 0 & 1 \\
0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & -c & 0 \\
-k & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The characteristic polynomial is:

$$
(\mu + a)(\mu + c)(\mu - b)
$$
$$
\times [\mu^3 + (a - b)\mu^2 - ab\mu + ak] = 0.
$$

This polynomial has six roots, three of them $\mu_1 = -a$, $\mu_2 = -c$, and $\mu_3 = b$ and the other three roots are the solutions of $\mu^3 + (a - b)\mu^2 - ab\mu + ak = 0$. It is clear that if $a > 0$, $b > 0$, $c > 0$ and $k > 0$, then E_0 is unstable. In the numerical calculations of this paper, we will use these conditions on system parameters with the condition $2a + c > 2b$.

3. Existence of Attractors of System (2)

This section is devoted to calculate Lyapunov exponents and their signs are used to classify attractors of system (2). Based on these exponents, we compute parameter values of our system (2) at which hyperchaotic, chaotic, periodic and quasi-periodic attractors and attractors that approach fixed point exist.

For the choice $a = 70$, $b = 15$, $c = 12$ and $k = 5$ and the initial conditions $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3$, $u_4(0) = 4$, $u_5(0) = 5$ and $u_6(0) = 6$, we calculate the Lyapunov exponents which are: $\lambda_1 = 1.262$, $\lambda_2 = 0.2124$, $\lambda_3 = 0$, $\lambda_4 = -3.245$, $\lambda_5 = -77.942$ and $\lambda_6 = -97.486$

(for more details about the calculations of Lyapunov exponents, see [Mahmoud *et al.*, 2009]). It is noticed that our choice of these parameters satisfies the necessary conditions to generate hyperchaotic behavior.

This means that our system (2) for this choice of a, b, c and k is a hyperchaotic system since two of the Lyapunov exponents λ_1 and λ_2 are positive and a dissipative system because their sum is negative. The hyperchaotic attractors of (2) using the same choice of the parameters and initial conditions are plotted in Figs. 1(a) and 1(b) in (u_3, u_4, u_5) and (u_1, u_2, u_6) spaces, respectively. For the case $a = 38$, $b = 15$, $c = 12$ and $k = 5$ with the same initial conditions as in Figs. 1(a) and 1(b), we plotted
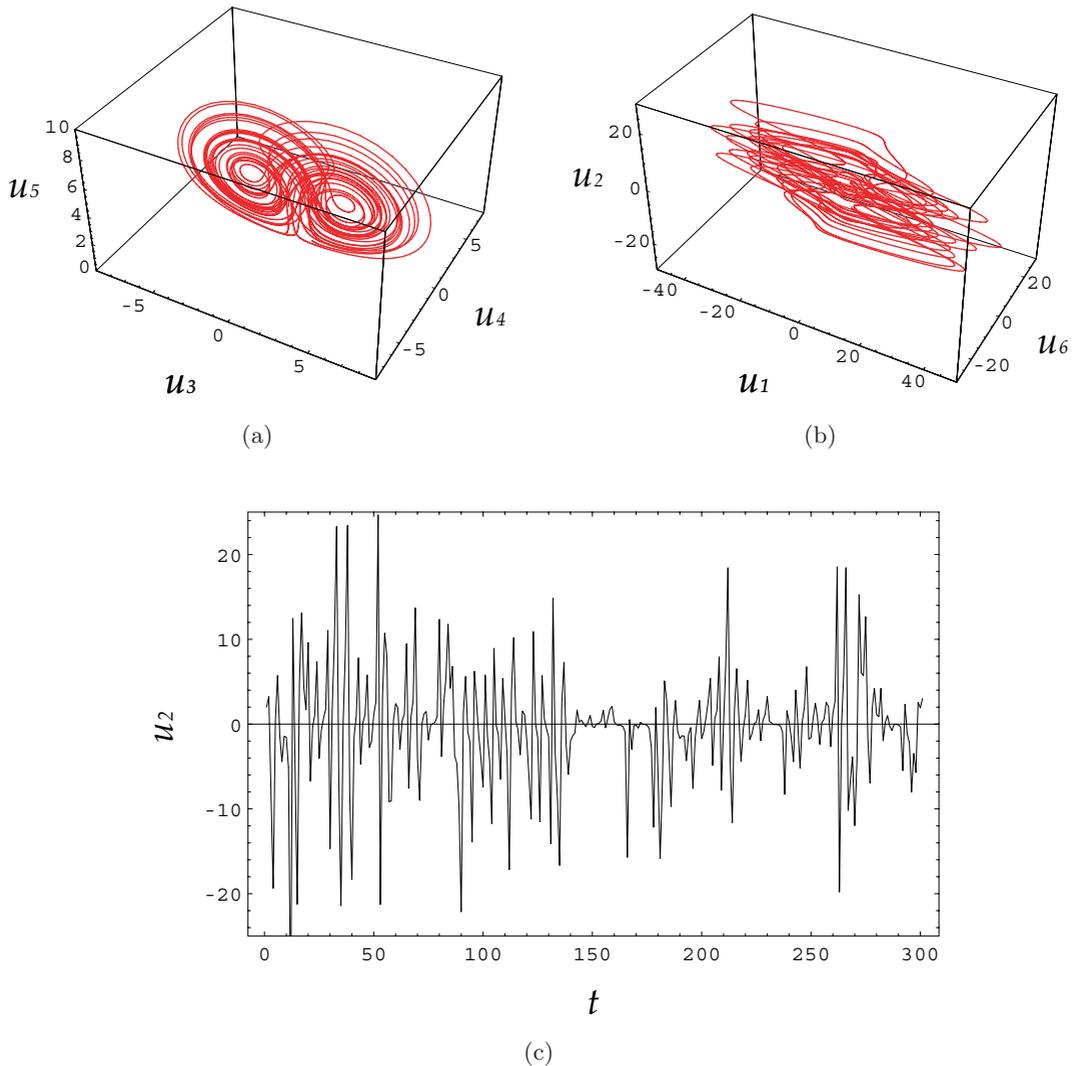

Fig. 1.   A hyperchaotic attractor of (2) for $a = 70$, $b = 15$, $c = 12$ and $k = 5$ with $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3$, $u_4(0) = 4$, $u_5(0) = 5$, $u_6(0) = 6$: (a) (u_3, u_4, u_5) space, (b) (u_1, u_2, u_6) space, (c) $t-u_2$ plane for $a = 38$, $b = 15$, $c = 12$ and $k = 5$.

u_2 versus t in Fig. 1(c), to show the hyperchaotic behavior of the state variable u_2, and other state variables can be similarly plotted.

The attractors of system (2) can be classified as:

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	Type of solutions
−	−	−	−	−	−	Solutions approach fixed points
0	−	−	−	−	−	Periodic solutions (limit cycles)
0	0	−	−	−	−	Quasi-periodic solutions (2-torus)
+	0	−	−	−	−	Chaotic attractors
+	+	0	−	−	−	Hyperchaotic attractors

3.1. *Fix $b = 15, c = 12, k = 5$ and vary a*

In Figs. 2(a) and 2(b) we plotted the corresponding Lyapunov exponents λ_i, $i = 1, 2, \ldots, 6$ of system (2) using the same initial conditions as in Fig. 1. It is clear that from Fig. 2(a), when $a \in (34.56, 40.64]$, $(41.36, 48.60]$ and $(51.32, 70]$ the new system (2) has hyperchaotic attractors, while it has chaotic

attractors for a lying in the intervals $(31.84, 32.1]$, $(33.04, 34.56]$ and $(72, 77.04)$. It also has a periodic attractor for $a \in (20, 31.50]$ and $(79.3, 290]$, while it has quasi-periodic attractors (2-torus) for a lying in the intervals $(31.50, 31.84]$, $(77.04, 77.2]$ and $(78.14, 79.24)$. As is shown in Fig. 2(b) the values of λ_4, λ_5 and λ_6 are negative.

3.2. *Fix $a = 38, c = 12, k = 5$ and vary b*

From Fig. 2(c) one can conclude that (2) has hyperchaotic attractors for $b \in (9.18, 11.66]$, $(12.16, 13]$ and $(13.60, 19.58]$, chaotic attractors for $b \in (5.4, 9.18]$, $(11.66, 12.16]$, $(19.58, 20.36]$ and $(20.52, 20.72]$, and quasi-periodic attractors for b lying in $(0.7, 1]$, and $(20.72, 21.16]$. The limit cycles of (2) are found for $b \in (0, 0.7]$, $(21.19, 32.82]$ and $(33.1, 35]$. Attractors of our system (2) approach nontrivial fixed points for $b \in (32.82, 33.1)$.

3.3. *Fix $a = 38, b = 15, k = 5$ and vary c*

As we did before, we plot only λ_1, λ_2 and λ_3 in Fig. 2(d) and we see that (2) has hyperchaotic

(a)

(b)

(c)

(d)

(e)
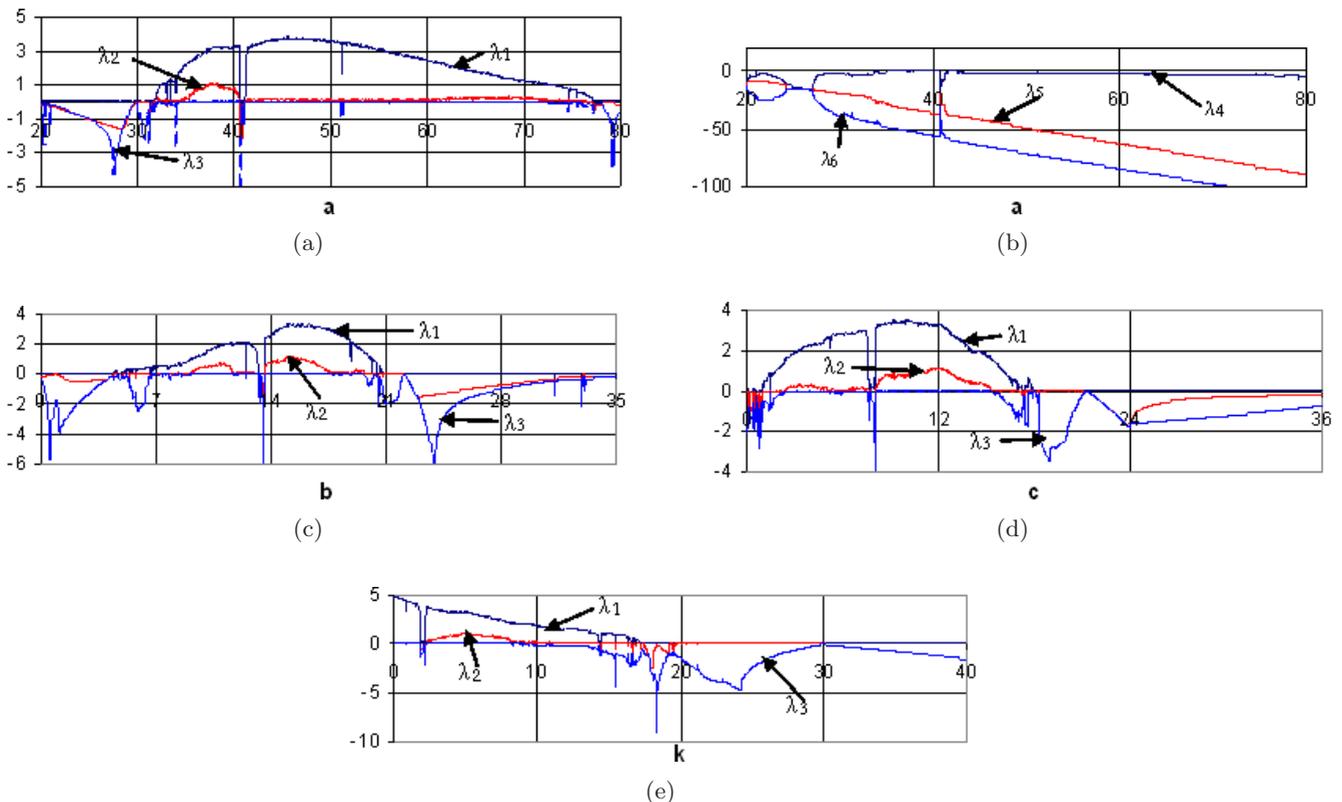
Fig. 2. Lyapunov exponents of (2) at the same initial conditions as in Fig. 1: (a) λ_1, λ_2 and λ_3 versus a, (b) λ_4, λ_5 and λ_6 versus a, (c) λ_1, λ_2 and λ_3 versus b, (d) λ_1, λ_2 and λ_3 versus c, (e) λ_1, λ_2 and λ_3 versus k.

attractors for $c \in [2.60, 4.50]$, $(6.44, 7.60]$, and $(8.06, 15.28]$. Attractors of (2) approach nontrivial fixed points for $c \in (37.28, 37.32]$ and $(37.58, 64]$. In between the above values of c, our system has periodic, quasi-periodic and chaotic attractors as one sees from Fig. 2(d).

3.4. *Fix* $a = 38, b = 15, c = 12$ *and vary* k

As is shown in Fig. 2(e), the hyperchaotic attractors exist for $k \in [2.1, 10.70]$, and $(10.80, 11.30]$, while the chaotic attractors for k lie in the intervals $(0, 1.88]$, $(14.44, 16.52]$ and $(16.70, 17.10]$. The periodic attractors exist for $k \in (1.88, 2.10]$, $(17.10, 19.58]$ and $(30.30, 40]$. The solutions of (2) are quasi-periodic attractors for k lying in $(19.52, 30.30]$.

4. Autonomous Modified Hyperchaotic Complex Lü Systems

In this section, we propose different forms of autonomous modified hyperchaotic complex Lü systems by adding the state feedback controller ω to system variables. The Lyapunov exponents of these proposed systems are calculated and two of them are positive. Their dynamics can be similarly studied as we did for system (2). These systems with their Lyapunov exponents are:

$$
\begin{aligned}
\dot{x} &= a(y - x + yz), \\
\dot{y} &= -xz + by + w, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{6}
$$

$\lambda_1 = 3.739$, $\lambda_2 = 0.876$, $\lambda_3 = 0.0$, $\lambda_4 = -7.181$, $\lambda_5 = -34.99$, $\lambda_6 = -53.27$, for the case $a = 38$, $b = 15$, $c = 12$ and $k = 5$,

$$
\begin{aligned}
\dot{x} &= a(y - x + yz) + w, \\
\dot{y} &= -xz + by, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{7}
$$

$\lambda_1 = 4.833$, $\lambda_2 = 0.255$, $\lambda_3 = 0.0$, $\lambda_4 = -7.21$, $\lambda_5 = -30.25$, $\lambda_6 = -52.49$, $a = 38$, $b = 18$, $c = 12$ and $k = 5$,

$$
\begin{aligned}
\dot{x} &= a(y - x + yz) + (1 + i)w, \\
\dot{y} &= -xz + by, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{8}
$$

$\lambda_1 = 4.62$, $\lambda_2 = 0.29$, $\lambda_3 = 0.0$, $\lambda_4 = -7.21$, $\lambda_5 = -29.26$, $\lambda_6 = -50.61$, $a = 38$, $b = 18$, $c = 12$ and $k = 5$,

$$
\begin{aligned}
\dot{x} &= a(y - x + yz), \\
\dot{y} &= -xz + by + (1 + i)w, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{9}
$$

$\lambda_1 = 3.95$, $\lambda_2 = 1.26$, $\lambda_3 = 0.0$, $\lambda_4 = -7.181$, $\lambda_5 = -31.08$, $\lambda_6 = -49.135$, $a = 38$, $b = 18$, $c = 12$ and $k = 5$,

$$
\begin{aligned}
\dot{x} &= a(y - x + yz) + w, \\
\dot{y} &= -xz + by + w, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{10}
$$

$\lambda_1 = 3.84$, $\lambda_2 = 0.80$, $\lambda_3 = 0.0$, $\lambda_4 = -7.181$, $\lambda_5 = -34.79$, $\lambda_6 = -53.51$, $a = 38$, $b = 15$, $c = 12$ and $k = 5$,

$$
\begin{aligned}
\dot{x} &= a(y - x + yz) + iw, \\
\dot{y} &= -xz + by + iw, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{11}
$$

$\lambda_1 = 3.97$, $\lambda_2 = 1.01$, $\lambda_3 = 0.0$, $\lambda_4 = -7.17$, $\lambda_5 = -31.88$, $\lambda_6 = -48.31$, $a = 38$, $b = 18$, $c = 12$ and $k = 5$

$$
\begin{aligned}
\dot{x} &= a(y - x + yz) + w, \\
\dot{y} &= -xz + by, \\
\dot{z} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - cz + w, \\
\dot{w} &= \frac{1}{2}(\bar{x}y + x\bar{y}) - kw,
\end{aligned} \tag{12}
$$

$\lambda_1 = 4.22$, $\lambda_2 = 0.1$, $\lambda_3 = 0.0$, $\lambda_4 = -9.071$, $\lambda_5 = -24.67$, $\lambda_6 = -44.12$, $a = 38$, $b = 21$, $c = 12$ and $k = 5$,

$$\dot{x} = a(y - x + yz),$$
$$\dot{y} = -xz + by + w,$$
$$\dot{z} = \frac{1}{2}(\overline{x}y + x\overline{y}) - cz + w, \qquad (13)$$
$$\dot{w} = \frac{1}{2}(\overline{x}y + x\overline{y}) - kw,$$

$\lambda_1 = 4.34$, $\lambda_2 = 1.02$, $\lambda_3 = 0.0$, $\lambda_4 = -8.85$, $\lambda_5 = -30.74$, $\lambda_6 = -47.970$, $a = 38$, $b = 18$, $c = 12$ and $k = 5$.

5. Nonautonomous Systems

This section contains the dynamics of our proposed nonautonomous modified hyperchaotic complex Lü system (3). For this system, we have added the complex periodic control signal to the complex variable y. The real version of (3) by using $u_6 = \omega t$ reads:

$$\begin{aligned}
\dot{u}_1 &= a(u_3 - u_1 + u_3 u_5), \\
\dot{u}_2 &= a(u_4 - u_2 + u_4 u_5), \\
\dot{u}_3 &= -u_1 u_5 + b u_3 + k \cos(u_6), \\
\dot{u}_4 &= -u_2 u_5 + b u_4 + k \sin(u_6), \\
\dot{u}_5 &= u_1 u_3 + u_2 u_4 - c u_5, \\
\dot{u}_6 &= \omega.
\end{aligned} \qquad (14)$$

System (14) is dissipative under the condition $2a - 2b + c > 0$, and its Lyapunov exponents are $\lambda_1 = 4.02$, $\lambda_2 = 0.41$, $\lambda_3 = -0.02$, $\lambda_4 = -29.48$, $\lambda_5 = -49.46$, $\lambda_6 = 0.0$, for $a = 38$, $b = 18$, $c = 12$, $\omega = 10$ and $k = 5$ and the initial conditions $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3$, $u_4(0) = 4$, $u_5(0) = 5$, $u_6(0) = 0$.

This means that our system (14) for this choice of a, b, c, ω and k is a hyperchaotic one.

Using Lyapunov exponents we calculate the parameter values at which attractors of (3) exist as follows.

5.1. *Fix* $b = 15, c = 12, k = 5, \omega = 10$, *and vary* a

In Figs. 3(a) and 3(b) we plotted the corresponding Lyapunov exponents λ_i, $i = 1, 2, \ldots, 6$ of system (3) using the same initial conditions as in Fig. 1. It is clear from Fig. 3(a), when $a \in (35.30, 41.60]$ system (3) has hyperchaotic attractors, while it has

chaotic attractors for a lying in the intervals $(32.10, 35.20]$, $(42.42, 58]$ and $(59.88, 60.38)$. It also has quasi-periodic solutions (2-torus) for a lying in the intervals $(25, 32.10]$, $(58, 59.88]$ and $(60.38, 80)$. In Fig. 3(b) the values of λ_4 and λ_5 are negative and $\lambda_6 = 0.0$.

5.2. *Fix* $a = 38, c = 12, k = 5, \omega = 10$, *and vary* b

From Fig. 3(c) one can conclude that (3) has hyperchaotic attractors for $b \in (11.42, 12.46]$ and $(16.14, 19.30]$, chaotic attractors for $b \in (6.24, 6.88]$, $(8.66, 11.42]$, $(13.62, 15.72]$ and $(21.04, 22.58]$, and quasi-periodic attractors for b lying in $(0.0, 6.24]$, and $(22.58, 36]$.

5.3. *Fix* $a = 38, b = 15, k = 5$ *and vary* c

As we did before, we plot only λ_1, λ_2 and λ_3 in Fig. 3(d) and we see that (3) has hyperchaotic attractors for $c \in (8.50, 14]$. The quasi-periodic attractors exist for $c \in (1.78, 3.16]$ and $(16.60, 35]$, while the system has periodic solution for c lying in the interval $(0, 1.12]$.

5.4. *Fix* $a = 38, b = 15, c = 12, \omega = 10$, *and vary* k

As is shown in Fig. 3(e), the hyperchaotic attractors exist for $k \in [0.84, 1.80]$, and $(2.54, 9.44]$, while the chaotic attractors for k lie in the intervals $(2.12, 2.54]$, $(9.44, 13.98]$, $(14.28, 24.74]$ and $(26.06, 40.42]$. The solutions of (3) are quasi-periodic attractors for k lying in $(1.80, 12.12]$, and $(40.42, 50]$.

5.5. *Fix* $a = 38, b = 15, c = 12, k = 5$, *and vary* ω

As is shown in Fig. 3(f), the hyperchaotic attractors exist for $\omega \in [0, 1.16]$, $(2.88, 5.52]$, $(5.60, 12.66]$, $(18.90, 24.90]$ and $(34.60, 38.20]$, while the chaotic attractors for ω lie in the intervals $(1.16, 2.88]$, $(12.66, 16.82]$, $(24.90, 27]$ and $(38.20, 50]$. The quasi-periodic solutions exist for $\omega \in (16.82, 17.52]$, $(27.0, 27.36]$ and $(31.74, 32]$.

By adding the complex periodic control signal to other variables of system (3) that we have proposed, for example, different models of nonautonomous modified hyperchaotic complex Lü
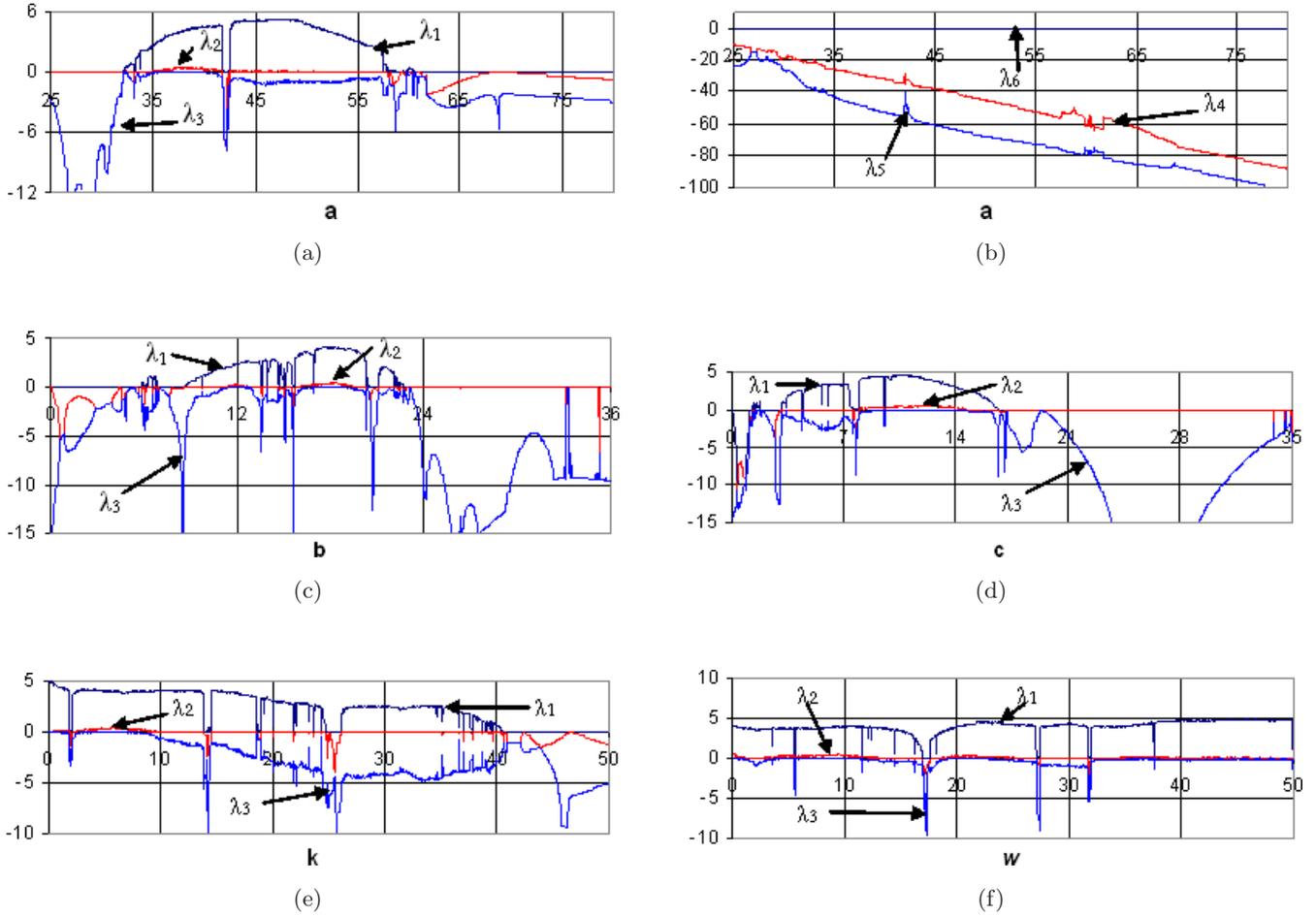
Fig. 3.   Lyapunov exponents of (3) with $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3, u_4(0) = 4, u_5(0) = 5, u_6(0) = 0$: (a) λ_1, λ_2 and λ_3 versus a, (b) λ_4, λ_5 and λ_6 versus a, (c) λ_1, λ_2 and λ_3 versus b, (d) λ_1, λ_2 and λ_3 versus c, (e) λ_1, λ_2 and λ_3 versus k, (f) λ_1, λ_2 and λ_3 versus w.

systems, these systems with their Lyapunov exponents are:

$$\dot{x} = a(y - x + yz) + k\exp(i\omega t),$$
$$\dot{y} = -xz + by,$$
$$\dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \tag{15}$$

$\lambda_1 = 3.41$, $\lambda_2 = 0.23$, $\lambda_3 = -0.004$, $\lambda_4 = -57.84$, $\lambda_5 = -78.15$, $\lambda_6 = 0$, for the case $a = 58$, $b = 18$, $c = 12$, $k = 5$ and $\omega = 10$,

$$\dot{x} = a(y - x + yz) + k(i + 1)\cos(\omega t),$$
$$\dot{y} = -xz + by,$$
$$\dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \tag{16}$$

$\lambda_1 = 3.791$, $\lambda_2 = 0.186$, $\lambda_3 = -0.13$, $\lambda_4 = -57.80$, $\lambda_5 = -78.58$, $\lambda_6 = 0$, $a = 58$, $b = 18$, $c = 12$, $k = 5$

and $\omega = 10$,

$$\dot{x} = a(y - x + yz) + k(i + 1)\sin(\omega t),$$
$$\dot{y} = -xz + by,$$
$$\dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \tag{17}$$

$\lambda_1 = 3.78$, $\lambda_2 = 0.19$, $\lambda_3 = -0.19$, $\lambda_4 = -57.80$, $\lambda_5 = -78.51$, $\lambda_6 = 0$, $a = 58$, $b = 18$, $c = 12$, $k = 5$ and $\omega = 10$,

$$\dot{x} = a(y - x + yz),$$
$$\dot{y} = -xz + by + k(i + 1)\sin(\omega t),$$
$$\dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \tag{18}$$

$\lambda_1 = 4.46$, $\lambda_2 = 0.627$, $\lambda_3 = -0.30$, $\lambda_4 = -30.87$, $\lambda_5 = -52.24$, $\lambda_6 = 0$, $a = 39$, $b = 18$, $c = 12$, $k = 5$
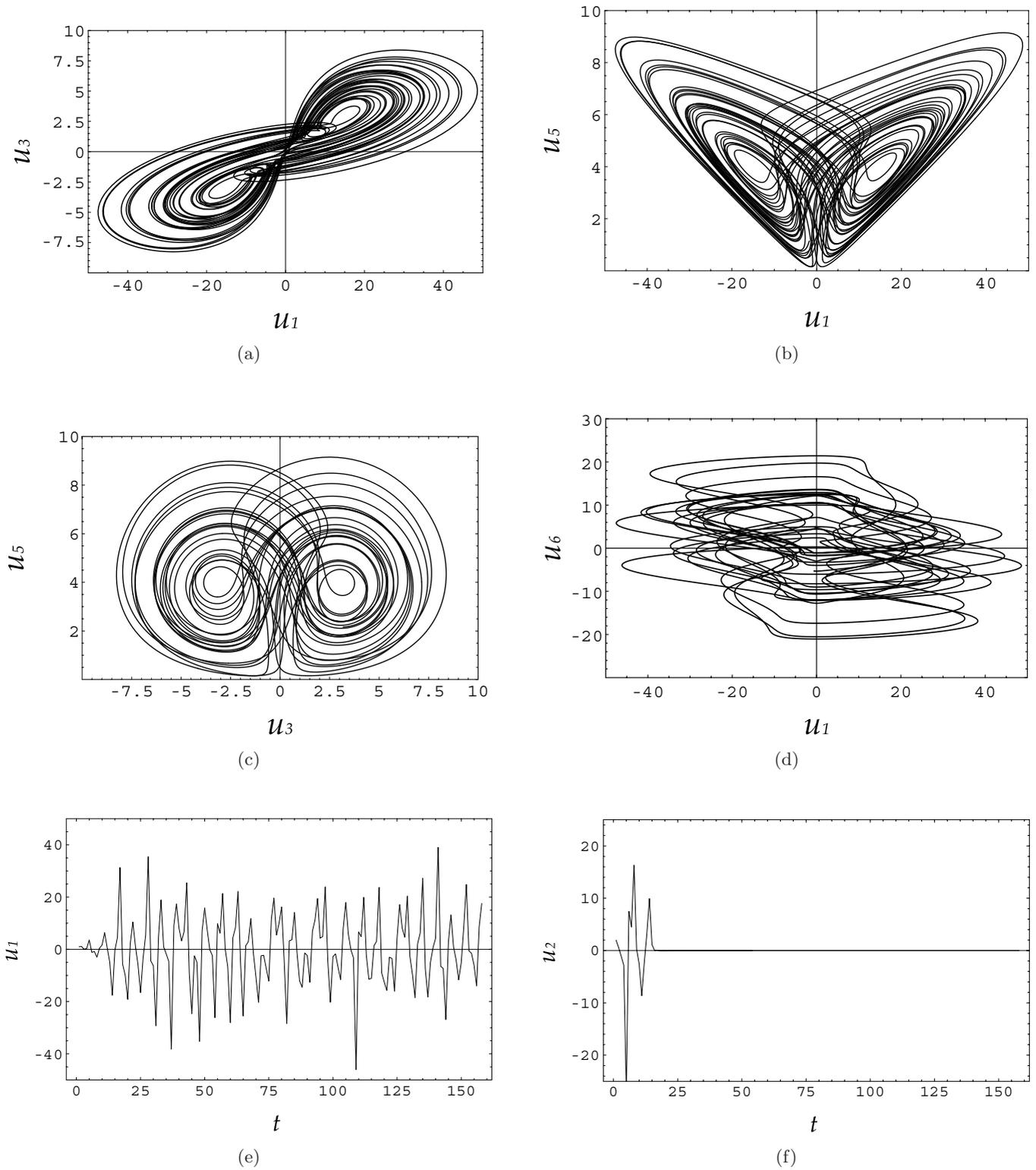
Fig. 4. Numerical calculations of the hyperchaotic attractor of (2) (or (4)) with the same initial conditions and system parameters as in Fig. 1: (a) u_1–u_3 plane, (b) u_1–u_5 plane, (c) u_3–u_5 plane, (d) u_1–u_6 plane, (e) t–u_1 plane, and (f) t–u_2 plane.

and $\omega = 10$,

$$\dot{x} = a(y - x + yz),$$
$$\dot{y} = -xz + by + k(i + 1)\cos(\omega t),$$
$$\dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - cz,$$

(19)

$\lambda_1 = 4.92$, $\lambda_2 = 0.64$, $\lambda_3 = -0.27$, $\lambda_4 = -30.91$, $\lambda_5 = -52.23$, $\lambda_6 = 0$, $a = 39$, $b = 18$, $c = 12$, $k = 5$ and $\omega = 10$.

6. Implementaion and Simulation of System (2)

In this section, we design an electronic circuit for the modified hyperchaotic complex Lü system (2) and compare its simulation results with those obtained numerically in Sec. 3. In Fig. 4, we plotted numerically different projections on the plane of the hyperchaotic attractor in Fig. 1. The designed circuitry realizing equations (4) are shown in Fig. 5. This circuit contains six channels which realize the state variables $u_1, u_2, u_3, u_4, u_5, u_6$, respectively. Moreover, all original devices shown in Fig. 5 are operational amplifiers of type TL082 and operational multiplier of type AD633 with voltage supply ± 15 V. Figure 6 shows the simulation of system (4) using Matlab/Simulink with real variables $u_1, u_2, u_3, u_4, u_5, u_6$. Simulation observations are displayed in Fig. 7, which agree with the numerical results of Fig. 4.
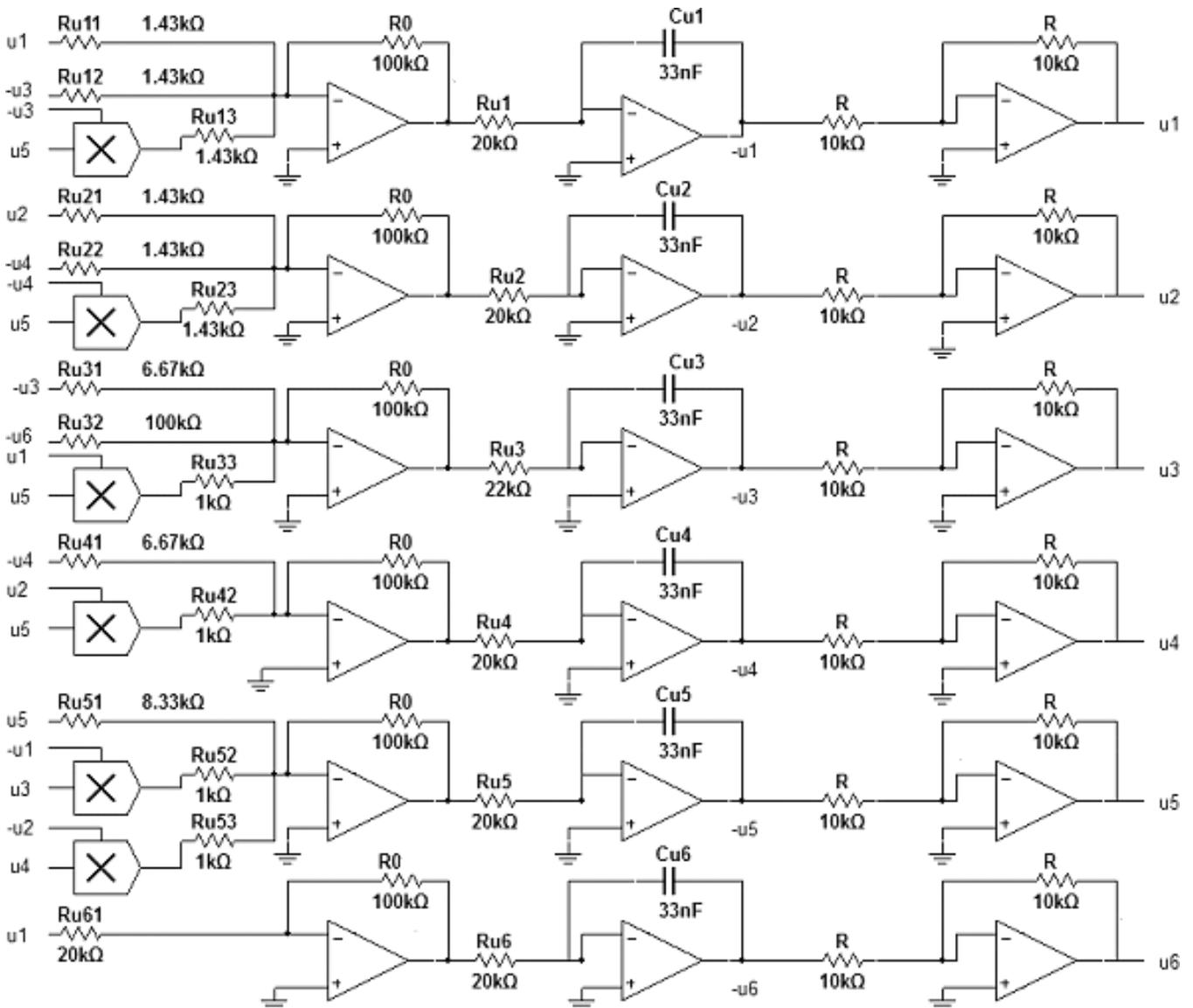

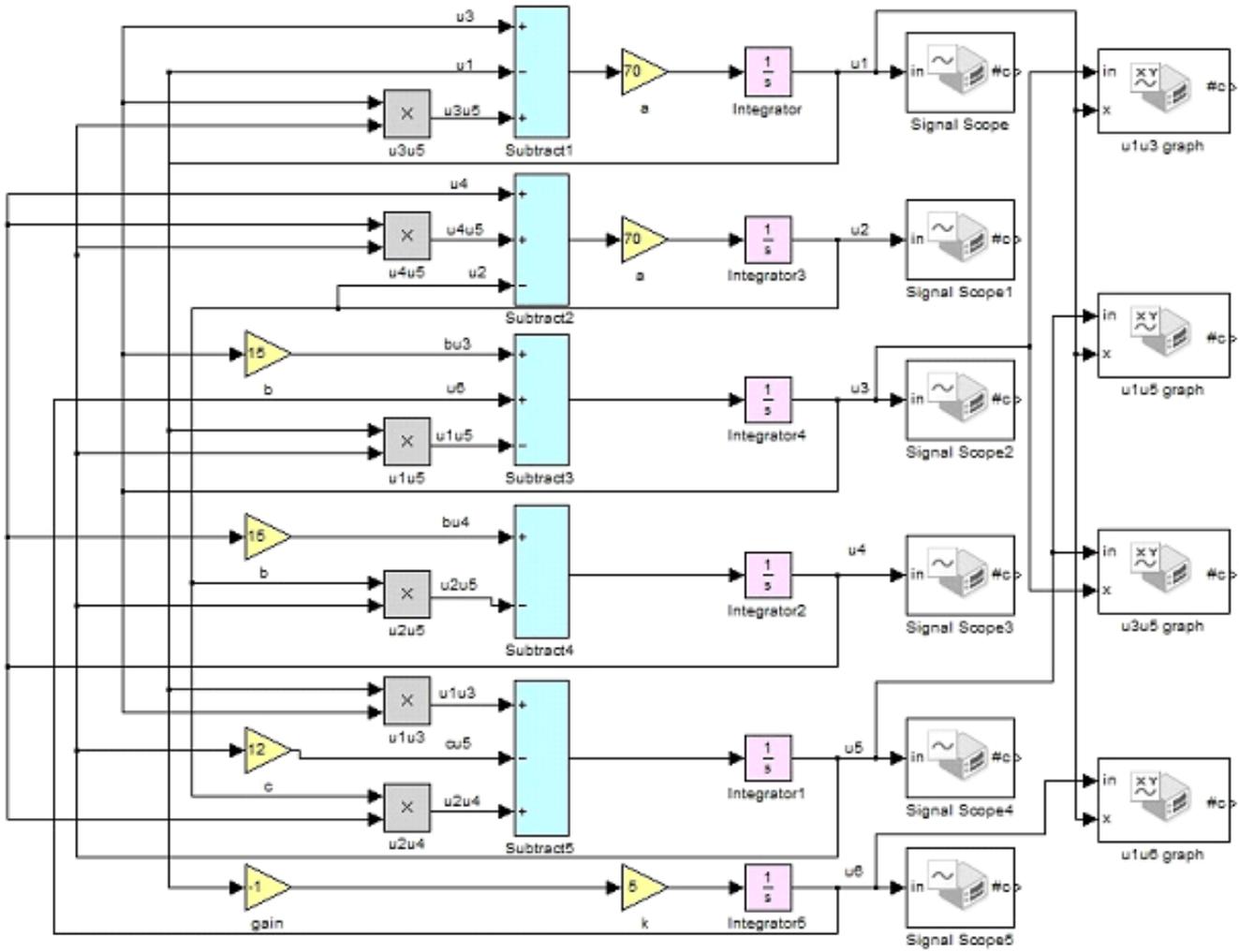Fig. 5.   Circuit diagram of the modified hyperchaotic complex Lü system (4).

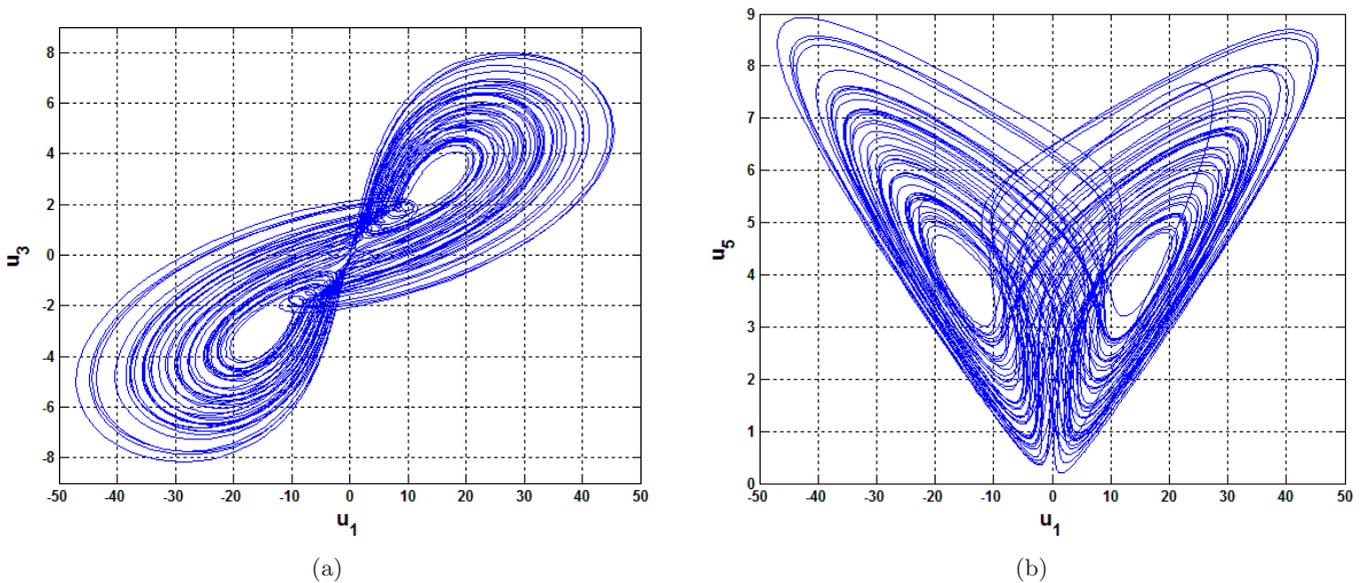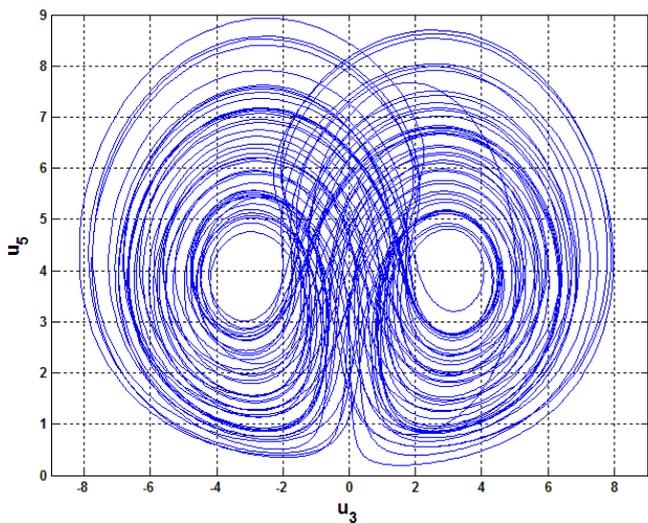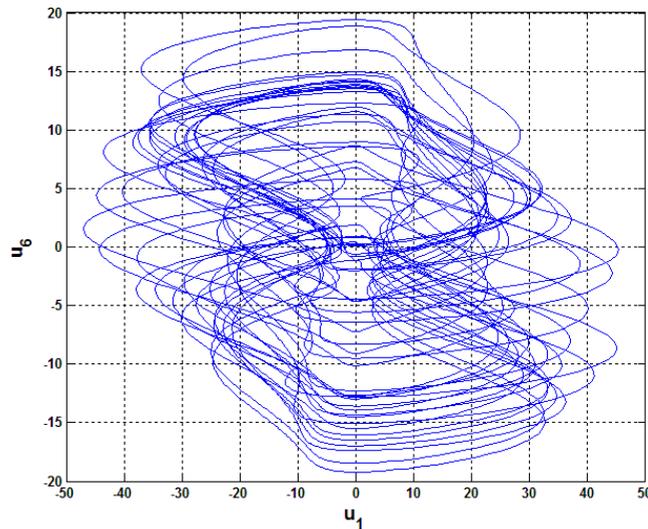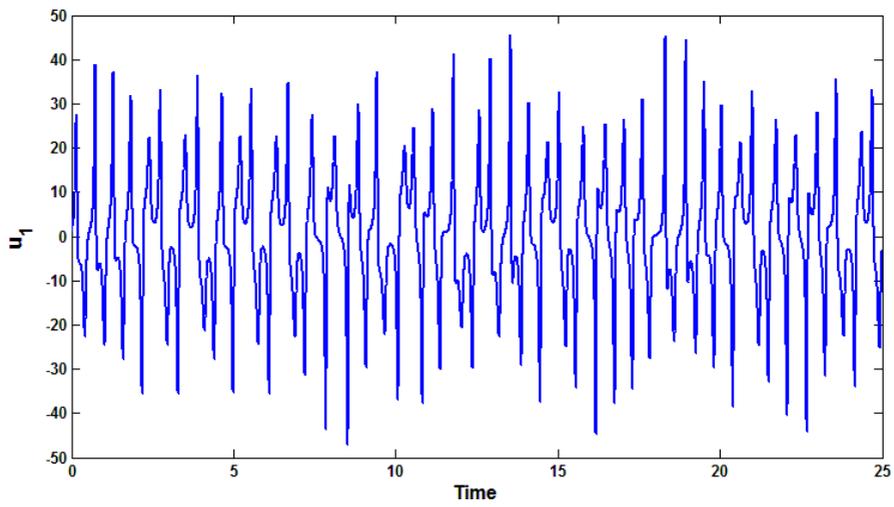Fig. 6.   Block diagram of system (4) using Matlab/Simulink.

(a)

(b)

Fig. 7.   Simulation observations of the hyperchaotic attractor of system (4): (a) u_1-u_3 plane, (b) u_1-u_5 plane, (c) u_3-u_5 plane, (d) u_1-u_6 plane, (e) $t-u_1$ plane, and (f) $t-u_2$ plane.
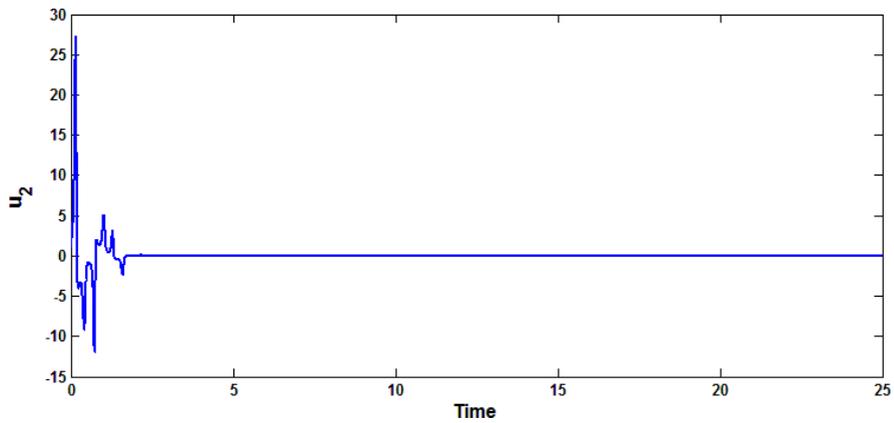
(c)

(d)

(e)

(f)

Fig. 7.   (*Continued*)

7. Conclusions

In this paper, we proposed both autonomous and nonautonomous modified hyperchaotic complex Lü systems. The hyperchaotic behaviors have been generated by using the state feedback controller and the complex periodic control signal. Their dynamics are rich and very complicated as shown in Figs. 2 and 3. It is clear that from these figures, our system exhibit chaotic, hyperchaotic attractors, periodic, quasi-periodic solutions and solutions approach fixed points. The stability analysis of the trivial fixed points E_0 of system (2) and dissipation condition are used to compute parameter values of systems (2) and (3) at which hyperchaotic attractor exist. An electronic circuit is designed for system (2), so we believe that our proposed systems (2), (3), (6), ..., (13), (15), ... and (19) will have broad applications in different fields of physics, engineering, and computer sciences, such as secure communication, information science and laser systems. Figures 4 and 7 shown good agreement between simulation observations of our new electronic circuit of the hyperchaotic attractor of system (2) and those of numerical calculations. Some basic dynamical behaviors of systems (2) and (3) are investigated, while the other proposed systems of this paper can be similarly studied and leave room for further studies in the near future. Other circuit diagrams can be similarly designed for systems ((6), ..., (19)) of this paper. The real counter parts of one of these systems have been studied in recent years in the literature.

Acknowledgment

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