Space-charge effects and gain in Cherenkov free-electron lasers

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Abstract
This paper presents a quantum mechanical treatment to study the growth rate characteristics of Cherenkov free-electron laser. For this purpose, we basically use the single-particle model in which the dynamics of a single electron in the presence of the laser field is analyzed. The inclusions of the space-charge (collective) effects are considered by taking into account the static electric field of neighboring electrons in the dynamics formulations. An analytical expression for the gain per pass in the Cherenkov laser is derived. It is shown that the space-charge effects depend mainly on the operating wavelength, the electron density, and the electron beam neutralization due to the possible presence of positive ions. We discuss the validity boundaries of the interaction mode evolved from the single-particle regime to the collective regime. Using quantum mechanical concepts, we finally present a formula for calculating the inclusion of the spontaneous emission power.

1. Introduction

In the Cherenkov laser devices [1,2], an electron beam passes close to the surface of a dielectric-lined waveguide. The so-called spontaneous Cherenkov radiation is generated when the electrons' velocity approaches the phase velocity of the radiation in the waveguide. The phase velocity of the light in the waveguide is determined by the dispersion relation for the specific guide model. Therefore, the operating wavelength can be tuned by changing the accelerating voltage of electrons and the geometry and material of the dielectric film. By feeding the radiation back onto the beam by the help of a resonator configuration, the existing radiation causes a bunching of the electrons and results in further stimulated Cherenkov emission.

Most fundamental molecules (e.g., water, oxygen and carbon monoxide) have their rotational and vibrational absorption lines in the far infrared (FIR) range that is defined as roughly 30–1000 μm in wavelength. FIR radiation can pass many non-metallic materials such as clothing, paper, woods and plastics. Furthermore, FIR radiation is non-ionizing and is not harmful for living cells. Such characteristic features attract the rising interest for FIR applications in many areas such as basic science, manufacturing, nondestructive inspection, medicine and broadband FIR communications [3–5]. Tunable coherent electromagnetic (EM) radiation sources do not exist for FIR spectral range. The successful production of microwave radiation by Cherenkov laser [6–8] drives immense research activities for exploring their feasibility for the generation of FIR radiation. Many reports [9–11] show theoretical and experimental investigations of a compact Cherenkov free-electron laser (CFEL) that fits on a tabletop, uses a low power electron beam, and is continuously tunable in the FIR. Cherenkov emission with a remarkable low output power has been observed in these experiments.

Some experiments showed that the CFEL in the optical spectral range (i.e., micro-Cherenkov FEL) is also achievable [12–14]. The electron beam couples to an existing transverse magnetic (TM) mode in the waveguide through the evanescent part of the wave above the waveguide surface. The distance that the guided mode extends into the vacuum is very small in the optical range, and that the beam diameter should be comparable to this length. This represents a significant experimental constraint to obtain a good coupling between the electrons and the TM mode. In the experiments of micro-Cherenkov FEL [12,13], the Cherenkov laser can be spontaneously emitted. However, it might be difficult to obtain the stimulated emission in the optical range.

During the electron beam passage above the dielectric waveguide, a periodic density modulation is formed and a periodic longitudinal Coulomb field (i.e., space-charge field) is therefore generated. For most CFELs, the low-quality of the electron beam results in a low gain in radiation power (i.e., the ratio of the output to the input field is linear). However, an optical cavity is required to build up stimulated emission by feedback. In the low gain regime, the single-electron model [15,16] is used to study the amplification characteristics of Cherenkov laser. The single-electron approach deals directly with the dynamic of one electron in the EM field. The electric field of the EM wave performs work on the electrons allowing energy to be exchanged between the electrons and the EM field. In most previous works, the space-charge effects are neglected in the low gain regime assuming a negligible electron density modulation. Using very high quality of
the electron beam, high gain CFEL can be obtained where the collective effects of electrons cannot be neglected. In the collective model [17–19], the dispersion relation for the coupled EM and space-charge waves is developed by the Maxwell’s equations. The “lasing” operation in the collective mode occurs when the phase velocities of the EM and space-charge waves are matched as well as the EM wave satisfies the dispersion relation for the waveguide. At the best of our knowledge, what has not been done classical (or quantum) mechanically is to count the space-charge effects in the low gain limit. Considering the space-charge effect in the low gain is presented in [20–22] for the undulator FEL basing on a classical treatment.

In this paper, we present a quantum mechanical analysis to describe the amplification gain in CFELs evolving from the low gain regime to the high gain regime. An explicit expression for the inclusion of the spontaneous emission power is also derived. In the theoretical model, the EM wave is described classically as a polarized plane wave while the electron is described quantum mechanically as a wave packet. Our calculations are initially performed in an arbitrary relativistic moving frame, and then a simple non-relativistic quantum mechanics can be used. In the moving frame, Maxwell’s equations and the Schrödinger equation are used to describe the dynamics of the EM and electron waves, respectively. Then, we derive an expression for calculating the single-pass gain of CFEL. The space-charge effects are introduced by counting the scalar potential, generated by the surrounding modulated electrons, in the single-electron Hamiltonian. The expression of the gain obtained in the moving frame is transformed back to the original laboratory frame using the relativistic Lorentz transformations. We show that the space-charge effects depend not only on the beam density as proposed in the old classical theory but also on the radiation wavelength and the guide material. We also show that the gain per pass derived in this paper basing on the concept of the transformation of references frames well match with standard expression of previous relativistic classical treatments. In the last section, we present an expression for the inclusion of the spontaneous power into the total radiated power.

2. Basic formulas in the moving frame

In the CFEL, an electron beam travels in the longitudinal direction (i.e., z-direction) above a dielectric planar waveguide. For the oscillator configuration, the waveguide is placed between two mirrors at each end. In our model, it is assumed that a sufficiently intense magnetostatic field in the direction of the beam flow is applied. Then, the electron beam is considered to be thin where the transverse velocities of electrons in the direction normal to the electrons propagation can be neglected. The initially unmodulated driving electron beam moves with a velocity which is close, but slightly greater, than the phase velocity of the forward propagating component of the radiation field. Therefore, the TM mode characterized by the z-component of the electric field \( E_z \) interacts most strongly with the electrons. Electron bunching occurs in the retarding phase of the wave and energy is transferred from the beam to the EM fields. In the limit of a thin electron beam, we neglect the effects of the self-magnetic fields in the transverse directions on the longitudinal modulations of electrons. It is noted that in the lowest-order approximation, the self-magnetic field is assumed to be generated from the average unmodulated component of the current density. In the cylindrical coordinates, the self-magnetic field is in the azimuthal direction when the electrons motion is in the longitudinal direction (z-axis). The self-electric field \( \mathbf{E}_z \) is assumed to be periodic function in the longitudinal direction, and that the space-charge field negligibly varies in the transverse directions.

To formulate the gain in CFELs, the relativistic dynamics of electrons should be analyzed in the laboratory frame, and then complicated calculations is required. In this paper, we use an arbitrary frame moving with a fixed velocity nearly equal to the phase velocity of light in the waveguide. In such frame, the velocity of electrons is non-relativistic for which we simply deal with a simple non-relativistic quantum mechanical treatments. From here on we will carry out all calculations in the moving frame to get an expression for the radiated power. Finally, the results obtained in the moving frame are Lorentz-transformed back to the laboratory frame. The method of frame transformation had been used to study the radiation growth characteristics in the undulator FEL [23,24]. Also, the moving frame is proved very useful in analyzing the operation of the Compton scattering [25].

The wavelength of the laser in the laboratory frame is expanded in a frame moving with a velocity \( \beta \) due to the Lorentz transformation. In this paper, we use the notation \([\ldots]_{\text{lab}}\) to indicate that the parameters in the bracket are given as measured in the laboratory frame. The laser wavelength in the moving frame \( \lambda \) is expressed in terms of the wavelength in the laboratory frame \([\lambda]_{\text{lab}}\) by [26]:

\[
\lambda = \frac{\gamma(1+\beta)}{\sqrt{1-(\beta/c)^2}} [\lambda]_{\text{lab}}.
\]

Therefore, the Lorentz transformation transforms the laser frequency in the laboratory frame \( \omega \) into

\[
\omega = \frac{\gamma(1+\beta)}{\sqrt{1-(\beta/c)^2}} \omega_{\text{lab}}.
\]

Using the Coulomb gauge, the evolution of the vector potential of the laser \( \mathbf{A}_l \) is described by the classical wave equation

\[
\frac{\partial^2 \mathbf{A}_l}{\partial t^2} - \mathbf{V}_{\text{loss}} - \sigma_l \mathbf{J} - \mu_l \mathbf{E}_l = - \mu_l \mathbf{J} + \mu_l \mathbf{V} \left( \frac{\partial \mathbf{U}}{\partial t} \right),
\]

where \( J \) is the current density and \( U \) is the scalar potential which corresponds to the space-charge field. \( \sigma_l \) and \( \mu_l \) are the conductivity representing the loss mechanism and the dielectric constant of the \( i \)-th layer (i.e., vacuum or dielectric layer) in the CFEL device.

The laser field is assumed to be a plane wave, and then the vector potential \( \mathbf{A}_l \) is

\[
\mathbf{A}_l = F(t,z) \mathbf{T}(x,y) e^{i \omega t - \mathbf{k} \cdot \mathbf{r}} + \text{c.c.},
\]

where \( \omega \) and \( \beta \) are the angular frequency and the wavenumber of the radiation. In Eq. (4), \( F(t,z) \) and \( \mathbf{T}(x,y) \) are the complex field amplitude and the transverse distribution of the vector potential, respectively. \( \mathbf{T}(x,y) \) satisfies the relation of

\[
(\nabla^2 + \mu_l \sigma_l \omega^2) \mathbf{T}(x,y) e^{-i \beta z} = 0,
\]

and the following normalization condition:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \mathbf{T}(x,y) \right|^2 dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \left| \mathbf{T}_x(x,y) \right|^2 + \left| \mathbf{T}_y(x,y) \right|^2 \right) dxdy = 1.
\]

By substituting Eq. (4) into Eq. (3), using the approximation of the slowly varying amplitude \( \mathbf{J} = \mathbf{J}_s \) \( T^s \approx J_s T^s \approx J_s T^s \) in the thin-electron beam limit, we get

\[
\frac{dF(t,z)}{dz} = \frac{1}{\omega} \frac{dF(t,z)}{dt} \mathbf{E}_l + \frac{1}{\omega} \mathbf{V}_{\text{loss}} F(t,z) - \frac{\mu_l}{\omega} \int_{-\infty}^{\infty} \left( J_z - \sigma_l \mathbf{J} \right) \mathbf{E}_l dxdy - \frac{\sigma_{\text{loss}}}{2} F(t,z).
\]

\[ \mathbf{E}_l = \mathbf{E}_0 \mathbf{J} \mathbf{T}(x,y) e^{i \omega t - \mathbf{k} \cdot \mathbf{r}} + \text{c.c.}, \]

\[ \mathbf{V}_{\text{loss}} = \frac{\sigma_{\text{loss}}}{\omega} \mathbf{J} \mathbf{E}_0 e^{i \omega t - \mathbf{k} \cdot \mathbf{r}} + \text{c.c.}, \]

\[ F(t,z) = \frac{1}{\omega} \frac{dF(t,z)}{dt} + \frac{1}{\omega} \mathbf{V}_{\text{loss}} F(t,z) - \frac{\mu_l}{\omega} \int_{-\infty}^{\infty} \left( J_z - \sigma_l \mathbf{J} \right) \mathbf{E}_l dxdy - \frac{\sigma_{\text{loss}}}{2} F(t,z), \]

\[ \mathbf{E}_l = \mathbf{E}_0 \mathbf{J} \mathbf{T}(x,y) e^{i \omega t - \mathbf{k} \cdot \mathbf{r}} + \text{c.c.}, \]

\[ \mathbf{V}_{\text{loss}} = \frac{\sigma_{\text{loss}}}{\omega} \mathbf{J} \mathbf{E}_0 e^{i \omega t - \mathbf{k} \cdot \mathbf{r}} + \text{c.c.}, \]

\[ F(t,z) = \frac{1}{\omega} \frac{dF(t,z)}{dt} + \frac{1}{\omega} \mathbf{V}_{\text{loss}} F(t,z) - \frac{\mu_l}{\omega} \int_{-\infty}^{\infty} \left( J_z - \sigma_l \mathbf{J} \right) \mathbf{E}_l dxdy - \frac{\sigma_{\text{loss}}}{2} F(t,z), \]
where the subscript “z” refers to the longitudinal component, \( v_z \) is the velocity of the light in the waveguide, and \( \alpha_{liss} \) is the loss coefficient given as
\[
\alpha_{liss} = \frac{\mu_0 \mu_0}{\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_i |T(x, y)|^2 dx dy. \tag{8}
\]

By substituting the complex field amplitude \( F(t, z) \) given in the form of
\[
F(t, z) = |F(t, z)| e^{i\theta(t, z)},
\]
into Eq. (7), we obtain the real part of the field amplitude
\[
diF(t, z) = (\frac{\sigma_{eff}}{\sigma}) F(t, z)^2 + \beta \sigma_{fp}, \tag{9}
\]
where \( g \) is the gain coefficient given by
\[
g = \frac{\mu_0 \gamma}{\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( 1 - e^{-\frac{\sigma_{eff}}{\sigma}} \right) |T(x, y)e^{i\omega t} e^{-j\omega t} dx dy, \tag{10}
\]
and \( \beta \sigma_{fp} \) is the inclusion rate of the spontaneous emission into the intensity of the EM wave.

The electric field of the radiation \( E_L \) is given in terms of the vector potential by
\[
E_L = -\frac{\partial A_L}{\partial t}. \tag{12}
\]

From Eqs. (4) and (12), considering the fact that the stored energy of the radiation magnetic field is almost equal to that of the electric field \( E_L \), the stored energy density of the EM wave is given by
\[
e_0 E_L^2 = 2\sigma_0 \mu_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(t, z)|^2. \tag{13}
\]

By the help of Eqs. (6) and (13), the propagation power of the radiation in the z-direction \( P(t, z) \) is then given by
\[
P(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\left( \frac{\mu_0 \mu_0}{\beta} \right)^{1/2}} e_0 E_L^2 dx dy = 2\sigma_0 \mu_0 \int_{-\infty}^{\infty} |F(t, z)|^2. \tag{14}
\]

3. The dynamic of electrons in the moving frame

In this work, the electron beam dynamics are calculated in the quantum mechanical framework. In this section, we will embark to calculate the expectation value of the integrand in Eq. (11). In the Section 3.1, we introduce the basic formulas of the density matrix method that is used to calculate the expectation value of any quantum mechanical operator. In the Section 3.2, we will apply the density matrix method to obtain the expectation value of \( \langle [\hat{J}_z - e_0 \hat{\rho} \partial (\partial U/\partial x) / \alpha] \rangle \). The dynamic of electrons in the moving frame

3.1. Density matrix for calculating the expectation value

To take into account the statistical properties of electrons, let us assign a number \( \nu \) for each electron in the electron beam. During the application of the electric field, the state vector of the \( \nu \)-th electron \( |\psi_{\nu}(t)\rangle \) evolves according to
\[
|\psi_{\nu}(t)\rangle = \sum_{m} \phi_{nm}(t) |\phi_{mn}(t)\rangle e^{-j\omega nt}, \tag{15}
\]
where \( \phi_{mn}(t) \) is the electron wave function of the \( m \)-th energy state and \( \alpha_{nm} \) is the angular frequency of the electron wave. The coefficient \( \alpha_{nm}(t) \) weights the contribution of the \( m \)-th energy state to the \( \nu \)-th electron.

Due to the wave-particle duality, a free electron can be considered as a single plane wave with a finite length corresponding to the finite localization in the space as [27]
\[
\phi_{mn}(t) = \frac{1}{\sqrt{V}} e^{ik_m r}, \tag{16}
\]
where \( k_m \) is the electron wavenumber of the \( m \)-th energy state. From Eq. (16), it is understood that an electron is described as a plane wave confined in a box (i.e., box-like plane wave model) with volume \( V = \ell_x \ell_y \ell_z \), \( \ell_x \), \( \ell_y \), and \( \ell_z \) are the spreading lengths of the electron wave in the x, y, and z directions, respectively. The longitudinal spreading length \( \ell_z \) is understood as the coherent length of the electron wave and corresponds to the separating distance between electrons.

In this paper, the density matrix method is used to calculate the expectation value of a quantum operator. For the interested reader, the density matrix method is reviewed in the Appendix of this paper. Further discussions on the density matrix can be found in Ref. [28]. Here, we quote the basic formulations shown in the Appendix. Using the density matrix formalism, the expectation value of an arbitrary quantum mechanical operator \( \mathcal{R} \) is given by (i.e., see Eq. (A8) in the Appendix)
\[
\langle \mathcal{R} \rangle = \sum_{n \neq m} \rho_{nm}(t) \mathcal{R}_{nm}, \tag{17}
\]
where \( \rho_{nm}(t) \) is the off-diagonal elements of the density matrix, and the dynamic equation of the density matrix is (i.e., see Eq. (A12) in the Appendix)
\[
\frac{d\rho}{dt} = \frac{i}{\hbar} [\mathcal{H} , \rho] - \frac{1}{2} \{ (\rho - \tilde{\rho}) \mathcal{I} + \mathcal{I} (\rho - \tilde{\rho}) \}, \tag{18}
\]
where \( [\mathcal{H} , \rho] \) is the commutator bracket between the total Hamiltonian operator of an electron \( \mathcal{H} \) and the density matrix operator \( \rho \). \( \mathcal{I} \) is an operator whose expectation value is the electron relaxation time \( \tau \) that characterizes the relaxation of electrons to their equilibrium positions. \( \rho \) is a distribution function of electrons at the thermal equilibrium. The operator \( \mathcal{I} \) and the distribution function \( \tilde{\rho} \) are assumed to have only the diagonal elements with eigenvalues as
\[
\mathcal{I} |\phi_{mn}(t)\rangle = \frac{1}{\ell} |\phi_{mn}(t)\rangle, \tag{19}
\]
\[
\tilde{\rho} |\phi_{mn}(t)\rangle = \rho_m |\phi_{mn}(t)\rangle. \tag{20}
\]
Here, it is instructive to comment on the electron relaxation process characterized by the relaxation time \( \tau \). When the initial velocity of electrons is perturbed, the separating distance between an electron and its neighbors will not be isotropic. Therefore, an electron will be subjected to asymmetric Coulomb’s forces from the neighboring electrons, and then the electron tends to relax to its initial state. The presence of the positive ions in (or around) the electron beam greatly influences the electron relaxation. In the current quantum mechanical analysis, the electron relaxation process is viewed as a phase distortion in the electron wave and represents the Coulomb repulsion forces between electrons [29–31].

Here, we introduce an expression for the Hamiltonian of an electron shown in Eq. (18). The Hamiltonian \( \mathcal{H} \) of the electron traveling in the presence of the laser field is given by
\[
\mathcal{H} = \frac{1}{2m_e} [\mathbf{p}_e + e\mathbf{A}_e(t)]^2 - eU, \tag{21}
\]
where \( \mathbf{p}_e = -j\hbar \nabla_z \) is the electron momentum operator. In some sense, considering the scalar potential in the single-electron Hamiltonian is almost similar to what is done in Ref. [20] for UFEs where the static field of the space-charge is considered in the Lorentz equation.

The total Hamiltonian \( \mathcal{H} \) shown in Eq. (21) can be divided into a principal Hamiltonian \( \mathcal{H}_0 \) and an interaction Hamiltonian \( \mathcal{H}_{int} \)
\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}. \tag{22}
\]
In Eq. (22), the principal Hamiltonian \( \mathcal{H}_0 \) is
\[
\mathcal{H}_0 = \frac{\mathbf{p}_e^2}{2m_e} - \frac{\hbar^2 \mathbf{V}_s^2}{2m_e}. \tag{23}
\]
The principal Hamiltonian \( \mathcal{H}_0 \) acts on the eigen energy state \( |\phi_{mn}(t)\rangle \) and results in an energy eigen value \( E_m \) for the state \( m \) in
the form of
\[ H_0|\phi_m(r)\rangle = E_m|\phi_m(r)\rangle, \]  
where
\[ E_m = \frac{\hbar^2 k_m^2}{2m_0} = \hbar \omega_m. \]  

During the interaction between the EM wave and electrons, the interaction Hamiltonian \( H_{\text{int}} \) is created. \( H_{\text{int}} \) is obtained from Eq. (21) in the form of
\[ H_{\text{int}} = \frac{e}{2m_0}[p_2A_{2z} + A_{2z}p_2] - eU. \]  

### 3.2. The expectation value of electrons current

Using the definition of the state vector \( |\psi_n^{\text{el}}(r, t)\rangle \), the electron density \( N \) is given by
\[ N = \sum_v \sum p^{2v}(1)|\psi_n^{\text{el}}(r, t)\rangle^2 = \sum_v \frac{p^{2v}}{V}. \]  

Recalling Eq. (15) and considering the relation \( i\hbar f_{\phi_n^{\text{el}}}(r)\phi_m(r) \) of \( \phi_m(r) \) as \( \delta_{nm} \), the electron density defined in Eq. (27) can be divided into two components
\[ N = N_1 + N_{\text{mod}} \]  
where \( N_1 \) is the average electron density and is given in terms of the diagonal elements of the energy eigen functions where
\[ N_1 = \sum_v \sum p^{2v}(1)|\psi_n^{\text{el}}(r, t)\rangle^2 = \sum_v \frac{p^{2v}}{V}. \]  

\( N_{\text{mod}} \) is the modulated electron density and is given in terms of the off-diagonal elements of the energy eigen functions in the form of
\[ N_{\text{mod}} = \sum_v \sum_{n \neq m} \sum \left\{ \frac{p^{2v}(1)|\psi_n^{\text{el}}(r, t)\rangle^2}{\rho_{nm}} \right\} |\phi_m(r)\rangle \langle \phi_n^{\text{el}}(r)|. \]  

The current density \( J_z \) is given by
\[ J_z = -\frac{eN}{m_0} \left[ p_2 + eA_{2z} \right]. \]  

Using Eq. (17) together with Eq. (31), considering the fact that \( eA_{2z} = eA_{2z} \), the expectation value \( \langle J_zT_x^2(x, y)\rangle \) is written as
\[ \langle J_zT_x^2(x, y)\rangle = \frac{eN}{m_0} \sum_{n \neq m} \rho_{nm}(t) \kappa_m |\phi_m(r)\rangle \langle T_x^2(x, y)| \phi_n^{\text{el}}(r)|. \]  

From Eq. (32), it is noted that the off-diagonal elements of the density matrix \( \rho_{nm} \) is the crucial parameter for obtaining \( \langle J_zT_x^2(x, y)\rangle \). Once we obtain \( \rho_{nm} \), the expectation value \( \langle J_zT_x^2(x, y)\rangle \) can be calculated. Next, we consider the motion equation of the density matrix given by Eq. (18).

Using Eq. (26), the off-diagonal element of \( H_{\text{int}} \) is given by
\[ \langle \phi_m(r)|H_{\text{int}}|\phi_n^{\text{el}}(r)\rangle = \frac{e\hbar k_n}{m_0} \langle \phi_m(r)|A_{2z}|\phi_n^{\text{el}}(r)\rangle - \langle \phi_m(r)|U|\phi_n^{\text{el}}(r)\rangle. \]  

In the Coulomb gauge framework, the scalar potential is obtained using Poisson equation that is
\[ \nabla^2 U = \frac{eN}{\varepsilon_0}. \]  

Considering that the spatial variation of the scalar potential \( U \) must be similar to that of \( N_{\text{mod}} \) and by substituting Eq. (30) into Eq. (34) with the help of Eq. (16), \( U \) is expressed by
\[ U = -\frac{e}{\varepsilon_0} \sum_v \sum_{n \neq m} \sum \left\{ \frac{p^{2v}(1)|\psi_n^{\text{el}}(r, t)\rangle^2}{(k_n - k_m)^2} \right\} \langle \phi_m(r)| \phi_n^{\text{el}}(r)|. \]  

Using Eqs. (29) and (35) and recalling the definition of \( \rho_{nm} \) shown in Eq. (A7) of the appendix, the off-diagonal element of the scalar potential is
\[ U_{nm} = \langle \phi_m(r)|U|\phi_n^{\text{el}}(r)\rangle \approx \frac{-eN_t\rho_{nm}}{\varepsilon_0(k_n - k_m)^2}. \]  

Using Eqs. (4) and (36), \( \langle \phi_m(r)|H_{\text{int}}|\phi_n^{\text{el}}(r)\rangle \) given by Eq. (33) becomes
\[ \langle \phi_m(r)|H_{\text{int}}|\phi_n^{\text{el}}(r)\rangle = \frac{e\hbar k_n}{m_0} \left\{ F(t, z)T_{mn}e^{\text{int}} + c.c \right\} + \frac{e^2N_t\rho_{nm}}{\varepsilon_0(k_n - k_m)^2}. \]  

where \( T_{mn} \) is an off-diagonal matrix element that represents the coupling efficiency between the EM wave and the electron wave, being given by
\[ T_{mn} = \langle \phi_m(r)|T_z(x, y)e^{-ij|\phi_n^{\text{el}}(r)|}. \]  

and its complex conjugate is
\[ T_{mn}^* = \langle \phi_m(r)|T_z^*(x, y)e^{ij|\phi_n^{\text{el}}(r)|}. \]  

Using Eq. (37), the exchange term between the interaction Hamiltonian \( H_{\text{int}} \) and the density matrix \( \rho \) in Eq. (18) becomes
\[ \frac{1}{J_{\rho}} \langle \phi_m(r)|H_{\text{int}} - \rho H_{\text{int}}|\phi_n^{\text{el}}(r)\rangle = -\frac{g(k_n + k_m)(\rho_{mn} - \rho_{nm})}{2m_0} \times \left\{ F(t, z)T_{mn}e^{\text{int}} + c.c \right\} - jG_{mn}\rho_{nm}. \]  

where \( G_{mn} \) is defined by
\[ G_{mn} = \frac{e^2N_t\rho_{mn} - e^2N_t\rho_{nm}}{\varepsilon_0(k_n - k_m)^2}. \]  

The coefficient \( G_{mn} \) obtained by counting the scalar potential in the Hamiltonian of an electron represents a crucial correction to our analytical models in the previously published work [32].

Using Eq. (24), the exchange term between \( H_0 \) and \( \rho \) in Eq. (20) is simply given by
\[ \frac{1}{J_{\rho}} \langle \phi_m(r)|H_0 - \rho H_0|\phi_n^{\text{el}}(r)\rangle = j\omega_{nm}\rho_{mn}. \]  

where
\[ \omega_{nm} = \frac{E_n - E_m}{\hbar}. \]  

By the help of Eqs. (19), (20), (40) and (42), the dynamic equation for the off-diagonal elements of the density matrix in Eq. (18) becomes
\[ \frac{d\rho_{mn}}{dt} = \left\{ j\omega_{mn} - j\frac{g(k_n + k_m)(\rho_{mn} - \rho_{nm})}{2m_0} \right\} \times \left\{ F(t, z)T_{mn}e^{\text{int}} + c.c \right\} - jG_{mn}\rho_{mn}. \]  

By assuming the solution of \( \rho_{mn} \) has the form
\[ \rho_{mn} = u(t)e^{i|\phi_n^{\text{el}}(r)|-\frac{1}{2}|t|^2}, \]  

we get
\[ \rho_{mn} = -\frac{g(k_n + k_m)(\rho_{mn} - \rho_{nm})}{2m_0} \left\{ \frac{F(t, z)e^{\text{int}} - e^{i|\phi_n^{\text{el}}(r)|-\frac{1}{2}|t|^2}}{\omega_{nm} + G_{mn} + 1/\varepsilon} + c.c \right\} + \frac{F(t, z)T_{mn}e^{\text{int}}}{|\omega_{nm} + G_{mn} + 1/\varepsilon|}. \]  

Using Eqs. (32) and (39), the expectation value of the term \( \langle J_zT_x^2(x, y)e^{\text{int}} \rangle \) can be rewritten as
\[ \langle J_zT_x^2(x, y)e^{\text{int}} \rangle = -\frac{eN_t}{m_0} \sum_{n \neq m} \sum \rho_{mn}(t) k_m T_{mn}^*. \]  

By substituting Eq. (46) into Eq. (47), Eq. (47) becomes
\[ \langle J_zT_x^2(x, y)e^{\text{int}} \rangle = \frac{e^2N_t}{2m_0} \sum_{n \neq m} \left\{ \frac{(k_n + k_m)(\rho_{mn} - \rho_{nm})}{\varepsilon_0(k_n - k_m)^2} \right\} \times \left\{ F(t, z)T_{mn}e^{\text{int}} + c.c \right\}. \]
In deriving Eq. (53), we use the relations of $k_\text{m} = k_\text{m}\text{a} \approx k_\text{m}\text{y}$ and $\omega_n - \omega_\text{m} = \omega$ and consider the first order perturbation where the diagonal element of the scalar potential in Eq. (35) is only taken into account. Using Eq. (46), Eq. (53) is rewritten as

$$\frac{\partial}{\partial t} T_2^2(x, y)e^{j\omega t} = \sum_{n = m} \frac{\rho_n}{\rho_m} \sum_{m \neq n} (k_m + k_n)(\rho_m - \rho_m)(|T_m|^2 F(t, z)) \times \left[ e^{j \omega t} - e^{j(\omega_n - \omega_m) - 1/\tau} \right].$$

Using similar notations of the energy levels as those used to calculate the expectation value in Eq. (51), Eq. (54) becomes

$$\left\langle \frac{\partial}{\partial t} T_2^2(x, y)e^{j\omega t} \right\rangle = \sum_{n = m} \sum_{m \neq n} \frac{\rho_n}{\rho_m} \sum_{m \neq n} \frac{(k_m + k_n)(\rho_m - \rho_m)(|T_m|^2 F(t, z)) \times \left[ e^{j \omega t} - e^{j(\omega_n - \omega_m) - 1/\tau} \right]}{[j(\omega - \omega_\text{m}) + \tau/2]} + (\rho_\text{cc} - \rho_\text{ba}) \text{Sinc}^2 \left[ \frac{k_\text{c} - k_\text{ba} - \beta (\omega_\text{c} - \omega_\text{ba})}{2} \right] \times \left[ e^{j \omega t} - e^{j(\omega_\text{c} - \omega_\text{ba}) - 1/\tau} \right]/[j(\omega - \omega_\text{ba} + \tau/2)].$$

### 4. The radiated power in the Cherenkov FEL

#### 4.1. Gain coefficient in the moving and laboratory frames

As evident from Eqs. (51) and (55), there are three parameters that can produce a broadening in the gain spectrum. These parameters are the spreading length of an electron wave $\epsilon_z$, the relaxation time $\tau$, and the interaction time $t$. In experiments of Refs. [12,13], the author and his group demonstrate that the inclusion of $\epsilon_z$ is an essential quantum correction to the classical theory where $\epsilon_z$ results in a wider broadening in the gain profile. The quantum contribution of the finite size of an electron becomes significant when $\epsilon_z$ is larger than the radiation wavelength. In [12,13], it is stated that the effect of the spreading length is more dominant for the CFELs operated in short wavelength range such as in the optical range. However, the broadness of the gain spectrum is determined by the interaction time (i.e., or by the relaxation time if it is shorter than the interaction time) when the spreading length of an electron wave is much shorter than the radiation wavelength. In this classical limit, the electron can be considered as a point-like particle and the classical theory is adequate to describe the spectrum broadening of the emission. The later case is typically valid for the CFELs operated from the microwave to the FIR range. In this paper, we mainly interest with the effects of the space-charges on Cherenkov radiation at millimeter and far infrared wave ranges. In these spectral ranges, the spreading length of an electron should be shorter than the radiation wavelength. Therefore, we will neglect the impacts of $\epsilon_z$ on the spectrum broadening.

Let us consider the interesting case of the efficient operation in CFELs that occurs at the maximum value of $|T_{mn}|^2$ given by Eq. (49) when

$$k_\text{m} - k_\text{m}\text{a} = \beta.$$ (56)

As a result of applying the momentum conservation rule shown in Eq. (56), the Sinc-squared functions in Eqs. (51) and (55) become equal to 1, and then the spreading length of electron wave has no contributions. In this case, the relaxation time and the interaction time, characterizing the uncertainty in the electron energy, determine the spectrum broadening. By neglecting the thermal distribution of the electron beam, we can assume

$$\rho_\text{nn} = \rho_\text{cc} = 0.$$ (57-a)
\( \rho_{bb} = \frac{1}{2} \)  

(57 – b)

Eq. (57-a) shows that there is a zero probability of finding an electron at final energy levels \( a \) and \( c \) before the transition process. Here, it is assumed that the energy separation of \( \hbar \omega \) is much larger than the thermal energy of the electron beam. We only use the forward exponent \( \exp(\hbar k_{rec}r) \) to represent the electron wave function in Eq. (16). Therefore, since the term that represents the backward electron wave (i.e., \( \exp(-\hbar k_{rec}r) \)) is dropped, we assume that the probability of finding an electron at the initial energy level \( b \) is \( 1/2 \), not 1.

From Eqs. (41), (56) and (57), the inclusion coefficients of the space-charge effects become

\[
G_{ba} = \frac{e^2 N_i}{2 \hbar \omega_{bb}^2} = G, \tag{58-a}
\]

\[
G_{cb} = \frac{-e^2 N_i}{2 \hbar \omega_{bb}^2} = -G. \tag{58-b}
\]

Using Eq. (56), the frequency difference between the transition states \( \omega_{ba} \) is

\[
\omega_{ba} = \frac{\hbar k^2_{ba}}{2m_0} \doteq \frac{\hbar \beta^2}{2m_0} (2k_{ba} - \beta) \approx \rho \tau_e - \frac{\hbar \beta^2}{2m_0} \tag{59-a}
\]

Similarly, we get

\[
\omega_{cb} = \frac{\hbar k^2_{cb}}{2m_0} \doteq \frac{\hbar \beta^2}{2m_0} (2k_{cb} + \beta) \approx \rho \tau_e + \frac{\hbar \beta^2}{2m_0}. \tag{59-b}
\]

Note that although \( k_{ba} \to \beta \) (or \( (\hbar \beta^2/2m_0) \approx \rho \tau_e \)), but the second term \( \hbar \beta^2/2m_0 \) in Eq. (59) is crucial in determining the gain. If the term \( \hbar \beta^2/2m_0 \) is neglected, no gain can be obtained.

By substituting Eqs. (51) and (55) into Eq. (11) and recalling Eqs. (56)-(58), the gain coefficient in the moving frame becomes

\[
\mathcal{G} = \frac{\mu_e^2 e^2 N_i \tau_e}{2 \hbar \omega_{ba}^2} [1 - \kappa] D. \tag{60}
\]

Where \( D \) is the dispersion function that determines the spectrum broadening in the amplification gain and is given by

\[
D = \text{Re} \left\{ \frac{\tau}{[(\omega - \omega_{ba} + \kappa \gamma \tau + 1)]} - \frac{\tau}{[(\omega - \omega_{cb} - \kappa \gamma \tau + 1)]} \right\}. \tag{61}
\]

In deriving Eq. (60), we define the averaged velocity by

\[
\tau_e = \frac{\hbar k_0}{m_0}. \tag{62}
\]

The scalar potential term given by Eq. (55) has the same value with an opposite sign with the term of the electron current density given by Eq. (51). We introduce the coefficient \( \kappa \) in Eqs. (60) and (61) to take into account the effects of the induced positive charges due to the passing electrons. Hence, \( \kappa \) must depend on the waveguide material and the gap distance between the dielectric and the beam. In the collective regime when the bunching of electrons continues to increase, the energy loss of electrons is distributed between the radiation and the space-charge field. It is understood that the coefficient \( \kappa \) determines the energy loss amount of electrons for the buildup of the space-charge field. In other words, \( \kappa \) characterizes the gain degradation due to the space-charge de-bunching mechanism. The coefficient \( \kappa \) takes a value from 0 to 1 where the smaller value leads to the more efficient operation. \( \kappa = 0 \) means that the number of positive charges is equal to that of the electron charges while \( \kappa = 1 \) means that there are no induced positive charges in the waveguide. Note that the scalar potential of the collective effects is appeared two times, first one in Eq. (7) (or Eq. (55)) and the second one in Eq. (33) (or Eq. (58)). Therefore, the coefficient \( \kappa \) also appears two times in Eqs. (60) and (61) to denote the effects of positive charges generation on the absolute value of the scalar potentials.

We end this section by introducing an expression for the gain coefficient in the laboratory frame. It is known that the radiated power is Lorentz invariant. However, as we go back to the laboratory frame, the involved parameters in the gain coefficient must be transformed to the laboratory frame using Lorentz transformations. For this purpose, we apply Lorentz transformations to the gain expression obtained in the moving frame and shown in Eq. (60). By using Eq. (1) and performing the transformations, \( \mathcal{G}_{lab} = \mathcal{G}_{rec} \), \( \omega = \omega / (1 + \mathcal{G}_{lab}/C_0) \), \( N_i \rightarrow N_i / \gamma \), and \( \tau \rightarrow \tau / \gamma \) in Eq. (60), we obtain the gain coefficient in the laboratory frame as

\[
\mathcal{G}_{lab} = \left[ \frac{\mu_e^2 e^2 N_i \tau_e^{2} \left( 1 + \mathcal{G}_{lab}/C_0 \right)}{2\hbar \omega_{ba}^2} \right] \left[ 1 - \kappa \right] D_{lab}. \tag{63}
\]

In Eq. (63), using Eqs. (58-b) and (59) and defining the interaction time as \( \Delta T = \tau / \gamma \), where \( \Delta T \) is the waveguide length, \( \mathcal{D}_{lab} \) is given by

\[
\mathcal{D}_{lab} = \left[ \text{Re} \left\{ \frac{\tau}{[(\omega - \omega_{ba} - \kappa \gamma \tau + 1)^{2}]^{1/2}} - \frac{\tau}{[(\omega - \omega_{cb} + \kappa \gamma \tau + 1)^{2}]^{1/2}} \right\} \right]. \tag{64}
\]

In Eq. (65), \( \tau = \tau / \gamma \) where \( \gamma \approx 1 \) for most practical cases. In Eq. (65), \( \omega_{rec} \) and \( \omega_{sp-ch} \) represent the change in the electron energy due to the electron recoil and the static potential, respectively. It is noted that the gain coefficient is influenced by the relaxation time \( \tau \) when \( \tau \ll \Delta T \), while it depends only on the interaction time \( \Delta T \) when \( \Delta T \ll \tau \). The effects of the relaxation time had been discussed in [33].

In the previous classical treatments of CFELs, the electron recoil \( \omega_{rec} \) and the potential energy of an electron \( \omega_{sp-ch} \) are neglected. In the current quantum mechanical treatment, the gain is critically determined by the summation term of the electron recoil and potential energies (i.e., \( \omega_{rec} + \omega_{sp-ch} \)). In Eqs. (65-a) and (65-b), it is understood that the single-electron model is dominant when \( \omega_{rec} > \omega_{sp-ch} \) while the collective model is dominant when \( \omega_{sp-ch} > \omega_{rec} \). In the collective limit when \( \omega_{sp-ch} > \omega_{rec} \), the gain coefficient no longer increases linearly with the electron density where a large number of electrons radiate coherently. From the expression of \( \omega_{sp-ch} \) shown in Eq. (65-c), the gain coefficient increases superlinearly with increasing the electron density and the radiation wavelength. Then, the high-gain CFEL features a much larger amplification than the low-gain CFEL. These predictions match with those of the UFELs [34]. In [34], it is stated that the UFELs are operated in the low gain regime when the electron beam current is low and the wavelength is short while the high gain regime is characterized by long wavelength and dense beam.

We close this section by summarizing the above results. The space-charge effects depend on the coefficient \( \kappa \) that reflects the degree of
the neutralization of the space-charges. The ion-neutralized electron beam can be induced as a natural response to the moving electrons where positive mirror charges are produced in the guide. Experimentally, the injection of positive ions into the electron beam enhances the neutralization of the space-charges [35]. Then, the values of $\kappa$ can be greatly modified. The large values of $\kappa$ (i.e., low neutralization degree) cause degradation in the amplification gain due to the electrons de-bunching. However, as we discussed above, the collective mode associates with the increase in the electron density where the CFELs could have an exponential growing wave (i.e., high gain limit). Therefore, the collective effects relate to two competitive processes; the first one is the enhancement in the bunching process that increases the gain. The second process characterized by the coefficient $\kappa$ is the loss of electron energy for building up of the electrons bunching. For the practical range of the electron density (i.e., $10^{12} \text{m}^{-3} < [N_{\text{lab}}] < 10^{18} \text{m}^{-3}$) and assuming $0.1 < \kappa < 1$, the single-electron (with no-space-charge effects) and collective interaction modes are typically applicable in the optical and microwave ranges, respectively. Mostly in the FIR spectral range, a transition regime between the single-particle and collective modes is observed. In the limit of $[\omega_{\text{sp}} - \omega_{\text{lab}}]_{\text{ch}}$, we will show in below that our results well match with the results based on the classical theory.

4.2. Numerical examples and discussions

A numerical example of the dispersion function $[D]_{\text{lab}}$ that determines the line shape of the gain coefficient is shown in Fig. 1. We define the propagation constant of radiation as $[\beta = 2n_{\text{eff}}(\omega)/\lambda]_{\text{lab}}$ where $n_{\text{eff}}(\omega)$ is the so-called the effective refractive index and is obtained from the dispersion function of the waveguide. In this example, we assume that $[\lambda]_{\text{lab}} = 10 \mu\text{m}$, $[N_{\text{lab}}] = 10^{12} \text{m}^{-3}$, $[n_{\text{eff}}]_{\text{lab}} = 3$, $[\kappa]_{\text{lab}} = 10^{-4}$, and the device length $[L]_{\text{lab}} = 5 \text{cm}$. Here, we neglect the effect of the relaxation time where we assume that $[\tau/\Delta T]_{\text{lab}} > 1$. In Fig. 1, the dispersion function $[D]_{\text{lab}}$ is drawn vs the dimensionless coefficient $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}}$. We vary $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}}$ in Fig. 1 by changing the accelerating voltage (or $V_{ch}$) of electrons. In Fig. 1 shows that the amplification gain is positive when $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}} > 0$ and it is negative when $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}} < 0$. The resonance condition at which no gain is obtained occurs when $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}} = 0$ or $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}} = 0$.

In Fig. 2 (a)–(c), we present the variation of the maximum values of the single-pass gain $[g_{\text{max}}(\omega) \times L]_{\text{lab}}$ with the electron density $[N_{\text{lab}}]$ for different values of radiation wavelength $[\lambda_{\text{lab}}]$. $[g_{\text{max}}(\omega)]_{\text{lab}}$ denotes the peak value of the gain coefficient $[g]_{\text{lab}}$ obtained by drawing $[g]_{\text{lab}}$ vs $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}}$. All other parameters values are same as given in Fig. 1. In Fig. 2 (a)–(c), the solid lines represent the cases for which the contribution of space-charge effects are counted (i.e., $[\omega_{\text{sp}} - \omega_{\text{ch}}]_{\text{lab}}$ is taken into account in Eqs. (64) and (65)). The dotted lines represent the no-space-charge single-electron approximation where the inclusion coefficient of the space-charges $[\omega_{\text{sp}} - \omega_{\text{ch}}]_{\text{lab}}$ is assumed to be 0 in Eqs. (74) and (75). The difference between the solid and dotted curves represents the influence degree of the collective effects. Fig. 2(a) shows that the space-charge contribution to the gain values is crucial when the radiation wavelength of 100 $\mu\text{m}$. This regime is defined as the collective regime. In general, we can say the collective effects are superior in the millimeter and submillimeter wavelengths range (i.e., when $[\lambda_{\text{lab}}] > 100 \mu\text{m}$). In the optical regime, say when $[\lambda_{\text{lab}}] = 1 \mu\text{m}$ as shown in Fig. 2(b), the collective effects have no contributions and then the no-space-charge single-electron model can be sufficiently applied for the Cherenkov laser. This regime is defined as the single-electron regime. At the wavelength of $[\lambda_{\text{lab}}] = 10 \mu\text{m}$ as shown in Fig. 2(c), the single-electron model is well applicable with no important contribution of the space-charge till the electron density reaches $10^{13} \text{m}^{-3}$. With increasing the electron density, the space-charge can significantly modify the values of the amplification gain predicted from the single-electron model. This regime is named as a transition regime between the single-particle mode and the collective mode. At high electron densities $> 10^{14} \text{m}^{-3}$, the space-charge corrections become unavoidable where $[\omega_{\text{sp}} - \omega_{\text{ch}}] = 200 \omega_{\text{rec}}$ and the collective model become dominant. It is noted that the gain when the beam passes becomes much larger than 1, the decrease in the average electron velocity causes a bunch shift from the optimum driving phase. In this work, the decrease in the amplification gain due to this mechanism of out-of-phase synchronization (i.e., saturation phenomena) is not considered. It is instructive to note that in Fig. 2 we consider the interesting case of high electron beam neutralization where $[\kappa]_{\text{lab}}$ takes a small value. In this case, the enhancement in the gain due to the increase of the electron density and the radiation wavelength becomes dominant. On the other hand, the gain degradation due to the de-bunching process (i.e., the Coulomb repulsive forces) is of less importance. Consequently, it is observed in Fig. 2(a) and (c) that the values of the gain coefficient in the collective mode can be higher than those predicted in the single-electron mode. These results are in agreement with those discussed in Refs. [35–37]. In [35–37], it is proposed that the partial “or full” neutralization of the electron space-charges weakens the repulsive forces among electrons and can generate higher output power with a noticeably enhanced growth rate.

Fig. 3 demonstrates the criteria by which the validity boundaries between the single-electron, collective, and transition regimes are determined. In Fig. 3, we show the variation of $[g \times L]_{\text{lab}}$ vs $[\beta_{\text{lab}} - \omega_{\text{rec}} - \omega_{\text{sp}} - \chi_{\text{lab}}]/C_{\text{lab}}$ for different values of the electron density. We assume the operating wavelength is $[\lambda]_{\text{lab}} = 10 \mu\text{m}$ corresponding to the case depicted in Fig. 2(c) and all other parameters values are kept unchanged. In the single-electron regime when $[N_{\text{lab}}] = 10^{12} \text{m}^{-3}$, the positive and negative peaks of the gain which associate with the emission and absorption processes occur when $[\beta_{\text{lab}} - \omega_{\text{rec}} = \pm \omega_{\text{rec}}]_{\text{lab}}$ where $[\omega_{\text{rec}}]_{\text{lab}}$ becomes comparable to $\omega_{\text{rec}}$ and the resonance peaks of the emission and absorption mechanisms are located at $[\beta_{\text{lab}} - \omega_{\text{rec}} = \pm \omega_{\text{rec}} + \omega_{\text{sp}} - \omega_{\text{ch}}]_{\text{lab}}$. In the collective regime when $[N_{\text{lab}}] > 10^{16} \text{m}^{-3}$, $[\omega_{\text{sp}} - \omega_{\text{ch}}]_{\text{lab}}$. Therefore, the positive and negative peaks of the gain always occur when $[\beta_{\text{lab}} - \omega_{\text{rec}} = \pm \omega_{\text{rec}} + \omega_{\text{sp}} - \omega_{\text{ch}}]_{\text{lab}}$.
In this section, we compare our obtained results based on quantum mechanical treatment with those given by Yariv and Shih [15] who used a classical treatment. Neglecting the effects of the space-charge, the authors in [15] determine the relative change in the single-electron and collective models become dominant are much less than and greater than \( \Delta \), respectively.

4.3. Comparison with the classical theory

In this section, we compare our obtained results based on quantum mechanical treatment with those given by Yariv and Shih [15] who used a classical treatment. Neglecting the effects of the space-charge, the authors in [15] determine the relative change in the electron beam power \( \Delta P/P \) during the interaction time \( \Delta t \lab \) as

\[
\frac{\Delta P}{P} \lab = \left[ \frac{n_0 e^2\gamma N_t L}{\tau^3 m_n} \times D_{\text{cl}} \right] \lab ,
\]

where \( D_{\text{cl}} \) is the no-space-charge dispersion function of the gain given by

\[
[D_{\text{cl}}] \lab = -\frac{d}{d\Omega} \left( \frac{\sin(\Omega/2)\sin(\Delta T/2)}{\Omega/2} \right) \lab ,
\]

and \( [D] \lab = [(\partial^2 \epsilon - \omega)\Delta T] \lab \) for the purpose of comparison with the classical theory, it is worth to express the quantum dispersion function \( D \) in terms of the classical dispersion function \( D_{\text{cl}} \) considering some approximations. In Eq. (64), assuming \( [\epsilon] \lab = -[(\partial^2 \epsilon - \omega)\Delta T] \lab \) and \( [\epsilon] \lab = [(\partial^2 \epsilon - \omega)\Delta T] \lab \) as

\[
[D] \lab = \left[ \frac{\epsilon - \partial T}{T} \right] \lab ,
\]

where \( D \lab \) becomes

\[
[D] \lab = \left[ \frac{2\Delta T}{T} \right] \lab \approx \left[ \frac{2\Delta T^2}{T} \right] \lab .
\]

By taking the approximation of \( d\sin(\Omega/2)/d\Omega \approx (8/5) [\sin(\Omega/2)/\Omega/2]^2 \) at the resonance peak of \([D] \lab \), we can express \([D] \lab \) in terms of \([D_{\text{cl}}] \lab \) as

\[
[D] \lab \approx \left[ \frac{8}{5} \right] \lab .
\]
Therefore, the gain coefficient in our model given by Eq. (63) can be reformulated as

$$g(\Delta z)_{lab} = \left[ g_{lab} e_{o}^{2} N_{t} e_{i} (1 + \frac{e_{i}}{e_{r}}) \right] \frac{1}{5^2 \mu_{0} \mu_{e} \mu_{r}^2} \left( 1 - \frac{\varepsilon_{r} m_{e}^{2} N_{t}}{h^2 \varepsilon_{0} \mu_{0}^{2}} \right) D_{c} \right]_{lab}. $$

(71)

In Ref. [15], the gain is defined as the power ratio between the power gained from electrons and the existing power such as emission except that it is independent of the existing radiation. Therefore, by comparing Eq. (76) with Eq. (71), the classical gain $g$ shown in Ref. [15] is almost equal to $[(4/5) (1 + \frac{e_{i}}{e_{r}}) g] \times L_{lab}$ in the case of neglecting the inclusions of the space-charge field (i.e., when $\kappa = 0$). In other words, the ratio between the gain in [15] and that obtained in our work is $[(4/5) (1 + \frac{e_{i}}{e_{r}})]_{lab}$. This is very satisfying, if one considers the vastly different approximations performed in these different calculations.

5. Inclusion of the spontaneous emission

From Eq. (10) and Eq. (14), the spatial growth rate of the radiated power is written as

$$\left[ \frac{dP}{dT} \right]_{lab} = \left[ g_{lab} - a_{lab} \right] P + C_{sp} \right]_{lab}, $$

(72)

where $C_{sp}$ corresponds to $\tilde{C}_{sp}$ in Eq. (10) and indicates the inclusion of the spontaneous emission power.

By defining the length $|\tilde{Z}|_{lab}$ during which the number of photons $S$ is emitted, the radiated power is given by

$$|P|_{lab} = \frac{h o S c}{|\tilde{Z}|_{lab}}. $$

(73)

It is noted that the number of photon $S$ is confined in finite volume having the cross-sectional area of the electron beam and length $|\tilde{Z}|_{lab}$. Neglecting the waveguide loss and using Eqs. (72) and (73), the time variation of the photon number takes the form

$$\left[ \frac{dS}{dT} \right]_{lab} = \left[ g_{lab} c S_{lab} + \frac{\tilde{C}_{sp}}{h o} \right]_{lab}. $$

(74)

Here, it is favorable to divide the gain coefficient $g_{lab}$ shown in Eq. (64) into two parts $[g_{0a} \right]_{lab}$ and $[g_{ch} \right]_{lab}$ corresponding to the emission and absorption mechanisms, respectively, in the form of

$$g_{lab} = \left[ g_{0a} \right]_{lab} - \left[ g_{ch} \right]_{lab}, $$

(75)

where

$$[g_{0a}]_{lab} = \left[ \frac{\mu_{0} e^{2} N_{t} e_{i} (1 + \frac{e_{i}}{e_{r}}) \left( 1 - k \right) \Re \left\{ \frac{1 - e^{i (\omega_{0a} - \kappa G - \omega) + \kappa G - \omega) + \kappa G - \omega \kappa G - \omega) + \kappa G - \omega \right]}{\left( \omega_{0a} - \kappa G - \omega \right) + \kappa G - \omega + \kappa G - \omega \right]} \right]_{lab}, $$

(76-a)

and

$$[g_{ch}]_{lab} = \left[ \frac{\mu_{0} e^{2} N_{t} e_{i} (1 + \frac{e_{i}}{e_{r}}) \left( 1 - k \right) \Re \left\{ \frac{1 - e^{i (\omega_{0a} - \kappa G - \omega) + \kappa G - \omega) + \kappa G - \omega \kappa G - \omega) + \kappa G - \omega \right]}{\left( \omega_{0a} - \kappa G - \omega \right) + \kappa G - \omega + \kappa G - \omega \right]} \right]_{lab}. $$

(76-b)

In the quantum mechanics, the emission and absorption processes are known as stimulated transitions when the electron transitions are proportional to the photon number $S$. Then, by recalling Eq. (75), the first term on the right-hand side of Eq. (74) represents both of the stimulated emission and absorption processes. The second term on the right-hand side of Eq. (74) represents the spontaneous emission process. The spontaneous emission has almost the same characteristics of the stimulated emission except that it is independent of the existing radiation field and corresponds to the zero-point energy for which $S = 1$. Hence, by comparing the first term and the second term on the right-hand side of Eq. (74), selecting the gain coefficient of the stimulated emission $[g_{0a}]_{lab}$ and putting $S = 1$, $[C_{sp}]_{lab}$ is obtained by

$$[C_{sp}]_{lab} = \frac{\hbar o c}{|\tilde{Z}|_{lab}}. $$

(77)

Since the length $|\tilde{Z}|_{lab}$ is defined as the length required to emit (or absorb) a photon number $S$, we can write

$$[\tilde{Z}]_{lab} = \frac{1}{|g_{lab}|}. $$

(78)

Substituting Eq. (78) into Eq. (77) with the help of Eq. (76-a), the spontaneous emission coefficient $[C_{sp}]_{lab}$ becomes

$$[C_{sp}]_{lab} = \frac{\hbar o c^{2} \left[ \frac{\mu_{0} e^{2} N_{t} e_{i} (1 + \frac{e_{i}}{e_{r}})}{2\tau} \left( \frac{1}{2} - k \right) \right]}{\left[ \frac{1}{(\omega_{0a} - \kappa G - \omega) + \kappa G - \omega + \kappa G - \omega \right]}^{2}}. $$

(79)

In Eq. (79), we use the relation of $\beta = 2n_{eff}/\lambda$.

In Fig. (4), numerical examples of the spontaneous emission coefficient $[C_{sp}]_{lab}$ is shown for different radiation wavelength. The horizontal axis of the figure is $[\tilde{Z}]_{lab}$, and putting $[N_{t}]_{lab} = 10^{15} m^{-3}$. As depicted in Fig. (4), $[C_{sp}]_{lab}$ has a similar line shape as that of the squared sinc function and shows a peak intensity under the condition of $[\tilde{Z}]_{lab} = 10^{15} m^{-3}$. The broadband and the absolute value of $[C_{sp}]_{lab}$ are determined by the radiation wavelength as well as by the interaction time.

6. Conclusions

The Cherenkov free-electron laser is studied using a treatment closely similar to that of a single-electron approximation. We basically treat the Schrödinger equation of a single electron where the electron is subjected to an electric force from the laser field and a static field from the surrounding modulated electrons. It is newly shown that the inclusions of the space-charge effects depend critically on the radiation wavelength and the dielectric material, not only on the electron density as known from the previous classical theory. A coefficient $k$ is introduced to parameterize the effects of space-charge de-bunching due to the presence of positive charges in the waveguide and/or in the electron beam. The space-charge effects increase with increasing this coefficient that takes a value from 0 to 1. For the practical range of the electron density and assuming $k \rightarrow 0$ (i.e., strong electron beam neutralization), it is shown that the single-electron model is well applicable in short-wavelength range such as in the optical region. On the other hand, the collective interaction

Fig. 4. Inclusion coefficient of the spontaneously emitted power per unit length $|C_{sp}]_{lab}$ vs dimensionless coefficient $(\tilde{Z})_{lab}$ for different values of operating wavelength.
mode becomes dominant in the microwave region (long-wavelength radiation) where the characteristic of the wave–wave interaction arises. In the FIR spectral range, the single-electron interaction mode evolves to the collective mode by increasing the electron density. In the last section, we introduce an expression for the inclusion of the spontaneous radiation power.

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Appendix

Assigning a number \( \nu \) for each electron in the electron beam. The expectation value of an arbitrary quantum mechanical operator \( \mathcal{R} \) is given by

\[
\langle \mathcal{R} \rangle = \sum_{\nu} \langle \Psi^{(\nu)}(r, t)|\mathcal{R}|\Psi^{(\nu)}(r, t) \rangle ,
\]

where \( P^{(\nu)} \) is the probability of finding the \( \nu \)-th electron in the ensemble where

\[
\sum_{\nu} P^{(\nu)} = 1.
\]

\( |\Psi^{(\nu)}(r, t)\rangle \) is the state vector of the \( \nu \)-th electron and satisfies the normalization condition

\[
\langle \Psi^{(\nu)}(r, t)|\Psi^{(\nu)}(r, t) \rangle = \int \sqrt{P^{(\nu)}(r, t)}dr = 1.
\]

Also, the eigen vector \( |\phi_{\nu}(r)\rangle \) satisfies the normalization and orthogonal conditions

\[
\langle \phi_{\nu}(r)|\phi_{\mu}(r) \rangle = \int [\phi_{\nu}(r)]^* \phi_{\mu}(r)dr = \delta_{\nu\mu}.
\]

Using Eq. (15), Eq. (A3), and Eq. (A4), we obtain

\[
\sum_{\nu} P^{(\nu)}(r, t) = 1.
\]

By substituting Eq. (15) into Eq. (A1), the expectation value of the operator \( \mathcal{R} \) is given by

\[
\langle \mathcal{R} \rangle = \sum_{\nu} \sum_{\mu} \sum_{n} \sum_{m} \left( P^{(\nu)}(r, t) |\phi_{\nu}(r) \rangle \langle \phi_{n}(r)|\mathcal{R}|\phi_{m}(r) \rangle \right) \delta_{\nu m}.
\]

By defining the matrix \( \rho \) whose matrix element is given by

\[
\rho_{nm}(t) = \sum_{\nu} \sum_{\mu} P^{(\nu)}(r, t) |\phi_{\nu}(r) \rangle \langle \phi_{n}(r)|\mathcal{R}|\phi_{m}(r) \rangle ,
\]

the expectation value of \( \mathcal{R} \) can be written as

\[
\langle \mathcal{R} \rangle = \sum_{n} \sum_{m} \rho_{nm}(t) \mathcal{R} \delta_{nm}.
\]

From Eq. (A8), the diagonal element of the density matrix \( \rho_{nm}(t) = \sum_{\nu} P^{(\nu)}(r, t) |\phi_{\nu}(r) \rangle \langle \phi_{n}(r)|\mathcal{R}|\phi_{m}(r) \rangle \) represents the total probability of finding the electron described by the state vector \( |\psi^{(\nu)}\rangle \) in the specific state \( |\phi_{n}\rangle \).

By using Eq. (15), we rewrite Eq. (A7) in the form of

\[
\rho_{nm}(t) = \left\langle \phi_{n}(r)|\left( \sum_{\nu} |\psi^{(\nu)}\rangle \langle \psi^{(\nu)}| \right)|\phi_{m}(r) \rangle.
\]

Therefore, the abstract density matrix \( \rho \) is given by

\[
\rho = \sum_{\nu} |\psi^{(\nu)}\rangle \langle \psi^{(\nu)}| \rho^{(\nu)}(t).
\]

The Schrödinger equation of the eigen vector \( |\psi^{(\nu)}(r, t)\rangle \) is

\[
\frac{\partial |\psi^{(\nu)}(r, t)\rangle}{\partial t} = \left( H - \frac{i\hbar}{\gamma} \right) |\psi^{(\nu)}(r, t)\rangle ,
\]

where the expectation value of the operator \( \gamma \) is the electron relaxation time \( r_0 \).

Using Eq. (A10) with the help of Eq. (A11), the dynamic equation of the density matrix is

\[
\frac{d\rho}{dt} = \frac{i}{\hbar} [H, \rho] - \frac{1}{2} \left( \rho - \rho^\dagger \right) \gamma + \gamma \left( \rho - \rho^\dagger \right).
\]

The second term on the right-hand side of Eq. (A12) is a phenomenological term that represents the relaxation of electrons to their equilibrium positions.

References