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Results on generalized fuzzy soft topological spaces

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In this manuscript, the concept of a generalized fuzzy soft point is introduced and some of its basic properties were studied. Also, the concepts of a generalized fuzzy soft base (subbase) and a generalized fuzzy soft subspace were introduced and some important theorems were established. Finally, the relationship between fuzzy soft set, intuitionistic fuzzy soft set, generalized fuzzy soft set and generalized intuitionistic fuzzy soft set were investigated.

Key words: Fuzzy soft set, generalized fuzzy soft set, generalized fuzzy soft topology, generalized fuzzy soft base (subbase), generalized fuzzy soft subspace, intuitionistic fuzzy soft set.

INTRODUCTION

Most of our real life problems in engineering, social and medical science, economics, environment, etc., involve imprecise data and their solutions involve the use of mathematical principles based on uncertainty and imprecision. To handle such uncertainties, Zadeh (1965) introduced the concept of fuzzy set (FS) and fuzzy set operations. The analytical part of fuzzy set theory was practically started with the paper of Chang (1968) who introduced the concept of fuzzy topological spaces. However, this theory is associated with an inherent limitation, which is the inadequacy of the parametrization tool associated with this theory as it was mentioned by Molodtsov (1999). Molodtsov (1999) introduced the concept of the soft set (SS) theory which is free from the aforementioned problems and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. Shabir and Naz (2011) studied the topological structures of soft sets. Intuitionistic fuzzy set theory was introduced by Atanassov (1986). In recent times, the process of fuzzification of soft set theory is rapidly progressed. Maji et al. (2001a, b) combined the theory of SS with the fuzzy and intuitionistic fuzzy set theory and called as fuzzy soft set (FSS) and intuitionistic fuzzy soft set (IFSS). Topological structure of fuzzy soft sets was started by Tanay and Burc Kandemir (2011). The study was pursued by some others researchers (Chakraborty et al., 2014; Gain et al., 2013; Mukherjee et al., 2015). Majumdar and Samanta (2010) introduced generalized fuzzy soft set (GFSS) and successfully applied their notion in a decision making problem. Yang (2011) pointed out that some results of Majumdar and Samanta (2010) are not valid in general. Chakraborty and Mukherjee (2015) introduced generalized fuzzy soft union, generalized fuzzy soft intersection and several

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other properties of generalized fuzzy soft sets. Also, they introduced generalized fuzzy soft topological spaces, generalized fuzzy soft closure, generalized fuzzy soft interior and studied some of their properties. Arora and Garg (2017a, b) solved the MCDM problem and established the IFSWA operator and the IFSWG operator under the IFSS environment. Garg (2017) introduced some series of averaging aggregation operators have been presented under the intuitionistic fuzzy environment. Garg and Arora (2017a, b) introduced distance and similarity measures for dual hesitant fuzzy soft sets i

**Preliminaries**

Here, the basic definitions and results which will be needed in the sequel were presented.

**Definition 1**

Let $X$ be a non-empty set. A fuzzy set $A$ in $X$ is defined by a membership function $\mu_A : X \rightarrow [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of $x$ in $A$ for $x \in X$. The set of all fuzzy sets in a set $X$ is denoted by $I^X$, where $I$ is the closed unit interval $[0,1]$.

**Theorem**

If $A, B \in I^X$, then, we have:

(i) $A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \forall x \in X$;
(ii) $A = B \iff \mu_A(x) = \mu_B(x), \forall x \in X$;
(iii) $C = A \cup B \iff \mu_C(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X$;
(iv) $D = A \cap B \iff \mu_D(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X$;
(v) $E = A^C \iff \mu_E(x) = 1 - \mu_A(x), \forall x \in X$.

**Definition 2**

Let $X$ be an initial universe and $E$ be a set of parameters (Molodtsov, 1999). Let $P(X)$ denotes the power set of $X$ and $A \subseteq E$. A pair $(f, A)$ is called a soft set over $X$ if $f$ is a mapping from $A$ into $P(X)$, that is, $f : A \rightarrow P(X)$.

In other words, a soft set is a parameterized family of subsets of the set $X$. For $e \in A$, $f(e)$ may be considered as the set of $e$—approximate elements of the soft set $(f, A)$.

**Definition 3**

Let $X$ be an initial universe and $E$ be a set of parameters. Let $I^X$ be the set of all fuzzy sets in $X$ and $A \subseteq E$. A pair $(F, A)$ is called a fuzzy soft set over $X$, where $F : A \rightarrow I^X$ is a function, that is, for each $e \in A$, $F(e) = F_\mu : X \rightarrow I$ is a fuzzy set in $X$.

**Definition 4**

Let $X$ be a universal set of elements and $E$ be a universal set of parameters for $X$ (Majumdar and Samanta, 2010). Let $F : E \rightarrow I^X$ and $\mu$ be a fuzzy subset of $E$, that is, $\mu : E \rightarrow I$. Let $F_\mu$ be the mapping $F_\mu : E \rightarrow I^X \times I$ defined as follows:

$$F_\mu(e) = (F(e), \mu(e)), \quad \text{where } F(e) \in I^X \text{ and } \mu(e) \in I.$$  

Then $F_\mu$ is called a generalised fuzzy soft set (GFSS in short) over $(X, E)$.

**Definition 5**

Let $F_\mu$ and $G_\delta$ be two GFSSs over $(X, E)$ (Majumdar and Samanta, 2010). Now $F_\mu$ is said to be a GFS subset of $G_\delta$, or $G_\delta$ is said to be a GFS super set of $F_\mu$ if:

(i) $\mu$ is a fuzzy subset of $\delta$;
(ii) $F(e)$ is also a fuzzy subset of $G(e), \forall e \in E$.

In this case, we write $F_\mu \subseteq G_\delta$.

**Definition 6**

Let $F_\mu$ be a GFSS over $(X, E)$. The complement of $F_\mu$, denoted by $F_\mu^c$, is defined by $F_\mu^c = G_\delta$, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e), \forall e \in E$ (Majumdar and Samanta, 2010). Obviously

$$F_\mu^c \subseteq G_\delta.$$  

**Definition 7**

Let $F_\mu$ and $G_\delta$ be two GFSSs over $(X, E)$ (Chakraborty and Mukherjee, 2015). The union of $F_\mu$ and $G_\delta$, denoted by $F_\mu \cup G_\delta$, is the GFSS $H(v)$, defined as
\[ H_\nu : E \rightarrow I^X \times I \text{ such that } H_\nu(e) = (H(e), \nu(e)), \]
where \[ H(e) = F(e) \lor G(e) \text{ and } \nu(e) = \mu(e) \lor \delta(e), \forall e \in E. \]

Let \( \{F_\mu^\lambda, \lambda \in \Lambda\} \), where \( \Lambda \) is an index set, be a family of GFSSs. The union of these family, denoted by \( \bigcup_{\lambda \in \Lambda}(F_\mu^\lambda) \), is the GFSS \( H_\nu \), defined as \[ H_\nu : E \rightarrow I^X \times I \text{ such that } H_\nu(e) = (H(e), \nu(e)), \]
where \[ H(e) = \vee_{\lambda \in \Lambda}(F(e))^\lambda, \text{ and } \nu(e) = \bigvee_{\lambda \in \Lambda}(\mu(e))^\lambda, \forall e \in E. \]

**Definition 8**

Let \( F_\mu \) and \( G_\delta \) be two GFSSs over \((X, E)\) (Chakraborty and Mukherjee, 2015). The intersection of \( F_\mu \) and \( G_\delta \), denoted by \( F_\mu \cap G_\delta \), is the GFSS \( M_\sigma \), defined as \[ M_\sigma : E \rightarrow I^X \times I \text{ such that } M_\sigma(e) = (M(e), \sigma(e)), \]
where \[ M(e) = F(e) \land G(e) \text{ and } \sigma(e) = \mu(e) \land \delta(e), \forall e \in E. \]

Let \( \{F_\mu^\lambda, \lambda \in \Lambda\} \), where \( \Lambda \) is an index set, be a family of GFSSs. The intersection of these family, denoted by \( \bigcap_{\lambda \in \Lambda}(F_\mu^\lambda) \), is the GFSS \( M_\sigma \), defined as \[ M_\sigma : E \rightarrow I^X \times I \text{ such that } M_\sigma(e) = (M(e), \sigma(e)), \]
where \[ M(e) = \wedge_{\lambda \in \Lambda}(F(e))^\lambda, \text{ and } \sigma(e) = \bigwedge_{\lambda \in \Lambda}(\mu(e))^\lambda, \forall e \in E. \]

**Definition 9**

A GFSS is said to be a generalized null fuzzy soft set, denoted by \( \emptyset_\nu \), if \( \emptyset_\nu : E \rightarrow I^X \times I \) such that \( \emptyset_\nu(e) = (\emptyset(e), \emptyset(e)) \) where \( \emptyset(e) = \emptyset, \forall e \in E \) and \( \emptyset(e) = 0, \forall e \in E \) (Where \( \emptyset(x) = 0, \forall x \in X \)) (Majumdar and Samanta, 2010).

**Definition 10**

A GFSS is said to be a generalized absolute fuzzy soft set, denoted by \( \text{Id}_\nu \), if \( \text{Id}_\nu : E \rightarrow I^X \times I \), where \( \text{Id}_\nu(e) = (\text{Id}(e), \text{Id}(e)) \) is defined by \( \text{Id}(e) = 1, \forall e \in E \) and \( \text{Id}(e) = 1, \forall e \in E \) (Where \( \text{Id}(x) = 1, \forall x \in X \)) (Majumdar and Samanta, 2010).

**Definition 11**

Let \( T \) be a collection of generalized fuzzy soft sets over \((X, E)\). Then \( T \) is said to be a generalized fuzzy soft topology (GFST, in short) over \((X, E)\) if the following conditions are satisfied:

(i) \( \emptyset_\nu \) and \( \text{Id}_\nu \) are in \( T \);
(ii) Arbitrary unions of members of \( T \) belong to \( T \);
(iii) Finite intersections of members of \( T \) belong to \( T \).

The triplet \((X, T, E)\) is called a generalized fuzzy soft topological space (GFST- space, in short) over \((X, E)\). The members of \( T \) are called a GFS open sets in \((X, T, E)\). The complement of a GFS open set is called GFS closed.

**Definition 12**

Let \((X, T, E)\) be a GFST-space and \( F_\mu \) be a GFSS over \((X, E)\). Then the generalized fuzzy soft closure of \( F_\mu \), denoted by \( \overline{F_\mu} \), is the intersection of all GFS closed supper sets of \( F_\mu \). Clearly, \( \overline{F_\mu} \) is the smallest GFS closed set over \((X, E)\) which contains \( F_\mu \) (Chakraborty and Mukherjee, 2015).

**Definition 13**

Let \( F_\mu \) be a GFSS over \((X, E)\) (Chakraborty and Mukherjee, 2015). We say that \((x_\alpha, e_\lambda) \in F_\mu \) read as \( x_\alpha, e_\lambda \) belongs to the GFSS \( F_\mu \) if \[ F(e)(x) = \alpha (0 < \alpha \leq 1) \text{ and } F(e)(y) = 0, \forall y \in X \setminus \{x\}, \mu(e) > \lambda. \]

**Definition 14**

A GFSS \( F_\mu \) in a GFST-space \((X, T, E)\) is called a generalized fuzzy soft neighborhood [GFS-nbd, in short] of \((x_\alpha, e_\lambda) \in F_\mu \) if there exists a GFS open set \( H_\nu \), such that \( G_\delta \subseteq H_\nu \subseteq F_\mu \) (Chakraborty and Mukherjee, 2015).

**Definition 15**

A GFSS \( F_\mu \) in a GFST-space \((X, T, E)\) is called a generalized fuzzy soft neighborhood of the generalized fuzzy soft point \((x_\alpha, e_\lambda) \in F_\mu \) if there exists a GFS open set \( G_\delta \) such that \( (x_\alpha, e_\lambda) \in G_\delta \subseteq F_\mu \).
Definition 16

Difference of two GFSS $F_\mu$ and $G_\delta$, denoted by $F_\mu \setminus G_\delta$, is a GFSS $H_\nu = F_\mu \setminus G_\delta^c$, defined as $H(e) = F(e) \wedge G^c(e)$ and $\nu(e) = \mu(e) \wedge \delta^c(e)$, $\forall e \in E$ (Mukherjee, 2015).

Definition 17

Let a set $E$ be fixed. An intuitionistic fuzzy set or IFS 'A' in $E$ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in E\}$ where the functions $\mu_A : E \rightarrow [0, 1]$ & $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and non-membership, respectively, of the element $x \in E$ to the set $A$ and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ (Atanassov, 1986).

Definition 18

Let $X$ be an initial universe and $E$ be a set of parameters (Maji et al., 2001a,b). Let $IF^X$ be the set of all intuitionistic fuzzy subsets of $X$ and $A \subseteq E$. Then, the pair $(F, A)$ is called an intuitionistic fuzzy soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow IF^X$.

For any $e \in A$, $F(e)$ is an intuitionistic fuzzy subset of $X$. Let us denote $\mu_{F(e)}(x)$ and $\nu_{F(e)}(x)$ by the membership degree and non-membership degree, respectively, that object $x$ holds parameter $e$, where $x \in X$ and $e \in A$. Then, $F(e)$ can be written as an intuitionistic fuzzy set such that $F(e) = \{(x, \mu_{F(e)}(x), \nu_{F(e)}(x)) : x \in X\}$.

Definition 19

Let $X$ be an initial universe and $E$ be a set of parameters. (Dinda et al., 2012). Let $IF^X$ be the set of all intuitionistic fuzzy subsets of $X$ and $A \subseteq E$. Let $F$ be a mapping given by $F : A \rightarrow IF^X$ and $\mu$ be a mapping given by $\mu : A \rightarrow [0, 1]$. Let $F_\mu$ be a mapping given by $F_\mu : A \rightarrow IF^X \times [0, 1]$ and defined by $F_\mu(e) = (F(e), \mu(e)) = ((\{x, \mu_{F(e)}(x), \nu_{F(e)}(x)\} : x \in X), \mu(e))$, where $e \in A$ and $x \in X$. Then, the pair $(F, A)$ is called a generalized intuitionistic fuzzy soft set over $(X, E)$.

GENERALIZED FUZZY SOFT POINTS AND NEIGHBORHOOD SYSTEMS

Here, a generalized fuzzy soft point was introduced and some of its basic properties were studied. Also, we discuss the concept of a neighborhood of a generalized fuzzy soft point in a generalized fuzzy soft topological space.

Definition 1

The generalized fuzzy soft set $F_\mu \in GFS(X, E')$ is called generalized fuzzy soft point (GFS point in short) if there exists the element $e \in E$ and $x \in X$ such that $F_\mu(e)(x) = \alpha \ (0 < \alpha \leq 1)$ and $F_\mu(e)(y) = 0$ for all $y \in X - \{x\}$ and $\mu(e) = \lambda \ (0 < \lambda \leq 1)$. This generalized fuzzy soft point was denoted $F_\mu = (x_\alpha, e_\lambda)$.

$(x, e)$ and $(\alpha, \lambda)$ are called, respectively the support and the value of $(x_\alpha, e_\lambda)$.

Definition 2

The complement of a generalized fuzzy soft point $(x_\alpha, e_\lambda)$, denoted by $(x_\alpha, e_\lambda)^c$, is defined as follows $(x_\alpha, e_\lambda)^c = (x_{1-\alpha}, e_{1-\lambda})$.

Example 1

Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$ the set of parameters. Then $(x_3^c(0.1), e_2^c(0.6)) = ((\{x_1^c, x_2^c, x_3^c, x_4^c\}, 0, 0.6))$ is a generalized fuzzy soft point whose complement is $(x_3(0.1), e_2(0.6)) = (\{x_1^c, x_2^c, x_3^c, x_4^c\}, 0.4))$.

The belongingness of a generalized fuzzy soft point to a generalized fuzzy soft set in Definition 13 as follows was redefined:

Definition 3

Let $F_\mu$ be a GFSS over $(X, E)$. We say that $(x_\alpha, e_\lambda) \in F_\mu$ read as $(x_\alpha, e_\lambda)$ belongs to the GFSS $F_\mu$ if for the element $e \in E$, $\alpha \leq F_\mu(e)(x)$ and $\lambda \leq \mu(e)$.

Definition 4

Let $(x_\alpha, e_\lambda)$ and $(y_\beta, e'_\delta)$ be two generalized fuzzy soft points, we say that $(x_\alpha, e_\lambda) \in (y_\beta, e'_\delta)$ if and only if $(x, e) = (y, e')$ and $\alpha < \beta$.
\( \lambda < \delta. \)

**Theorem 1**

Let \( F_\mu \) be a GFSS over \((X, E)\), then:

1. \( F_\mu \cup F_\mu^C \neq \mathbb{1}_\Delta; \)
2. \( F_\mu \cap F_\mu^C \neq \emptyset; \)
3. \( F_\mu \cup F_\mu^C \neq \mathbb{1}_\Delta; \)
4. \( F_\mu \cap F_\mu^C \neq \emptyset. \)

\[ F_\mu = \left\{ F_\mu(e_1) = \left\{ \left[ \frac{x_1}{0.2}, \frac{x_2}{0.7} \right], 0.3 \right\}, F_\mu(e_2) = \left\{ \left[ \frac{x_1}{0.5}, \frac{x_2}{0.4} \right], 0.9 \right\} \right\}, \]

\[ F_\mu^C = \left\{ F_\mu(e_1) = \left\{ \left[ \frac{x_1}{0.8}, \frac{x_2}{0.3} \right], 0.7 \right\}, F_\mu(e_2) = \left\{ \left[ \frac{x_1}{0.5}, \frac{x_2}{0.6} \right], 0.1 \right\} \right\}. \]

Then

\[ F_\mu \cup F_\mu^C = \left\{ F_\mu(e_1) = \left\{ \left[ \frac{x_1}{0.8}, \frac{x_2}{0.7} \right], 0.7 \right\}, F_\mu(e_2) = \left\{ \left[ \frac{x_1}{0.5}, \frac{x_2}{0.6} \right], 0.9 \right\} \right\} \neq \mathbb{1}_\Delta, \]

\[ F_\mu \cap F_\mu^C = \left\{ F_\mu(e_1) = \left\{ \left[ \frac{x_1}{0.8}, \frac{x_2}{0.3} \right], 0.3 \right\}, F_\mu(e_2) = \left\{ \left[ \frac{x_1}{0.5}, \frac{x_2}{0.6} \right], 0.1 \right\} \right\} \neq \emptyset. \]

**Example 2**

Let \( X = \{x_1, x_2\} \) and \( E = \{e_1, e_2\} \), consider the GFSS \( F_\mu \) over \((X, E)\), as:

\[ F_\mu = \left\{ F_\mu(e_1) = \left\{ \left[ \frac{x_1}{0.2}, \frac{x_2}{0.7} \right], 0.3 \right\}, F_\mu(e_2) = \left\{ \left[ \frac{x_1}{0.5}, \frac{x_2}{0.4} \right], 0.9 \right\} \right\}. \]

**Definition 5**

Let \((X, T, E)\) be a GFST-space. The set of all GFS neighborhoods of a generalized fuzzy soft point \((x_\alpha, e_\lambda)\) is called the GFS neighborhoods system of \((x_\alpha, e_\lambda)\) and is denoted by \( N_\mathcal{T}(x_\alpha, e_\lambda) \).

**Theorem 2**

Let GFSS \((X, E)\) be a family of all generalized fuzzy soft sets over soft universe \((X, E)\) and \((X, T, E)\) be GFST-space. Then the following properties are satisfied:

1. \( F_\mu = \bigcup_{(x_\alpha, e_\lambda) \in F_\mu} (x_\alpha, e_\lambda); \)
2. \( (x_\alpha, e_\lambda) \in \bigcup_{i \in J} (F_\mu)_i \) \( \Rightarrow \exists i_0 \in J \) such that \( (x_\alpha, e_\lambda) \in (F_\mu)_{i_0}; \)
3. \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) \( \Rightarrow \forall i \in J, (x_\alpha, e_\lambda) \in (F_\mu)_i; \)
4. \( (x_\alpha, e_\lambda) \in (x_\beta, e_\delta) \) and \( (x_\beta, e_\delta) \in \bigcap_{i \in J} (F_\mu)_i \).

Then \((x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) if and only if \((x_\alpha, e_\lambda) \in (x_\beta, e_\delta) \) for some \( i_0 \in J \).

(2) \( F_\mu \subseteq \bigcup_{i \in J} (F_\mu)_i \) if and only if \( F_\mu \subseteq \bigcup_{i \in J} (F_\mu)^C_i \).

(3) \( F_\mu \subseteq \bigcap_{i \in J} (F_\mu)_i \) if and only if \( F_\mu \subseteq (F_\mu)^C_i \).

(4) \( F_\mu \subseteq \bigcap_{i \in J} (F_\mu)_i \) if and only if \( F_\mu \subseteq (F_\mu)^C_i \).

(5) \( F_\mu \subseteq \bigcap_{i \in J} (F_\mu)_i \) if and only if \( F_\mu \subseteq (F_\mu)^C_i \).

**Proof 1**

(1) \( F_\mu \subseteq \bigcup_{(x_\alpha, e_\lambda) \in F_\mu} (x_\alpha, e_\lambda) \) straightforward. Let \( (x_\beta, e_\delta) \in \bigcup_{(x_\alpha, e_\lambda) \in F_\mu} (x_\alpha, e_\lambda) \), then there exists \( (x_\alpha, e_\lambda) \in F_\mu \) such that \( (x_\alpha, e_\lambda) \in (x_\beta, e_\delta) \).

Therefore, \( \beta < \alpha \), \( \delta < \lambda \) put \( \alpha' \leq F(e)(x) \) and \( \lambda' \leq \mu(e) \), then \( \beta \leq F(e)(x) \) and \( \delta \leq F(e)(x) \) shows that \( F_\mu \subseteq \bigcup_{i \in J} (F_\mu)_i \).

(2) Let \( (x_\alpha, e_\lambda) \in \bigcup_{i \in J} (F_\mu)_i \) = \( H_\nu \). Then \( \alpha \leq H_\nu(e)(x) = \bigvee_{i \in J} F(e)(x)_i \) and \( \delta \leq H_\nu(e)(x) \). Then there exists \( i_0 \in J \) such that \( \alpha \leq F(e)(x)_{i_0} \) and \( \lambda \leq \nu(e) = \bigvee_{i \in J} \mu(e)_{i_0} \forall e \in E, x \in X \).

Conversely, let \( (x_\alpha, e_\lambda) \in (F_\mu)_{i_0} \) such that \( (x_\alpha, e_\lambda) \in (F_\mu)_{i_0} \) for some \( i_0 \in J \) and \( \alpha \leq (F(e)(x))_{i_0} \) which implies that \( \alpha \leq \bigvee_{i \in J} (F(e)(x))_{i_0} \) and \( \lambda \leq \bigvee_{i \in J} \mu(e)_{i_0} \). Therefore \( \alpha \leq (F(e)(x))_{i_0} \).

(3) Let \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) = \( M_\sigma \). Then \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) if and only if \( (x_\alpha, e_\lambda) \in (F_\mu)_i \) for some \( i_0 \in J \).

(4) Let \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) for some \( i_0 \in J \). Then \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) if and only if \( (x_\alpha, e_\lambda) \in (F_\mu)_i \) for some \( i_0 \in J \).

(5) Let \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) for some \( i_0 \in J \). Then \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) if and only if \( (x_\alpha, e_\lambda) \in (F_\mu)_i \) for some \( i_0 \in J \).

(6) Let \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) for some \( i_0 \in J \). Then \( (x_\alpha, e_\lambda) \in \bigcap_{i \in J} (F_\mu)_i \) if and only if \( (x_\alpha, e_\lambda) \in (F_\mu)_i \) for some \( i_0 \in J \).
\[ \alpha \leq M(e)(x) = \bigwedge_{i \in E}(F(e)(x))_i \quad \text{and} \quad \lambda \leq \sigma(e) = \bigwedge_{i \in E}(\mu(e))_i \; . \]

Thus \( \alpha \leq (F(e)(x))_i \) and \( \lambda \leq (\mu(e))_i \). Therefore \((x_{\alpha}, e_{\lambda}) \in (F_{\mu})_i \quad \forall i \in J \). Conversely, let \((x_{\alpha}, e_{\lambda}) \in (F_{\mu})_i \quad \forall i \in J \). Then \( \alpha \leq (F(e)(x))_i \) and \( \lambda \leq (\mu(e))_i \). Therefore \( \alpha \leq \bigwedge_{i \in E}(F(e)(x))_i \) and \( \lambda \leq \bigwedge_{i \in E}(\mu(e))_i \). This implies that \( \alpha \leq M(e)(x) \) and \( \lambda \leq \sigma(e) \). Hence \((x_{\alpha}, e_{\lambda}) \in M_{\sigma} = \bigwedge_{i \in E}(F_{\mu})_i \).

(4) Let \((x_{\alpha}, e_{\lambda}) \in (F_{\mu})_i \), then \((x_{\alpha}, e_{\lambda}) \in (F_{\mu})_i \). This implies that \( \alpha \leq (F(e)(x))_i \) and \( \lambda \leq (\mu(e))_i \).\( \forall i \in J \). This implies that \( \alpha \leq \min\{(F(e)(x))_i : i \in J\} \) and \( \lambda \leq \min\{\mu(e)_i : i \in J\} \). Therefore

\[ F_{\mu} = \left\{ F_{\mu}(e) = \left( \left\{ \begin{array}{c}
\frac{e}{0.6, 0.5, 0.4} \\
\frac{e}{0.2, 0.3, 0.5}
\end{array} \right) \right) , 0.2 \right\} , \quad F_{\mu}(d) = \left( \left\{ \begin{array}{c}
\frac{d}{0.7, 0.5, 0.6} \\
\frac{d}{0.1, 0.3, 0.5}
\end{array} \right) \right) , 0.3 \right\} , \quad F_{\mu}(h) = \left( \left\{ \begin{array}{c}
\frac{h}{0.3, 0.4, 0.5} \\
\frac{h}{0.5, 0.3, 0.6}
\end{array} \right) \right) , 0.3 \right\} \]

\[ G_{\delta} = \left\{ G_{\delta}(e) = \left( \left\{ \begin{array}{c}
\frac{e}{0.7, 0.5, 0.6} \\
\frac{e}{0.2, 0.3, 0.3}
\end{array} \right) \right) , 0.1 \right\} , \quad G_{\delta}(d) = \left( \left\{ \begin{array}{c}
\frac{d}{0.1, 0.3, 0.5} \\
\frac{d}{0.2, 0.4, 0.5}
\end{array} \right) \right) , 0.2 \right\} , \quad G_{\delta}(h) = \left( \left\{ \begin{array}{c}
\frac{h}{0.3, 0.4, 0.5} \\
\frac{h}{0.5, 0.3, 0.6}
\end{array} \right) \right) , 0.3 \right\} \]

Consider \( T = \{ 0_{\theta} , 1_{\Delta} , F_{\mu} , G_{\delta} \} \). Then \( T \) forms GFS topology over \((X, E)\). Consider the following GFS over \((X, E)\).

\[ H_{\nu} = \left\{ H_{\nu}(e) = \left( \left\{ \begin{array}{c}
\frac{e}{0.7, 0.5, 0.6} \\
\frac{e}{0.2, 0.3, 0.4}
\end{array} \right) \right) , 0.3 \right\} , \quad H_{\nu}(d) = \left( \left\{ \begin{array}{c}
\frac{d}{0.3, 0.4, 0.5} \\
\frac{d}{0.2, 0.3, 0.6}
\end{array} \right) \right) , 0.5 \right\} , \quad H_{\nu}(h) = \left( \left\{ \begin{array}{c}
\frac{h}{0.5, 0.3, 0.6} \\
\frac{h}{0.7, 0.3, 0.6}
\end{array} \right) \right) , 0.8 \right\} \]

If \((x_{(0.4)}, e_{(0.1)}) \in 1_{\Delta} \), then there exists GFS open set \( F_{\mu} \) such that \((x_{(0.4)}, e_{(0.1)}) \in F_{\mu} \subseteq H_{\nu} \), that is, \( H_{\nu} \) is a GFS neighborhood of \((x_{(0.4)}, e_{(0.1)})\). Also, if \( M_{\sigma} \subseteq F_{\mu} \subseteq H_{\nu} \), then \( H_{\nu} \) is a GFS neighborhood of \( M_{\sigma} \in GFFS(X, E) \), where GFFS \((X, E)\) the family of all generalized fuzzy soft sets over \((X, E)\).

**GENERALIZED FUZZY SOFT BASE AND GENERALIZED FUZZY SOFT SUBBASE**

**Definition 1**

Let \((X, T, E)\) be GFS topological space. A collection \( R \) of generalized fuzzy soft sets over \((X, E)\) is called a generalized fuzzy soft open base or simply a base for generalized fuzzy soft topology on \((X, E)\), if the following conditions hold:

1. \( \emptyset_{\theta} \in R \)
2. \( \bigcup R = 1_{\Delta} \) i.e. for each \( e \in E \) and \( \chi \in X \), there exists \( R_{\mu} \in R \) such that \( R_{\mu}(e)(\chi) = 1 \) and \( \mu(e) = 1 \).
3. \( \text{If } F_{\mu} , G_{\delta} \in R , \text{ then for each } e \in E \text{ and } \chi \in X , \text{ there exists } H_{\nu} \in R \text{ such that } H_{\nu} \subseteq F_{\mu} \cap G_{\delta} \text{ and } H_{\nu}(e)(\chi) = \min\{ F(e)(\chi) , G(e)(\chi) \} \text{ and } v(e) = \min\{ \mu(e) , \delta(e) \} . \)

**Example 1**

Let \( X = \{ x_1 , x_2 , x_3 , x_4 \} , E = \{ e_1 , e_2 , e_3 , e_4 \} \). Let us consider the collection \( R = \{ \emptyset_{\theta} , 1_{\Delta} , G_{\delta} , H_{\nu} , K_{\gamma} , J_{\alpha} , M_{\eta} , N_{\psi} \} \).

\( G_{\delta} = \{ G_{\delta}(e_2) = \left( \left\{ \begin{array}{c}
\frac{e_2}{0.5} \\
\frac{e_2}{0.2}
\end{array} \right) \right) , 0.2 \right\} , \quad H_{\nu} = ( H_{\nu}(e_4) = \left( \left\{ \begin{array}{c}
\frac{e_4}{0.7} \\
\frac{e_4}{0.4}
\end{array} \right) \right) , 0.4 \right\} . \)
We can see that $s$ satisfies the conditions 1 to 3 of Definition (4-1). Therefore, $s$ forms a GFS base for a topology on $(X, E)$. 

Definition 2

$T_R = \{ 0_θ^0, 1_Δ^0, F_0^0, G_0^0, H_0^0, F_0^\Delta \bigcup G_0^\Delta (e_1) = \left( \frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{1}, 1 \right), F_0^\Delta \bigcup H_0^\Delta (e_2) = \left( \frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{1}, 1 \right), F_0^\Delta \bigcup H_0^\Delta (e_3) = \left( \frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{1}, 1 \right) \}$.

Example 2

Let $E = \{e_1, e_2, e_3\}$, $X = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{R} = \{0_\theta, F_\mu, G_\mu, H_\mu\}$, where $F_\mu = \{f_\mu(e_1) = \left( \frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{1}, 1 \right), f_\mu(e_2) = \left( \frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{1}, 1 \right), f_\mu(e_3) = \left( \frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{1}, 1 \right) \}$. Then obviously, $\mathcal{R}$ is a GFS base for a topology on $(X, E)$. The GFS topology generated by $\mathcal{R}$ is $T_\mathcal{R}$, where

Theorem 1

Let $(X, T, E)$ be a GFST-space and $\mathcal{R}$ be a subcollection of $T$ such that every member of $T$ is union of some members of $\mathcal{R}$. Then $\mathcal{R}$ is GFS base for the GFS topology $T$ on $(X, E)$.

Proof 1

Since $0_\theta \in T$, $0_\theta \in \mathcal{R}$. Again since $1_\Delta \in T$, $1_\Delta = U \mathcal{R}$. Let $(R_\mu)_1, (R_\mu)_2 \in \mathcal{R}$. Then $(R_\mu)_1, (R_\mu)_2 \in T$ and so $(R_\mu)_1 \cap (R_\mu)_2 \in T$. Then there exists $(R_\mu)_\alpha \in \mathcal{R}$, $\alpha \in \mathcal{V}$ such that $\cap (R_\mu)_\alpha \in \mathcal{R} \cap \mathcal{V}$. Therefore, $(R_\mu)_1(e) \cap (R_\mu)_2(e) = \bigcup (R_\mu)_\alpha (e) \in \mathcal{V}$. For $e \in E$. That is, for each $e \in E$ and $x \in X$, min $\{ (R_\mu)(x) \}, (R_\mu)(x) = \min \{ (R_\mu)(x) \}$, $\alpha \in \mathcal{V}$ and $\min \{ (\mu(e))^{(1)}, (\mu(e))^{(2)} = \min \{ (\mu(e))^{(\alpha)} \}$, $\alpha \in \mathcal{V}$. Therefore there exists $\alpha \in \mathcal{V}$ such that: $\min \{ (R_\mu)(x) \}, (R_\mu)(x) = (R_\mu)(x)^{\alpha}$ and $\min \{ (\mu(e))^{(1)}, (\mu(e))^{(2)} = (\mu(e))^{\alpha}$.

Definition 3

A collection $\Omega$ of some members of GFST-space $(X, T, E)$ is said to be a subbase of $T$ if and only if the collection of all finite intersection of members of $\Omega$ is a base for $T$.

Example 3

Let $X = \{x_1, x_2, x_3, x_4\}, E = \{e_1, e_2, e_3\}$ and $\mathcal{R} = \{f_\mu, M_\eta, N_\psi\}$, where
The collection of all finite intersection of members of $\Omega$ is the base $R$ in Example 1. So $\Omega$ is a subbase for a GFS topology on $(X, E)$.

**Theorem 2**

A collection $\Omega$ of GFSSs over $(X, E)$ is a subbase for a suitable GFS topology $T$ if and only if:

1. $\partial_\theta \in \Omega$ or $\partial_\theta$ is the intersection of finite number of members of $\Omega$.
2. $\mathbf{1}_\Delta = \bigcup \Omega$.

**Proof 2**

First let $\Omega$ be a subbase for $T$ and $R$ be a base generated by $\Omega$. Since $\partial_\theta \in R$, either $\partial_\theta \in \Omega$ or $\partial_\theta$ is expressible as an intersection of many finite members of $\Omega$. Now let $x \in X$ and $e \in E$. Since $\bigcup R = \mathbf{1}_\Delta$, there exists $R^*_\mu \in R$ such that $R^*_\mu(x) = 1$ and $\mu(e) = 1$.

Since $R^*_\mu \in R$, there exists $K^*_\mu \in \bigcap_{i=1}^n (K_i)$. Therefore $R^*_\mu(x) = \min_{i=1}^n (K_i(x))$, $\mu(e) = \min_{i=1}^n (\nu(e))$, and so $R^*_\mu(x) = (K_i(x))^n_i$, for some $i \in \{1, 2, ..., n\}$, $\mu(e) = (\nu(e))^n_i$, for some $i \in \{1, 2, ..., n\}$.

Thus $K_i(x) = 1$ and $\nu(e) = 1$. Hence $\mathbf{1}_\Delta = \bigcup \Omega$.

Conversely, let $\Omega$ be collection of GFSSs over $(X, E)$ satisfying the conditions 1 and 2. Let $R$ be the collection of all finite intersection of members of $\Omega$. Now it enough to show that $R$ forms base for suitable GFS topology. Since $R$ is the collection of all finite intersection of members of $\Omega$, by assumption (1) we get $\partial_\theta \in R$ and by (2) we get $\bigcup R = \mathbf{1}_\Delta$. Again let $F^*_\mu, G^*_\delta \in R$ and $x \in X$, $e \in E$. Since $F^*_\mu \in R$, there exists $(F^*_\mu)_i \in \Omega$ for $i = 1, 2, ..., n$ such that $F^*_\mu = \bigcap_{i=1}^n (F^*_\mu)_i$. Again since $G^*_\delta \in R$, there exists $G^*_\delta_j \in \Omega$ for $j = 1, 2, ..., m$ such that $G^*_\delta = \bigcap_{j=1}^m (G^*_\delta)_j$. Therefore, $F^*_\mu \cap G^*_\delta = \bigcap_{i=1}^n (F^*_\mu)_i \cap (\bigcap_{j=1}^m (G^*_\delta)_j) \in R$.

That is, $F^*_\mu \cap G^*_\delta \in R$. This completes the proof.

**GENERALIZED FUZZY SOFT TOPOLOGICAL SUBSPACES**

**Definition 1**

Let $(X, T, E)$ be a GFS topological space. Let $Y$ be an ordinary subset of $X$ and $Y^\phi$ be GFSS over $(Y, E)$ such that:

$\forall e \in E, Y(e)(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \not\in Y \end{cases}$

$Y(e) = Y, \forall e \in E, \nu(e) = 1$.

Let $T_Y = \{ Y^\phi \cap G^*_\delta : G^*_\delta \in T \}$. We can show that $T_Y$ is a GFS topology on $(Y, E)$ as follows:

(i) Since $\partial_\theta, \mathbf{1}_\Delta \in T$, $(\partial_\theta) \cap Y^\phi = Y^\phi \cap \partial_\theta$, then $(\partial_\theta) \cap \mathbf{1}_\Delta \in T_Y$.

(ii) Suppose that $(F^*_\mu)_1, (F^*_\mu)_2 \in T_Y$. Then for each $i = 1, 2$, there exist, $(G^*_\delta)_i \in T$ such that $(F^*_\mu)_i = Y^\phi \cap (G^*_\delta)_i$. We have $(F^*_\mu)_1 \cap (F^*_\mu)_2 = [Y^\phi \cap (G^*_\delta)_1] \cap [Y^\phi \cap (G^*_\delta)_2] = Y^\phi \cap ((G^*_\delta)_1 \cap (G^*_\delta)_2)$.

(iii) Let $\{ (G^*_\delta)_i \cap \} \in J$ be a subfamily of $T_Y$. Then for each $i \in J$, there is a GFSS $(M^*_\sigma)_i$ of $T$ such that $(G^*_\delta)_i = Y^\phi \cap (M^*_\sigma)_i$. We have $\bigcup_{i \in J} (G^*_\delta)_i = \bigcup_{i \in J} [Y^\phi \cap (M^*_\sigma)_i] = Y^\phi \cap (\bigcup_{i \in J} (M^*_\sigma)_i)$.

Since $\bigcup_{i \in J} (M^*_\sigma)_i \in T$, we have $\bigcup_{i \in J} (G^*_\delta)_i \in T_Y$. Then
Let $Y = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$. Consider $F_\mu$ and $G_\delta$ as follows:

$F_\mu = \{e_1 = (x_1, 0.8, 0.1), e_2 = (x_2, 0.9, 0.1), e_3 = (x_3, 0.8, 0.2), e_4 = (x_4, 0.65, 0.2)\}$

$G_\delta = \{e_1 = (x_1, 0.8, 0.2), e_2 = (x_2, 0.5, 0.3), e_3 = (x_3, 0.5, 0.4), e_4 = (x_4, 0.7, 0.2)\}$

We consider the GFS topology $T$ on $(Y, E)$ as

$T = \{\emptyset, Y, E\}$

Let $\mathcal{T}_\delta = \{0_\emptyset \cap Y, 1_\emptyset \cap Y, Y, \mathcal{G}_\delta \cap Y\}$, where

(i) $0_\emptyset \cap Y = (0_\emptyset)_Y$
(ii) $1_\emptyset \cap Y = (1_\emptyset)_Y$
(iii) $\mathcal{F}_\mu \cap Y = \mathcal{F}_\mu$
(iv) $\mathcal{G}_\delta \cap Y = \mathcal{G}_\delta$

Thus, the collection $\mathcal{T}_\delta = \{(0_\emptyset)_Y, (1_\emptyset)_Y, \mathcal{F}_\mu \cap Y, \mathcal{G}_\delta \cap Y\}$ is GFS a topology on $(Y, E)$.

**Theorem 1**

Let $(Y, T, E)$ be a GFS subspace of $(X, T, E)$ and $F_\mu$ a GFSS over $(Y, E)$. Then

(i) $F_\mu$ is GFS closed in $(Y, E)$ if and only if $F_\mu = Y \setminus G_\delta$ for some GFS closed set $G_\delta$ in $(Y, E)$.
(ii) $\text{cl}(F_\mu)_Y = \overline{F}_\mu \cap Y$, where $\text{cl}(F_\mu)_Y$ is the closure of $F_\mu$ in $(Y, E)$ with respect to $T_Y$.

**Proof 1**

(i) If $F_\mu$ is GFS closed in $(Y, E)$ then we have $F_\mu = Y \setminus G_\delta$, for some $G_\delta \in T_Y$.

Now, $G_\delta = Y \setminus M_\sigma$, for some $M_\sigma \in T$.

$F_\mu = Y \setminus (Y \setminus M_\sigma) = Y \setminus (Y \setminus M_\sigma)^c = Y \setminus (Y \cup M_\sigma^c) = (Y \cap Y') \cup (Y \cap M_\sigma^c) = Y \setminus M_\sigma^c$ (since $Y \cap Y' = (0_\emptyset)_Y$, see Definition 1) where $M_\sigma^c$ is GFS closed in $(X, E)$ as $M_\sigma \in T$. Conversely, assume

Example 1

Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2, e_3, e_4\}$. Consider $F_\mu$ and $G_\delta$ as follows:

$F_\mu = \{e_1 = (x_1, 0.8, 0.1), e_2 = (x_2, 0.9, 0.1), e_3 = (x_3, 0.8, 0.2), e_4 = (x_4, 0.65, 0.2)\}$

$G_\delta = \{e_1 = (x_1, 0.8, 0.2), e_2 = (x_2, 0.5, 0.3), e_3 = (x_3, 0.5, 0.4), e_4 = (x_4, 0.7, 0.2)\}$

It is useful to investigate the relationship between types of sets that are generalized fuzzy set.

**GENERALIZED FUZZY SOFT SET (GFSS) AND INTUITIONISTIC FUZZY SOFT SET(IFSS)**

**Lemma 1**

The relationships between the sets: FS, SS, FSS, IFSS, GFSS and GFIFSS that generalized the crisp set (CS) notion are illustrated in Figure 1.

**Counter example 1**

Let $X = \{x_1, x_2, x_3, x_4\}$, and $E = \{e_1, e_2, e_3\}$.

1. Let $f(E) = \{e_1 = ((x_1, 0.8, 0.1), (x_2, 0.9, 0.1), (x_3, 0.8, 0.1), (x_4, 0.7, 0.2))\}$.
2. Let $f(E) = \{e_1 = ((x_1, 0.8, 0.2), (x_2, 0.5, 0.3), (x_3, 0.5, 0.4), (x_4, 0.7, 0.2))\}$.
3. Let $f(E) = \{e_1 = ((x_1, 0.8, 0.2), (x_2, 0.5, 0.3), (x_3, 0.5, 0.4), (x_4, 0.7, 0.2))\}$.
4. Let $f(E) = \{e_1 = ((x_1, 0.8, 0.2), (x_2, 0.5, 0.3), (x_3, 0.5, 0.4), (x_4, 0.7, 0.2))\}$.

Note: The counter example is not a complete proof but rather an illustrative example to demonstrate the concept.
Let \( e_2 = \{(x_1, 0.6,0.3), (x_2, 0.65,0.2), (x_3, 0.7,0.2), (x_4, 0.65,0.2)\}, 0.6\}, \)
\( e_3 = \{(x_2, 0.8,0.2), (x_2, 0.5,0.3), (x_3, 0.5,0.4), (x_4, 0.7,0.2)\}, 0.5\} \) is GIFSS but neither IFSS nor GFSS.

From the examples, we see that GFSS and IFSS are independent notions.

**Lemma 2**

Similarly, one can be deduce similar diagram of the relationship between analogues topologies.

**Conclusion**

In this paper, we have introduced generalized fuzzy soft point, generalized fuzzy soft open base and subbase. The generalized fuzzy soft topological subspaces is introduced. Finally, we concluded that GFSS and IFSS are independent notions, whereas each of them is GIFSS. So, one can try to introduce some special properties of compactness, some separation axioms, connected nessetc on generalized fuzzy soft topological spaces.

**CONFLICT OF INTERESTS**

The authors have not declared any conflict of interests.

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