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TWO-SIDED INEQUALITIES FOR THE STRUVE AND LOMMEL FUNCTIONS

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Abstract. Mathematical inequalities and other results involving such widely- and extensively-studied special functions of mathematical physics and applied mathematics as (for example) the Bessel, Struve and Lommel functions as well as the associated hypergeometric functions are potentially useful in many seemingly diverse areas of applications, especially in situations in which these functions are involved in solutions of mathematical, physical and engineering problems which can be modeled by ordinary and partial differential equations. With this objective in view, our present investigation is motivated by some open problems involving inequalities for a number of particular forms of the hypergeometric function ${}_1F_2(a; b, c; z)$. Here, in this paper, we apply a novel approach to such problems and obtain presumably new two-sided inequalities for the Struve function, the associated Struve function and the modified Struve function by first investigating inequalities for the general hypergeometric function ${}_1F_2(a; b, c; z)$. We also briefly discuss the analogous new inequalities for the Lommel function under some conditions and constraints. Finally, as special cases of our main results, we deduce several inequalities for the modified Lommel function and the normalized Lommel function.

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Key words: Generalized hypergeometric function ${}_pF_q$, gamma function, Pochhammer symbol, gamma function, Mellin-Barnes contour integral, confluent hypergeometric functions, Struve functions, associated and modified Struve functions, Lommel functions, modified and normalized Lommel functions.

1. Introduction and motivation. The need for inequalities for various families of the special functions of Mathematical Physics and Applied Mathematics arises frequently in the literature in the mathematical, physical and engineering. Recently, this kind of inequalities have attracted the attention of many researchers because some of these potentially useful inequalities have been applied in a number of different problems. Further details can be found in several recent papers on the subject including (for example) [2, 3, 7, 8, 9, 10, 14, 15, 17, 19, 20, 21, 22] (see also [5, 6, 18, 23, 24, 26]) as well as in the references cited in each of these earlier investigations. In particular, inequalities for the hypergeometric and the confluent hypergeometric functions play an important rôle in such fields as Mathematical Analysis and Applied Mathematics, Number Theory, Physics and Mathematical Physics, and in various other areas of the physical, engineering and statistical sciences. These special functions allow us to solve many interesting problems which are usually modeled by ordinary and partial differential equations.

Under certain additional conditions and constraints, some two-sided inequalities for the confluent hypergeometric function were studied by Joshi and Bissu [11]. On the other hand, the Struve functions are related to the more familiar Bessel functions. Thus, naturally, many of the properties of the Struve functions are also potentially useful in problems of Mathematical Physics. In fact, Joshi and Nalwaya [12] presented some two-sided inequalities for the modified Struve functions and their ratios.

With the above-mentioned objective in view, our main motivation in this paper is to investigate a number of (presumably new) two-sided inequalities for the Struve and Lommel functions. We begin by presenting some definitions and preliminaries in Section 2. In Section 3, we describe and use a novel method which allows us to deal extensively with the hypergeometric functions and leads us to various two-sided inequalities for the hypergeometric function ${}_1F_2$. In Section 4, our main aim is to establish some inequalities for the Struve functions. We obtain some new two-sided inequalities for the associated Struve functions $W_\nu(z)$ in Section 5. We introduce another interesting extension of the two-sided inequalities for the modified Struve function $L_\nu(z)$ in Section 6. In Section 7, we describe new two-sided inequalities for the Lommel function $s_{\mu,\nu}(z)$ of the first kind. In Section 8, we obtain new two-sided inequalities for the modified Lommel function $T_{\mu,\nu}(z)$. Finally, some inequalities for the normalized Lommel function $h_{\mu,\nu}(z)$ are given in Section 9.

2. Definitions and preliminaries. In the usual notation, we use $\mathbb{Z}$ to denote the set of integers, while $\mathbb{R}$ denotes the set of *real* numbers, and $\mathbb{C}$ is used for the set of *complex* numbers z whose *real* and *imaginary* parts are denoted as $\Re(z)$ and $\Im(z)$, respectively. We also denote by $(\lambda)_\nu$ the *general* Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0;\ \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}),$$

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which is defined for $\lambda, \nu \in \mathbb{C}$, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{2.1}$$

In the definition (2.1), it is understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists. In our present investigation, we shall make use of the various special cases of the generalized hypergeometric function ${}_pF_q$ involving p numerator parameters

$$\alpha_j \in \mathbb{C} \qquad (j = 1, \cdots, p)$$

and q denominator parameters

$$\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \qquad (j = 1, \cdots, q)\,,$$

where

$$\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0,-1,-2,\cdots\}.$$

In fact, we have (see, for example, [14, Chapter 3] and [16, Chapter 4]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{matrix}\ z\right] &:= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!} \\ &= {}_pF_q\left(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z\right) \end{aligned} \tag{2.2}$$

$$\Big(p, q \in \mathbb{N}_0;\ p \leqq q+1;\ p \leqq q \quad \text{and} \quad |z| < \infty;$$

$$p = q+1 \quad \text{and} \quad |z| < 1;\ p = q+1,\ |z| = 1 \quad \text{and} \quad \Re(\omega) > 0\Big),$$

where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \quad \big(\alpha_j \in \mathbb{C}\ (j = 1, \cdots, p);\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1, \cdots, q)\big).$$

We now make use of the Mellin-Barnes contour integral for the Gamma function (see, for details, [14, p. 17, Equation 2.7(4)] and [25, p. 219, Equation 4.1(5)]):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} e^s\, s^{-z}\, \mathrm{d}s \qquad \big(\sigma > 0;\ \Re(z) > 0\big), \tag{2.3}$$

where the contour in the complex s-plane is of the familiar Mellin-Barnes type. Indeed, in conjunction with the definition (2.2), we find from (2.3) that

$${}_pF_{q+1}\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \lambda, \beta_1, \cdots, \beta_q; \end{matrix}\ -z\right] = \frac{\Gamma(\lambda)}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} e^s\, s^{-\lambda}\, {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{matrix}\ -\frac{z}{s}\right] \mathrm{d}s \tag{2.4}$$

$$\big(\sigma > 0;\ \Re(\lambda) > 0;\ p \leqq q+1\big),$$

provided that each member of (2.4) exists.

Remark 1. Henceforth, in this paper, we consider the confluent cases of the definition (2.2) when $p \leqq q$ for which the hypergeometric series would converge absolutely whenever $|z| < \infty$. Moreover, all parameters $a, b, c, \cdots$ and the argument z occurring in each of the inequalities considered here are tacitly assumed to be in the set $\mathbb{R}$ of real numbers.

Some of the known inequalities for the confluent hypergeometric function ${}_1F_1$, which we need in our investigation, are recalled here as Lemma 1 and Lemma 2 below.

Lemma 1. (see [16, Equations (5.3), (5.5) and (5.8)]) *Each of the following assertions holds true for the confluent hypergeometric function* ${}_1F_1(a;c;-z)$:

(i) *If* $c \geqq a > 0$ *and* $z > 0$, *then*

$$\begin{aligned} -1+2\left(1+\frac{a}{2c}\,z\right)^{-1} &< {}_1F_1(a;c;-z) \\ &< 1-\frac{a(c+1)}{c(a+1)}+\frac{a(c+1)}{c(a+1)}\left(1+\frac{a+1}{c+1}\,z\right)^{-1}. \end{aligned} \tag{2.5}$$

(ii) *If* $c \geqq a > 0$ *and* $z > 0$, *then*

$$\exp\left(-\frac{a}{c}\,z\right) < {}_1F_1(a;c;-z) < 1-\frac{a}{c}+\frac{a}{c}\,e^{-z}. \tag{2.6}$$

(iii) *If* $c \geqq a > 0$ *and* $z > 0$, *then*

$$1+\frac{a}{c}\,z\,\exp\left(\frac{a+1}{2(c+1)}\,z\right) < {}_1F_1(a;c;z) < 1+\frac{a}{c}\,z\left(1-\frac{a+1}{2(c+1)}+\frac{a+1}{2(c+1)}\,e^{z}\right). \tag{2.7}$$

Lemma 2. (see Joshi and Bissu [10, Equations (1.1) and (1.2)] and [11, Equations (2.4) and (2.5)]) *The following assertions hold true for the hypergeometric function* ${}_0F_1$:

(i) *If* $c > 0$ *and* $z > 0$, *then*

$$1-\frac{z}{c} < {}_0F_1(\,\text{—}\,;c;-z) < 1-\frac{z}{c}+\frac{z^2}{2c(c+1)}. \tag{2.8}$$

(ii) *If* $c > 0$ *and* $0 < z < 1$, *then*

$$1+\frac{z}{c} < {}_0F_1(\,\text{—}\,;c;z) < 1+\frac{2z}{c}. \tag{2.9}$$

3. Inequalities for the hypergeometric function ${}_1F_2$. In this section, we apply Lemma 1 and Lemma 2 in order to derive several interesting two-sided inequalities for the hypergeometric function ${}_1F_2$.

THEOREM 1. *Each of the following assertions holds true for the hypergeometric function* ${}_1F_2$:

(i) *If* $\min\{b,c\} \geqq a > 0$ *and* $z > 0$, *then*

$$\begin{aligned}-3+4\left(1+\frac{a}{4bc}z\right)^{-1} &< {}_1F_2(a;b,c;-z)\\ &< 1-\frac{a(b+1)(c+1)}{2bc(a+1)}+\frac{a(b+1)(c+1)}{2bc(a+1)}\left(1+\frac{2(a+1)}{(b+1)(c+1)}z\right)^{-1}. \qquad (3.1)\end{aligned}$$

(ii) *If* $\min\{b,c\} \geqq a > 0$ *and* $z > 0$, *then*

$$1-\frac{az}{bc} < {}_1F_2(a;b,c;-z) < 1-\frac{az}{bc}+\frac{az^2}{2bc(b+1)}. \qquad (3.2)$$

(iii) *If* $\min\{b,c\} \geqq a > 0$ *and* $0 < z < 1$, *then*

$$1+\frac{az}{bc}+\frac{a(a+1)z^2}{2bc(b+1)(c+1)} < {}_1F_2(a;b,c;z) < 1+\frac{az}{bc}+\frac{a(a+1)z^2}{bc(b+1)(c+1)}. \qquad (3.3)$$

Proof. Since, in general,

$$(1-z)^{-\mu}=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,z^n =: {}_1F_0(-\mu;\ —\ ;z) \qquad (|z|<1;\ \mu\in\mathbb{C}),$$

if we replace z in (2.5) by $\frac{z}{s}$, multiply both sides of the resulting equation by

$$\frac{\Gamma(b)}{2\pi\mathrm{i}}\,e^s\,s^{-b}\,\mathrm{d}s$$

and integrate each term along the above-mentioned Mellin-Barnes contour, we obtain

$$\begin{aligned}&-\frac{\Gamma(b)}{2\pi\mathrm{i}}\int_{\mathcal{C}}e^s\,s^{-b}\,\mathrm{d}s+\frac{2\Gamma(b)}{2\pi\mathrm{i}}\int_{\mathcal{C}}e^s\,s^{-b}\,{}_1F_0\left(1;\ —\ ;-\frac{az}{2cs}\right)\mathrm{d}s\\ &\quad<\frac{\Gamma(b)}{2\pi\mathrm{i}}\int_{\mathcal{C}}e^s\,s^{-b}\,{}_1F_1\left(a;c;-\frac{z}{s}\right)\mathrm{d}s\\ &\quad<\left(1-\frac{a(c+1)}{c(a+1)}\right)\frac{\Gamma(b)}{2\pi\mathrm{i}}\int_{\mathcal{C}}e^s\,s^{-b}\,\mathrm{d}s\\ &\qquad+\frac{a(c+1)}{c(a+1)}\frac{\Gamma(b)}{2\pi\mathrm{i}}\int_{\mathcal{C}}e^s\,s^{-b}\,{}_1F_0\left(1;\ —\ ;-\frac{a+1}{c+1}\,\frac{z}{s}\right)\mathrm{d}s. \qquad (3.4)\end{aligned}$$

Now, upon evaluating each of the contour integrals in (3.4) by appealing appropriately to the general result (2.4), we find that

$$-1+2\,{}_1F_1\left(1;b;-\frac{a}{2c}\,z\right) < \,{}_1F_2(a;b,c;-z)$$
$$< 1-\frac{a(c+1)}{c(a+1)}+\frac{a(c+1)}{c(a+1)}\,{}_1F_1\left(1;b;-\frac{a+1}{c+1}\,z\right),$$

which, in view of the known inequality (2.5), yields the assertion (3.1) of Theorem 1.

Next, in order to prove the assertion (3.2) of Theorem 1, we replace z in (2.6) by $\frac{z}{s}$, multiply both sides of the resulting equation by

$$\frac{\Gamma(b)}{2\pi\mathrm{i}}\,e^s\,s^{-b}\,\mathrm{d}s$$

and integrate each term along the above-mentioned Mellin-Barnes contour. Since

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=:\,{}_0F_0(\,\text{—}\,;\,\text{—}\,;z) \qquad (z\in\mathbb{C}),$$

by suitably making use of (2.4) as well as the known inequality (2.8), we arrive at the assertion (3.2) of Theorem 1.

Finally, the assertion (3.3) of Theorem 1 can be proven similarly by using (2.7), (2.4) and the known inequality (2.9). This evidently completes the proof of Theorem 1. □

Our next set of two-sided inequalities for the hypergeometric function ${}_1F_2$ are given by Theorem 2 below.

THEOREM 2. *Each of the following assertions holds true for the hypergeometric function* ${}_1F_2$:

(i) *If* $\min\{b,c\}>0$ *and* $z>0$, *then*

$$1-\frac{az}{bc} < \,{}_1F_2(a;b,c;-z) < 1-\frac{az}{bc}+\frac{a(a+1)z^2}{2bc(b+1)(c+1)}. \tag{3.5}$$

(ii) *If* $\min\{b,c\}>0$ *and* $0<z<1$, *then*

$$1+\frac{az}{bc} < \,{}_1F_2(a;b,c;z) < 1+\frac{2az}{bc}. \tag{3.6}$$

Proof. Our demonstration of Theorem 2 is based rather heavily upon the following integral representation:

$${}_1F_2(\alpha;\beta,\gamma;-z)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}\,(1-t)^{\beta-\alpha-1}\,{}_0F_1(\,\text{—}\,;c;-zt)\,\mathrm{d}t$$

$$\left(\Re(\beta) > \Re(\alpha) > 0;\ |z| < \infty\right).$$

Indeed, in order to prove the assertions (3.5) and (3.6), we make use also of the known inequalities (2.8) and (2.9), respectively. The details involved are being left as an exercise for the interested reader. □

EXAMPLE 1. The numerical utility of the above inequalities for the hypergeometric function ${}_1F_2$ is shown to manifest in the following table.

Table 1. Inequalities for the hypergeometric function ${}_1F_2$

Eq.	z	a	b	c	L	Function	R
	$z = 0.1$	$a = 0.1$	$b = 0.2$	$c = 0.2$	0.764705882352941	${}_1F_2(a;b,c;z)$	0.783132530120482
Eq.(3.1)	$z = 0.2$	$a = 0.2$	$b = 0.23$	$c = 0.25$	0.407407407407407	${}_1F_2(a;b,c;z)$	0.469856149991919
	$z = 0.2$	$a = 0.2$	$b = 0.25$	$c = 0.23$	0.407407407407407	${}_1F_2(a;b,c;z)$	0.469856149991919
	$z = 0.1$	$a = 0.1$	$b = 0.2$	$c = 0.2$	0.750000000000000	${}_1F_2(a;b,c;z)$	0.760416666666667
Eq.(3.2)	$z = 0.2$	$a = 0.2$	$b = 0.25$	$c = 0.23$	0.304347826086956	${}_1F_2(a;b,c;z)$	0.360000000000000
	$z = 0.2$	$a = 0.2$	$b = 0.23$	$c = 0.25$	0.304347826086956	${}_1F_2(a;b,c;z)$	0.360904913396960
	$z = 0.1$	$a = 0.1$	$b = 0.2$	$c = 0.2$	1.259548611111111	${}_1F_2(a;b,c;z)$	1.269097222222222
Eq.(3,3)	$z = 0.2$	$a = 0.2$	$b = 0.25$	$c = 0.23$	1.749946977730647	${}_1F_2(a;b,c;z)$	1.804241781548250
	$z = 0.2$	$a = 0.2$	$b = 0.23$	$c = 0.25$	1.749946977730647	${}_1F_2(a;b,c;z)$	1.804241781548250
	$z = 0.1$	$a = 0.1$	$b = 0.2$	$c = 0.2$	0.750000000000000	${}_1F_2(a;b,c;z)$	0.759548611111111
Eq.(3.5)	$z = 0.2$	$a = 0.2$	$b = 0.25$	$c = 0.23$	0.304347826086956	${}_1F_2(a;b,c;z)$	0.358642629904560
	$z = 0.2$	$a = 0.2$	$b = 0.23$	$c = 0.25$	0.304347826086956	${}_1F_2(a;b,c;z)$	0.358642629904560
	$z = 0.1$	$a = 0.1$	$b = 0.2$	$c = 0.2$	1.250000000000000	${}_1F_2(a;b,c;z)$	1.500000000000000
Eq.(3.6)	$z = 0.2$	$a = 0.2$	$b = 0.25$	$c = 0.23$	1.695652173913044	${}_1F_2(a;b,c;z)$	2.391304347826087
	$z = 0.2$	$a = 0.2$	$b = 0.23$	$c = 0.25$	1.695652173913044	${}_1F_2(a;b,c;z)$	2.391304347826087

REMARK 2. Numerous other inequalities for ${}_1F_2(a;b,c;z)$ can be found by enlarging upon the techniques and ideas enunciated in this section. The theorems developed herein seem to be reasonably sufficient to indicate the general nature of the expected results. We, therefore, do not pursue the subject any further. In fact, various inequalities can be found in this manner for hypergeometric functions of two and more variables. We choose to defer a detailed discussion on this point to a future paper.

4. Inequalities for the Struve function $\boldsymbol{H_\nu(z)}$.

In this section, we propose to investigate some inequalities for the Struve function $H_\nu(z)$ by using the inequalities of the preceding sections involving hypergeometric functions.

The Struve function $H_\nu(z)$ may be defined in the following hypergeometric form (see, for example, [17, p. 413, Equation (10.1.3)] and [25, p. 44, Equation 1.4(16)]):

$$H_\nu(z) = \frac{2}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} {}_1F_2\left(1; \nu+\frac{3}{2}, \frac{3}{2}; -\frac{z^2}{4}\right) \qquad \left(\nu > -\frac{3}{2}\right). \quad (4.1)$$

In view of the relationship in (4.1), the following result is an immediate consequence of Theorem 1. It is of prime importance to our studies.

THEOREM 3. *Each of the following assertions holds true for the Struve function $H_\nu(z)$:*

(i) *If* $\nu \geqq -\frac{1}{2}$ *and* $z > 0$, *then*

$$\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left[-3+4\left(1+\frac{z^2}{12(2\nu+3)}\right)^{-1}\right]<H_\nu(z)$$
$$<\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left[1-\frac{5(2\nu+5)}{12(2\nu+3)}+\frac{5(2\nu+5)}{12(2\nu+3)}\left(1+\frac{4z^2}{5(2\nu+5)}\right)^{-1}\right]. \tag{4.2}$$

(ii) *If* $\nu \geqq -\frac{1}{2}$ *and* $z>0$, *then*

$$\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left(1-\frac{z^2}{3(2\nu+3)}\right)<H_\nu(z)$$
$$<\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left(1-\frac{z^2}{3(2\nu+3)}+\frac{z^4}{12(2\nu+3)(2\nu+5)}\right). \tag{4.3}$$

(iii) *If* $\nu \geqq -\frac{1}{2}$ *and* $z>0$, *then*

$$\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left(1-\frac{z^2}{3(2\nu+3)}\right)<H_\nu(z)$$
$$<\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left(1-\frac{z^2}{3(2\nu+3)}+\frac{z^4}{60(2\nu+3)}\right). \tag{4.4}$$

Proof. By using the inequalities (3.1) and (3.2) for the hypergeometric function ${}_1F_2$, we get the assertions (4.2), (4.3) and (4.4) of Theorem 3. Indeed, upon setting

$$a=1, \quad b=\nu+\frac{3}{2}, \quad c=\frac{3}{2} \qquad \text{and} \qquad z\mapsto\frac{z^2}{4}$$

in the inequality (3.1),

$$a=1, \quad b=\nu+\frac{3}{2}, \quad c=\frac{3}{2} \qquad \text{and} \qquad z\mapsto\frac{z^2}{4}$$

in the inequality (3.2), and

$$a=1, \quad b=\frac{3}{2}, \quad c=\nu+\frac{3}{2} \qquad \text{and} \qquad z\mapsto\frac{z^2}{4}$$

in the inequality (3.2), we readily complete the proof of Theorem 3. □

The following result (Corollary 1) follows from the assertion (3.5) of Theorem 2 upon putting

$$a = 1, \quad b = \nu + \frac{3}{2}, \quad c = \frac{3}{2} \qquad \text{and} \qquad z \mapsto \frac{z^2}{4}$$

in the inequality (3.5).

COROLLARY 1. *For* $\nu > -\frac{3}{2}$ *and* $z > 0$, *the following inequality holds true for the Struve function* $H_\nu(z)$:

$$\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \left(1 - \frac{z^2}{3(2\nu+3)}\right) < H_\nu(z)$$
$$< \frac{2}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \left(\frac{z}{2}\right)^{\nu+1} \left(1 - \frac{z^2}{3(2\nu+3)} + \frac{z^4}{15(2\nu+3)(2\nu+5)}\right). \tag{4.5}$$

EXAMPLE 2. The numerical utility of these inequalities for the Struve function $H_\nu(z)$ is shown to manifest by the following table.

Table 2. Inequalities for the Struve function $H_\nu(z)$

Eq.	z	ν	L	Function	R
Eq. (4.2)	$z = 0.1$	$\nu = -0.5$	0.251892905259572	$H_{-0.5}(0.1)$	0.251893569480456
		$\nu = 0$	0.063591261349797	$H_0(0.1)$	0.063591354702550
		$\nu = 0.5$	0.012605151747689	$H_{0.5}(0.1)$	0.012605163556664
Eq. (4.3)	$z = 0.1$	$\nu = 0$	0.251892730115013	$H_{-0.5}(0.1)$	0.251892992941317
		$\nu = 0$	0.063591241706495	$H_0(0.1)$	0.063591277074260
		$\nu = 0.5$	0.012605149557926	$H_{0.5}(0.1)$	0.012605153938364
Eq. (4.4)	$z = 0.1$	$\nu = -0.5$	0.251892730115013	$H_{-0.5}(0.1)$	0.251892940376056
		$\nu = 0$	0.063591241706495	$H_0(0.1)$	0.063591277074260
		$\nu = 0.5$	0.012605149557926	$H_{0.5}(0.1)$	0.012605154814452
Eq. (4.5)	$z = 0.1$	$\nu = -1.49$	0.041044009597028	$H_{-1.49}(0.1)$	0.041052137123681
		$\nu = -1$	0.634497706459690	$H_{-1}(0.1)$	0.634499121170295
		$\nu = -0.5$	0.251892730115013	$H_{-0.5}(0.1)$	0.251892940376056
		$\nu = 0$	0.063591241706495	$H_0(0.1)$	0.063591270000707
		$\nu = 0.5$	0.012605149557926	$H_{0.5}(0.1)$	0.012605153062276

5. Inequalities for the associated Struve function $W_\nu(z)$. Here, in this section, we deal with the associated Struve functions $W_\nu(z)$ defined by (see Baricz and Yağmur [1, p. 142])

$$W_\nu(z) = \sqrt{\pi}\, 2^\nu\, z^{\frac{1-\nu}{2}}\, \Gamma\left(\nu+\frac{3}{2}\right)\, H_\nu(\sqrt{z}) = z\, {}_1F_2\left(1; \nu+\frac{3}{2}, \frac{3}{2}; -\frac{z}{4}\right) \quad \left(\nu > -\frac{3}{2}\right). \tag{5.1}$$

Our main results in this section are contained in Theorem 4 below.

THEOREM 4. *Each of the following assertions holds true for the associated Struve function* $W_\nu(z)$:

(i) *If* $\nu \geqq -\frac{1}{2}$ *and* $z > 0$, *then*

$$z\left[-3+4\left(1+\frac{z}{12(2\nu+3)}\right)^{-1}\right] < W_\nu(z)$$
$$< z\left[1-\frac{5(2\nu+5)}{12(2\nu+3)}+\frac{5(2\nu+5)}{12(2\nu+3)}\left(1+\frac{4z}{5(2\nu+5)}\right)^{-1}\right]. \tag{5.2}$$

(ii) *If* $\nu \geqq -\frac{1}{2}$ *and* $z > 0$, *then*

$$z\left(1-\frac{z}{3(2\nu+3)}\right) < W_\nu(z) < z\left(1-\frac{z}{3(2\nu+3)}+\frac{z^2}{12(2\nu+3)(2\nu+5)}\right). \tag{5.3}$$

(iii) *If* $\nu \geqq -\frac{1}{2}$ *and* $z > 0$, *then*

$$z\left(1-\frac{z}{3(2\nu+3)}\right) < W_\nu(z) < z\left(1-\frac{z}{3(2\nu+3)}+\frac{z^2}{60(2\nu+3)}\right). \tag{5.4}$$

Proof. The assertions (5.2), (5.3) and (5.4) of Theorem 4 would follow, respectively, when we set

$$a=1, \quad b=\nu+\frac{3}{2}, \quad c=\frac{3}{2} \qquad \text{and} \qquad z \mapsto \frac{z}{4}$$

in the inequality (3.1),

$$a=1, \quad b=\nu+\frac{3}{2}, \quad c=\frac{3}{2} \qquad \text{and} \qquad z \mapsto \frac{z}{4}$$

in the inequality (3.2) and

$$a=1, \quad b=\frac{3}{2}, \quad c=\nu+\frac{3}{2} \qquad \text{and} \qquad z \mapsto \frac{z}{4}$$

in the inequality (3.2). Our proof of Theorem 4 is thus completed. □

Corollary 2 below can be deduced by putting

$$a=1, \quad b=\nu+\frac{3}{2}, \quad c=\frac{3}{2} \qquad \text{and} \qquad z=\frac{z}{4}$$

in the inequality (3.5) and using the definition (5.1).

Corollary 2. *If* $\nu > -\dfrac{3}{2}$ *and* $z > 0$, *then*

$$z\left(1-\frac{z}{3(2\nu+3)}\right) < W_\nu(z) < z\left(1-\frac{z}{3(2\nu+3)}+\frac{z^2}{15(2\nu+3)(2\nu+5)}\right). \quad (5.5)$$

Example 3. The numerical utility of the above inequalities for associated Struve function $W_\nu(z)$ is shown to manifest by the following table.

Table 3. Inequalities for the associated Struve function $W_\nu(z)$

Eq.	z	ν	L	Function	R
Eq. (5.2)	$z=0.1$	$\nu=-0.5$	0.098340248962656	$W_{-0.5}(0.1)$	0.098366013071895
		$\nu=0$	0.098891966759003	$W_0(0.1)$	0.098906386701662
		$\nu=0.5$	0.099168399168399	$W_{0.5}(0.1)$	0.099177631578947
Eq. (5.3)	$z=0.1$	$\nu=-0.5$	0.098333333333333	$W_{-0.5}(0.1)$	0.098343750000000
		$\nu=0$	0.098888888888889	$W_0(0.1)$	0.098894444444444
		$\nu=0.5$	0.099166666666667	$W_{0.5}(0.1)$	0.099170138888889
Eq. (5.4)	$z=0.1$	$\nu=-0.5$	0.098333333333333	$W_{-0.5}(0.1)$	0.098341666666667
		$\nu=0$	0.098888888888889	$W_0(0.1)$	0.098894444444444
		$\nu=0.5$	0.099166666666667	$W_{0.5}(0.1)$	0.099170833333333
Eq. (5.5)	$z=0.1$	$\nu=-1.49$	-0.066666666666667	$W_{-1.49}(0.1)$	-0.065016501650165
		$\nu=-1$	0.096666666666667	$W_{-1}(0.1)$	0.096688888888889
		$\nu=-0.5$	0.098333333333333	$W_{-0.5}(0.1)$	0.098341666666667
		$\nu=0$	0.098888888888889	$W_0(0.1)$	0.098893333333333
		$\nu=0.5$	0.099166666666667	$W_{0.5}(0.1)$	0.099169444444444

6. Inequalities for the modified Struve function $L_\nu(z)$. The modified Struve function $L_\nu(z)$ is defined by (see Luke [17, p. 413, Equation (10.1.5)])

$$\begin{aligned} L_\nu(z) &= -\mathrm{i}e^{-\frac{\mathrm{i}\pi\nu}{2}} H_\nu(\mathrm{i}z) \\ &= \frac{2}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1} {}_1F_2\left(1;\nu+\frac{3}{2},\frac{3}{2};\frac{z^2}{4}\right) \qquad \left(\nu > -\frac{3}{2}\right). \quad (6.1)\end{aligned}$$

The main inequalities for the modified Struve function $L_\nu(z)$ are stated and proved as Theorem 5 below.

Theorem 5. *Under the conditions* $\nu > -\dfrac{3}{2}$ *and* $0 < z^2 < 4$, *the following assertion holds true for the modified Struve function* $L_\nu(z)$:

$$\begin{aligned} &\frac{2}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left(1+\frac{z^2}{3(2\nu+3)}+\frac{z^4}{15(2\nu+3)(2\nu+5)}\right) < L_\nu(z) \\ &\quad < \frac{2}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\left(1+\frac{z^2}{3(2\nu+3)}+\frac{2z^4}{15(2\nu+3)(2\nu+5)}\right). \quad (6.2)\end{aligned}$$

Proof. The assertion (6.2) of Theorem 5 follows when we set

$$a = 1, \quad b = \nu + \frac{3}{2}, \quad c = \frac{3}{2} \qquad \text{and} \qquad z \mapsto \frac{z^2}{4}$$

in (3.3). □

If we put

$$a = 1, \quad b = \nu + \frac{3}{2}, \quad c = \frac{3}{2} \qquad \text{and} \qquad z \mapsto \frac{z^2}{4}$$

in (3.6) and make use of the definition (6.1), we are led to Corollary 3 below.

COROLLARY 3. *If* $\nu > -\frac{3}{2}$ *and* $0 < z^2 < 4$, *then*

$$\frac{2}{\sqrt{\pi}\Gamma\left(\nu + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \left(1 + \frac{z^2}{3(2\nu+3)}\right) < L_\nu(z)$$
$$< \frac{2}{\sqrt{\pi}\Gamma\left(\nu + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \left(1 + \frac{2z^2}{3(2\nu+3)}\right). \tag{6.3}$$

EXAMPLE 4. The numerical utility of these inequalities for the modified Struve function $L_\nu(z)$ is shown to manifest by the following table.

Table 4. Inequalities for the modified Struve function $L_\nu(z)$

Eq.	z	ν	L	Function	R
Eq. (6.2)	$z = 0.1$	$\nu = -1.49$	0.057469740962492	$L_{-1.49}(0.1)$	0.057477868489145
		$\nu = -1$	0.638743252986079	$L_{-1}(0.1)$	0.638744667696684
		$\nu = -0.5$	0.252733984550063	$L_{-0.5}(0.1)$	0.252734194811106
		$\nu = 0$	0.063732741061233	$L_0(0.1)$	0.063732769355445
		$\nu = 0.5$	0.012626179166627	$L_{0.5}(0.1)$	0.012626182670977
Eq. (6.3)	$z = 0.1$	$\nu = -1.49$	0.057461613435839	$L_{-1.49}(0.1)$	0.065670415355245
		$\nu = -1$	0.638741838275473	$L_{-1}(0.1)$	0.640863904183365
		$\nu = -0.5$	0.252733774289019	$L_{-0.5}(0.1)$	0.253154296376023
		$\nu = 0$	0.063732712767021	$L_0(0.1)$	0.063803448297284
		$\nu = 0.5$	0.012626175662276	$L_{0.5}(0.1)$	0.012636688714451

7. Inequalities for the Lommel function $s_{\mu,\nu}(z)$ of the first kind. In this section, we derive some inequalities for the Lommel function $s_{\mu,\nu}(z)$ of the first kind (see Eugen von Lommel [13]), which is connected with the hypergeometric function ${}_1F_2$ as follows (see Luke [17, p. 413, Equation (10.1.1)]; see also Srivastava and Manocha [25, p. 44, Equation 1.4(13)]):

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\, {}_1F_2\left(1; \frac{1}{2}(\mu-\nu+3), \frac{1}{2}(\mu+\nu+3); -\frac{z^2}{4}\right) \tag{7.1}$$

$$(\mu \pm \nu > -3;\ \mu \pm \nu \neq -1).$$

THEOREM 6. *Each of the following assertions holds true for the Lommel function* $s_{\mu,\nu}(z)$:

(i) *If* $\mu-\nu>-1,\ \mu+\nu>-1$ *and* $z>0$, *then*

$$\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left[-3+4\left(1+\frac{z^2}{4(\mu-\nu+3)(\mu+\nu+3)}\right)^{-1}\right]<s_{\mu,\nu}(z)$$
$$<\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left[1-\frac{(\mu-\nu+5)(\mu+\nu+5)}{4(\mu-\nu+3)(\mu+\nu+3)}\right.$$
$$\left.+\frac{(\mu-\nu+5)(\mu+\nu+5)}{4(\mu-\nu+3)(\mu+\nu+3)}\left(1+\frac{4z^2}{(\mu-\nu+5)(\mu+\nu+5)}\right)^{-1}\right]. \tag{7.2}$$

(ii) *If* $\mu-\nu>-1,\ \mu+\nu>-1$ *and* $z>0$, *then*

$$\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1-\frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right)<s_{\mu,\nu}(z)$$
$$<\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1-\frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right.$$
$$\left.+\frac{z^4}{4(\mu-\nu+3)(\mu-\nu+5)(\mu+\nu+3)}\right). \tag{7.3}$$

(iii) *If* $\mu-\nu>-1,\ \mu+\nu>-1$ *and* $z>0$, *then*

$$\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1-\frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right)<s_{\mu,\nu}(z)$$
$$<\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left[1-\frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right.$$
$$\left.+\frac{z^4}{4(\mu-\nu+3)(\mu+\nu+3)(\mu+\nu+5)}\right]. \tag{7.4}$$

Proof. The first assertion (7.2) of Theorem 6 follows upon setting

$$a=1,\quad b=\frac{1}{2}(\mu-\nu+3),\quad c=\frac{1}{2}(\mu+\nu+3)\qquad\text{and}\qquad z\mapsto\frac{z^2}{4}$$

in (3.1).

The second assertion (7.3) and the third assertion (7.4) of Theorem 6 would follow when we put

$$a=1,\quad b=\frac{1}{2}(\mu-\nu+3),\quad c=\frac{1}{2}(\mu+\nu+3)\qquad\text{and}\qquad z\mapsto\frac{z^2}{4}$$

and

$$a = 1, \quad b = \frac{1}{2}(\mu+\nu+3), \quad c = \frac{1}{2}(\mu-\nu+3) \qquad \text{and} \qquad z \mapsto \frac{z^2}{4},$$

respectively, in the two-sided inequality (3.2). □

The demonstration of the following result similarly uses Theorem 2. The details involved may, therefore, be omitted.

COROLLARY 4. *Let $\mu \pm \nu > -3$, $\mu \pm \nu \neq -1$ and $z > 0$. Then*

$$\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1 - \frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right) < s_{\mu,\nu}(z) < \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1 - \frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)} + \frac{z^4}{(\mu-\nu+3)(\mu+\nu+3)(\mu-\nu+5)(\mu+\nu+5)}\right). \tag{7.5}$$

EXAMPLE 5. For instance, for the set of values $z = 0.1$, $\mu = 0.1, 0.2, 1, 2$ and $\nu = 0.1, 0.2, 1$, we have the inequalities for the Lommel function $s_{\mu,\nu}(z)$ from (7.2), (7.3), (7.4) and (7.5). The following table shows the manifestation of the numerical utility of the above inequalities for the Lommel function $s_{\mu,\nu}(z)$.

Table 5. Inequalities for the Lommel function $s_{\mu,\nu}(z)$

Eq.	z	μ	ν	L	Function	R
Eq. (7.2)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.055493143705825	$s_{0.2,0.1}(0.1)$	0.055493210673923
		$\mu = 0.2$	$\nu = 0.2$	0.045024207887523	$s_{0.2,0.2}(0.1)$	0.045024262422561
		$\mu = 0.1$	$\nu = 0.2$	0.053871641803259	$s_{0.1,0.2}(0.1)$	0.053871713746972
		$\mu = 0.1$	$\nu = 1$	0.047563756997719	$s_{0.1,1}(0.1)$	0.047563829257585
		$\mu = 1$	$\nu = 0.1$	0.019895527487596	$s_{1,0.1}(0.1)$	0.019895539363441
Eq. (7.3)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.055493130439624	$s_{0.2,0.1}(0.1)$	0.055493157056566
		$\mu = 0.1$	$\nu = 0.2$	0.053871627086342	$s_{0.1,0.2}(0.1)$	0.053871655836890
		$\mu = 0.2$	$\nu = 0.2$	0.045024197060596	$s_{0.2,0.2}(0.1)$	0.045024219152940
Eq. (7.4)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.055493130439624	$s_{0.2,0.1}(0.1)$	0.055493156052153
		$\mu = 0.1$	$\nu = 0.2$	0.053871627086342	$s_{0.1,0.2}(0.1)$	0.053871653667038
		$\mu = 0.2$	$\nu = 0.2$	0.045024197060596	$s_{0.2,0.2}(0.1)$	0.045024217516471
		$\mu = 2$	$\nu = 1$	0.001249479166667	$s_{2,1}(0.1)$	0.001249479329427
Eq. (7.5)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.055493130439624	$s_{0.2,0.1}(0.1)$	0.055493150527882
		$\mu = 0.1$	$\nu = 0.2$	0.053871627086342	$s_{0.1,0.2}(0.1)$	0.053871648784869
		$\mu = 0.2$	$\nu = 0.2$	0.045024197060596	$s_{0.2,0.2}(0.1)$	0.045024213425296
		$\mu = 2$	$\nu = 1$	0.001249479166667	$s_{2,1}(0.1)$	0.001249479275174

8. Inequalities for the modified Lommel function $t_{\mu,\nu}(z)$. In this section, we first define the modified Lommel function $t_{\mu,\nu}(z)$ as follows (see [27, p. 143, Equation (7)]):

$$t_{\mu,\nu}(z) = \mathrm{i}^{1-\mu}\, s_{\mu,\nu}(\mathrm{i}z)$$
$$= \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\ {}_1F_2\left(1; \frac{1}{2}(\mu-\nu+3), \frac{1}{2}(\mu+\nu+3); \frac{1}{4}z^2\right) \tag{8.1}$$
$$(\mu \pm \nu > -3;\ \mu \pm \nu \neq -1).$$

Each of the following results can be derived for the modified Lommel function $t_{\mu,\nu}(z)$ by appealing appropriately to the corresponding inequalities involving the hypergeometric function ${}_1F_2(a; b, c; z)$:

THEOREM 7. *If* $0 < z < 1$ *and* $\mu \pm \nu > -1$, *then*

$$\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1 + \frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)} + \frac{z^4}{(\mu-\nu+3)(\mu-\nu+5)(\mu+\nu+3)(\mu+\nu+5)}\right) < t_{\mu,\nu}(z)$$
$$< \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1 + \frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)} + \frac{2z^4}{(\mu-\nu+3)(\mu-\nu+5)(\mu+\nu+3)(\mu+\nu+5)}\right). \tag{8.2}$$

Proof. Here, in proving the assertion (8.2) of Theorem 7, we apply the inequality (3.3) asserted by Theorem 1. □

COROLLARY 5. *Let* $\mu-\nu > -3$, $\mu+\nu > -3$, $\mu-\nu \neq -1$, $\mu+\nu \neq -1$ *and* $0 < z < 1$. *Then*

$$\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1 + \frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right) < t_{\mu,\nu}(z)$$
$$< \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\left(1 + \frac{2z^2}{(\mu-\nu+3)(\mu+\nu+3)}\right). \tag{8.3}$$

Proof. For proving the assertion (8.3) of Corollary 5, we make use of the inequality (3.6) asserted by Corollary 2. □

EXAMPLE 6. In particular, for the set of values $z = 0.1$, $\mu = 0.1, 0.2, 2$ and $\nu = 0.1, 0.2, 1$, we have the inequalities for the modified Lommel function $t_{\mu,\nu}(z)$ from (8.2) and (8.3). The following table shows the manifestation of the numerical utility of the above inequalities for the Lommel function $t_{\mu,\nu}(z)$.

Table 6. Inequalities for the modified Lommel function $t_{\mu,\nu}(z)$

Eq.	z	μ	ν	L	Function	R
Eq. (8.2)	$z=0.1$	$\mu=0.2$	$\nu=0.1$	0.055601747652030	$t_{0.2,0.1}(0.1)$	0.055601767740289
		$\mu=0.1$	$\nu=0.2$	0.053984350933585	$t_{0.1,0.2}(0.1)$	0.053984372632112
		$\mu=0.2$	$\nu=0.2$	0.045112582801273	$t_{0.2,0.2}(0.1)$	0.045112599165973
		$\mu=2$	$\nu=1$	0.001250520941840	$t_{2,1}(0.1)$	0.001250521050347
Eq. (8.3)	$z=0.1$	$\mu=0.2$	$\nu=0.1$	0.055601727563772	$t_{0.2,0.1}(0.1)$	0.055656026125846
		$\mu=0.1$	$\nu=0.2$	0.053984329235058	$t_{0.1,0.2}(0.1)$	0.054040680309416
		$\mu=0.2$	$\nu=0.2$	0.045112566436574	$t_{0.2,0.2}(0.1)$	0.045156751124563
		$\mu=2$	$\nu=1$	0.001250520833333	$t_{2,1}(0.1)$	0.001251041666667

9. Inequalities for the normalized Lommel function $h_{\mu,\nu}(z)$. The normalized Lommel function $h_{\mu,\nu}(z)$ of the first kind is defined, in terms of the hypergeometric function ${}_1F_2$, as follows (see Çağlar and Deniz [4, p. 1190, Equation (4)]):

$$\begin{aligned} h_{\mu,\nu}(z) &= (\mu-\nu+1)(\mu+\nu+1)z^{\frac{1-\mu}{2}}\ s_{\mu,\nu}(\sqrt{z}) \\ &= z\ {}_1F_2\left(1; \frac{1}{2}(\mu-\nu+3), \frac{1}{2}(\mu+\nu+3); -\frac{1}{4}\ z\right) \qquad (\mu \pm \nu > -3). \end{aligned} \tag{9.1}$$

For the normalized Lommel function $h_{\mu,\nu}(z)$ defined by (9.1), we have the two-sided inequalities asserted by Theorem 8 and Corollary 6 below.

THEOREM 8. *Each of the following assertions holds true for the normalized Lommel function $h_{\mu,\nu}(z)$ of the first kind*:

(i) *If $\mu-\nu \geqq -1$, $\mu+\nu \geqq -1$ and $z>0$, then*

$$\begin{aligned} z\left[-3+4\left(1+\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}\right)^{-1}\right] &< h_{\mu,\nu}(z) \\ < z\Bigg[1-\frac{(\mu+\nu+5)(\mu-\nu+5)}{4(\mu-\nu+3)(\mu+\nu+3)} & \\ +\frac{(\mu-\nu+5)(\mu+\nu+5)}{4(\mu-\nu+3)(\mu+\nu+3)}&\left(1+\frac{4z}{(\mu-\nu+5)(\mu+\nu+5)}\right)^{-1}\Bigg]. \end{aligned} \tag{9.2}$$

(ii) *If $\mu-\nu \geqq -1$, $\mu+\nu \geqq -1$ and $z>0$, then*

$$\begin{aligned} z\left(1-\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}\right) &< h_{\mu,\nu}(z) \\ < z\left(1-\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}\right.&\left.+\frac{z^2}{4(\mu-\nu+3)(\mu-\nu+5)(\mu+\nu+3)}\right). \end{aligned} \tag{9.3}$$

(iii) *If* $\mu-\nu \geqq -1$, $\mu+\nu \geqq -1$ *and* $z>0$, *then*

$$z\left(1-\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}\right) < h_{\mu,\nu}(z) < z\left(1-\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}+\frac{z^2}{4(\mu-\nu+3)(\mu+\nu+3)(\mu+\nu+5)}\right). \tag{9.4}$$

Proof. First of all, if we put

$$a=1, \quad b=\frac{1}{2}(\mu-\nu+3), \quad c=\frac{1}{2}(\mu+\nu+3) \qquad \text{and} \qquad z\mapsto\frac{z}{4}$$

in (3.1), we obtain (9.2). Next, upon setting

$$a=1, \quad b=\frac{1}{2}(\mu-\nu+3), \quad c=\frac{1}{2}(\mu+\nu+3) \qquad \text{and} \qquad z\mapsto\frac{z}{4}$$

in (3.2), we get (9.3). Finally, if we choose

$$a=1, \quad b=\frac{1}{2}(\mu+\nu+3), \quad c=\frac{1}{2}(\mu-\nu+3) \qquad \text{and} \qquad z\mapsto\frac{z}{4}$$

in (3.2), we readily arrive at the third assertion (9.4) of Theorem 8. □

In a like manner, derive the inequality (9.4) below.

COROLLARY 6. *Let* $\mu-\nu>-3$, $\mu+\nu>-3$ *and* $z>0$. *Then*

$$z\left(1-\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}\right) < h_{\mu,\nu}(z) < z\left(1-\frac{z}{(\mu-\nu+3)(\mu+\nu+3)}+\frac{z^2}{(\mu-\nu+3)(\mu+\nu+3)(\mu-\nu+5)(\mu+\nu+5)}\right). \tag{9.5}$$

EXAMPLE 7. Here, in this example, we present our computation for the set of values $z=0.1$, $\mu=0.1, 0.2, 1, 2$ and $\nu=0.1, 0.2, 1, 2$ in the inequalities (9.2) and (9.3).

Table 7. Inequalities for the normalized Lommel function $h_{\mu,\nu}(z)$

Eq.	z	μ	ν	L	Function	R
Eq. (9.2)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.099024865919064	$h_{0.2,0.1}(0.1)$	0.099036737608821
		$\mu = 0.1$	$\nu = 0.2$	0.098957790515894	$h_{0.1,0.2}(0.1)$	0.098970918236541
		$\mu = 0.2$	$\nu = 0.2$	0.099022004889976	$h_{0.2,0.2}(0.1)$	0.099033920137398
		$\mu = 2$	$\nu = 1$	0.099583766909469	$h_{2,1}(0.1)$	0.099586776859504
Eq. (9.3)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.099022482893451	$h_{0.2,0.1}(0.1)$	0.099027274643973
		$\mu = 0.1$	$\nu = 0.2$	0.098955067920585	$h_{0.1,0.2}(0.1)$	0.098960399206705
		$\mu = 0.2$	$\nu = 0.2$	0.099019607843137	$h_{0.2,0.2}(0.1)$	0.099024509803922
		$\mu = 2$	$\nu = 1$	0.099583333333333	$h_{2,1}(0.1)$	0.099585069444444
Eq. (9.4)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.099022482893451	$h_{0.2,0.1}(0.1)$	0.099027093823199
		$\mu = 0.1$	$\nu = 0.2$	0.098955067920585	$h_{0.1,0.2}(0.1)$	0.098959996845488
		$\mu = 0.2$	$\nu = 0.2$	0.099019607843137	$h_{0.2,0.2}(0.1)$	0.099024146695715
		$\mu = 2$	$\nu = 1$	0.099583333333333	$h_{2,1}(0.1)$	0.099584635416667
		$\mu = 2$	$\nu = 2$	0.099523809523810	$h_{2,2}(0.1)$	0.099525132275132
Eq. (9.5)	z= 0.1	$\mu = 0.2$	$\nu = 0.1$	0.099022482893451	$h_{0.2,0.1}(0.1)$	0.099026099308939
		$\mu = 0.1$	$\nu = 0.2$	0.098955067920585	$h_{0.1,0.2}(0.1)$	0.098959091532751
		$\mu = 0.2$	$\nu = 0.2$	0.099019607843137	$h_{0.2,0.2}(0.1)$	0.099023238925200
		$\mu = 2$	$\nu = 1$	0.099583333333333	$h_{2,1}(0.1)$	0.099584201388889
		$\mu = 1$	$\nu = 0.1$	0.099523809523810	$h_{1,0.1}(0.1)$	0.09952486772486

REMARK 3. A number of other interesting inequalities can be derived by appealing appropriately to the methodology and technique which we have applied in our present investigation.

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