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On the Omega Distribution: Some Properties and Estimation

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Abstract: Explicit expressions for single moments, recurrence relations for single and product moments of order statistics of the omega distribution are derived. The L-moments are also obtained. We also consider different methods for estimating the model parameters, namely: maximum likelihood, maximum product of spacings, ordinary least-squares and weighted least-squares, percentiles, Anderson–Darling and right–tail Anderson–Darling. For different parameter settings and sample sizes, various simulation results are performed to compare the performance of the proposed estimators. Further, the method of maximum likelihood is adopted to estimate the omega parameters under type-II censoring scheme. The flexibility of the omega distribution is proved by means of a real data set.

Keywords: L-moments; Omega distribution; Order statistics; Parameter estimation; Type-II censored samples

1. Introduction

In the past few years, developing new statistical distributions are much in trend and several authors have proposed and discussed various extended forms of classical distributions with their applications in many fields. Moreover, it was found that the extended forms of well-known distributions provide greater flexibility in modeling data in different areas such as lifetime analysis, engineering, economics, finance, demography, actuarial and medical sciences.

Dombi *et al*. (2019) pioneered the omega distribution with three positive parameters and showed that its asymptotic distribution is just the Weibull distribution. They also obtained some mathematical properties of this distribution and proved that it allows us to model bathtub-shaped hazard function (hf) in two ways. First, the curve of the omega hf with special parameters can be utilized to describe a complete bathtub-shaped hazard curve. Second, the omega distribution can be applied in the same way as the Weibull distribution to model each phase of a bathtub-shaped hf. From the practical perspective, there are two notable properties of this distribution, namely, simplicity and flexibility. The simplicity follows because its cumulative distribution function (cdf) and hf include only power functions and lack exponential terms. The flexibility follows from the fact that the omega function has bounded support $(0, d)$, which means that its hf can be more appropriately to follow changes of $d > 0$, while the exponential function tends to infinity over an unbounded support.

Furthermore, Dombi *et al*. (2019) proposed two approaches for practical statistical estimation of the omega parameters: the first one depends on the log-likelihood function, the so-called GLOBAL method to maximize it, and the second depends on fitting its cdf to an empirical cdf. They also illustrated how the omega distribution can be adopted to model the distribution of the time-to-first-failure random variable if its hf is bathtub shaped. The authors also introduced two novel models with bathtub-shaped hf and demonstrated how the omega distribution can be applied in reliability theory using a practical example.

The probability density function (pdf) and cdf of the omega distribution with three parameters (α, β, d), say Omg(α, β, d), are

$$f(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ \frac{\alpha\,\beta\,d^{2\beta} x^{\beta-1}}{d^{2\beta} - x^{2\beta}} \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{-\frac{\alpha\,d^\beta}{2}}, & \text{if} \quad 0 < x < d, \\ 0, & \text{if} \quad x \geq d \end{cases} \tag{1}$$

and

$$F(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ 1 - \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{-\frac{\alpha\,d^\beta}{2}}, & \text{if} \quad 0 < x < d, \\ 1, & \text{if} \quad x \geq d, \end{cases} \tag{2}$$

respectively, where $\alpha > 0$, $\beta > 0$ and $d > 0$. The parameter α is the scale parameter and the maximum value of the density function increases with it. The parameter β affects the shape of the density function, and it is strictly monotonously decreasing when $\beta \in (0, 1)$, and unimodal when $\beta > 1$. Clearly, the parameter d specifies the support.

Henceforth, we denote by X a random variable with density (1). Notice that

$$(d^{2\beta} - x^{2\beta}) f(x) = \alpha\,\beta\,d^{2\beta}\,x^{\beta-1}[1 - F(x)]. \tag{3}$$

Okorie and Nadarajah (2019) derived closed-form expressions for the raw moments and quantile function of the omega distribution. The applications of moments of order statistics are quite well-known in statistical literature. For example, they are useful in statistical modelling, statistical inferences, decision procedures, nonparametric statistics, among others. Recurrence relations for single and product moments of order statistics for specific distributions have been established by several authors such as Malik *et al*. (1988), Balakrishnan *et al*. (1988), and Balakrishnan and Sultan (1998). Explicit expressions for moments of order statistics of some distributions were determined by Nadarajah (2008). For more results in this context, one may also refer to Nagaraja (2013), Çetinkaya and Genç (2018), Akhter *et al*. (2019), Akhter *et al*. (2020) and references therein.

In recent years, the importance of order statistics has increased because of the more frequent use of nonparametric inferences and robust procedures. The aim in this paper is to complete the works of Dombi *et al*. (2019) and Okorie and Nadarajah (2019) by deriving explicit expressions for single moments, recurrence relations for single and product moments of the order statistics of the omega distribution. The L-moments are also obtained.

These results can be used to derive the best linear unbiased estimators (BLUEs) and best linear invariant estimators (BLIEs) of the scale and location-scale parameters of the omega distribution as well as the best linear unbiased predictors (BLUPs) and best linear invariant predictors (BLIPs) of future unobserved order statistic; see, for example, Balakrishnan and Cohen (1991). Though many properties of the omega distribution have been discussed in the literature, the estimation of the parameters and prediction of future observations based on some types of ordered data and similar topics for this distribution have not been discussed yet, and we hope that the results of this paper will be useful in a large variety of problems related to order statistics and inferential investigations in the future.

The characterizations of probability distributions based on the moments of order statistics have also been of great interest to the researcher for the past several decades. So, it is important to mention that the findings of this paper can also be useful in the characterization of the omega distribution; see, for example, Lin (1989) and Kamps (1998). This will encourage researchers to do further works in the field of order statistics, especially for the omega distribution.

We also consider different methods for estimating the omega parameters, namely: maximum likelihood, maximum product of spacings, ordinary least-square and weighted least-square, percentiles, Anderson–Darling and right–tail Anderson–Darling. These methods are important to develop guidelines for choosing the best estimation method for the parameters, which would be of great interest to applied statisticians and practitioners. We also provide numerical simulations to examine the mean square errors of the proposed estimators. Furthermore, the method of maximum likelihood is adopted for estimating the distribution parameters under type-II censored samples.

Furthermore, the flexibility of the omega distribution is illustrated using a real-life data set. This distribution provides a better fit than ten extensions of the Weibull distribution (with three and four parameters), namely the modified Weibull due to Sarhan and Zaindin (2009), transmuted complementary Weibull-geometric by Afify *et al*. (2014), Lindley Weibull by Cordeiro *et al*. (2018), power generalized Weibull by Bagdonovacius and Nikulin (2002), alpha power Weibull by Nassar *et al*. (2017), alpha power exponentiated-Weibull by Mead *et al*. (2019), exponentiated-Weibull due to Mudholkar and Srivastava (1993), extended odd Weibull exponential by Afify and Mohamed (2020), logarithmic transformed Weibull by Nassar *et al*. (2020), and Weibull distributions.

This paper is organized as follows. In Section 2, we derive explicit expressions and some recurrence relations for the single and product moments of order statistics from the omega distribution. Some of its statistical properties are obtained in Section 3. Five estimation methods for the omega parameters are discussed in Section 4. A simulation study is performed in Section 5, and some conclusions are offered in Section 6.

2. Single and product moments of order statistics

Let $X_1, \cdots, X_n$ be a random sample of size n from the omega distribution with the pdf $f(x)$ and cdf $F(x)$, given in (1) and (2), respectively, and let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Then, the pdf of the rth order statistic $X_{r:n}$, say $f_{r:n}(x)$, is (David and Nagaraja, 2003; Arnold *et al*., 2008) (for $1 \leq r \leq n$)

$$f_{r:n}(x) = C_{r:n}\, F(x)^{r-1}\,[1 - F(x)]^{n-r} f(x), \;\; 0 < x < d, \tag{4}$$

and the joint pdf of the rth ($X_{r:n}$) and sth ($X_{s:n}$) order statistics, say $f_{r,s:n}(x,y)$, can be expressed as (for $1 \leq r < s \leq n$)

$$f_{r,s:n}(x,y) = C_{r,s:n}\, F(x)^{r-1}\,[F(y) - F(x)]^{s-r-1}\,[1 - F(y)]^{n-s}\, f(x)\, f(y), \;\; 0 < x < y < d, \tag{5}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} \quad \text{and} \quad C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Next, the kth single moment of $X_{r:n}$ takes the form

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = \int_0^d x^k\, f_{r:n}(x)dx; \;\; 1 \leq r \leq n; \; k \in \mathbb{N}, \tag{6}$$

and the (k,l)th product moment of $X_{r:n}$ and $X_{s:n}$ reduces to

$$\mu_{r,s:n}^{(k,l)} = E(X_{r:n}^{k}X_{s:n}^{l}) = \int_0^d\int_x^d x^k y^l f_{r,s:n}(x,y)dydx, \;\; 1 \le r < s \le n, \; k,l \in \mathbb{N}. \tag{7}$$

Further, we use the integral formula of Gradshteyn and Ryzhik (2007)

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\eta x)^{-\nu}dx = B(\lambda,\nu)\; {}_2F_1\left(\nu,\lambda;\lambda+\mu;\eta\right), \; \lambda > 0, \; \mu > 0, \tag{8}$$

to prove some results of this paper, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \;\text{ and }\; {}_2F_1(a,b;c;x) = \sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(c)_p}\frac{x^p}{p!}$$

are the beta and Gauss hypergeometric functions, respectively, and $(e)_p = e(e+1)\cdots(e+p+1)$ is the ascending factorial.

2.1. Single moments

We obtain the single moments of the order statistics from the omega distribution. The results are presented in the following theorems.

Theorem 2.1: For the omega distribution (1) ($1 \le r \le n-1$, $k \in \mathbb{N}$), we have

$$\mu_{r:n}^{(k)} = \alpha\, d^{k+\beta}\sum_{i=r}^{n}\sum_{p=0}^{i-1}(-1)^{p+i-r}\binom{i-1}{r-1}\binom{n}{i}\binom{i-1}{p} iB\Big(\frac{k}{\beta}+1, \frac{\alpha(p+1)d^\beta}{2}+1\Big)$$

$$\times {}_2F_1\Big(\frac{\alpha(p+1)d^\beta}{2}+1,\; \frac{k}{\beta}+1;\; \frac{k}{\beta}+\frac{\alpha(p+1)d^\beta}{2}+1;\; -1\Big). \tag{9}$$

Proof: In view of (6) and the result given by David and Nagaraja (2003, page 45), we can write

$$\mu_{r:n}^{(k)} = \sum_{i=r}^{n}(-1)^{i-r}\binom{i-1}{r-1}\binom{n}{i}\mu_{i:i}^{(k)}, \; 1 \le r \le n-1, \; k \in \mathbb{N}, \tag{10}$$

where

$$\mu_{i:i}^{(k)} = i\int_0^d x^k[F(x)]^{i-1}f(x)dx \tag{11}$$

or, equivalently, from Equation (3),

$$\mu_{i:i}^{(k)} = i\alpha\beta d^{2\beta}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}\int_0^d\frac{x^{k+\beta-1}}{(d^{2\beta}-x^{2\beta})}[1-F(x)]^{p+1}dx$$

$$= i\alpha\beta d^{2\beta}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}\int_0^d\frac{x^{k+\beta-1}}{(d^\beta+x^\beta)(d^\beta-x^\beta)}\left(\frac{d^\beta+x^\beta}{d^\beta-x^\beta}\right)^{-\frac{\alpha(p+1)d^\beta}{2}}dx. \tag{12}$$

Setting $z = x^\beta/d^\beta$, Equation (12) can be rewritten as

$$\mu_{i:i}^{(k)} = i\alpha d^{k+\beta}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}\int_0^1 z^{\frac{k}{\beta}}\,(1-z)^{\frac{\alpha(p+1)d^\beta}{2}-1}(1+z)^{-\frac{\alpha(p+1)d^\beta}{2}-1}dz.$$

Using the integral formula (8), we obtain

$$\mu_{i:i}^{(k)} = i\alpha\, d^{k+\beta} \sum_{p=0}^{i-1} (-1)^p \binom{i-1}{p} B\Big(\frac{k}{\beta}+1, \frac{\alpha(p+1)d^\beta}{2}+1\Big)$$

$$\times\, {}_2F_1\Big(\frac{\alpha(p+1)d^\beta}{2}+1, \frac{k}{\beta}+1; \frac{k}{\beta}+\frac{\alpha(p+1)d^\beta}{2}+1; -1\Big).$$

Inserting $\mu_{i:i}^{(k)}$ in Equation (10), it follows (9).

An alternative equation for $\mu_{r:n}^{(k)}$ is addressed in the following theorem.

Theorem 2.2: For $1 \le r \le n$ and $k \in \mathbb{N}$, we obtain

$$\mu_{r:n}^{(k)} = C_{r:n}\, \alpha\, d^{\beta+k} \sum_{p=0}^{r-1} (-1)^p \binom{r-1}{p} B\Big(\frac{k}{\beta}+1, \frac{\alpha}{2}(p+n-r+1)d^\beta+1\Big)$$

$$\times {}_2F_1\Big(\frac{\alpha}{2}(p+n-r+1)d^\beta+1, \frac{k}{\beta}+1; \frac{k}{\beta}+\frac{\alpha}{2}(p+n-r+1)d^\beta+1; -1\Big), \qquad (13)$$

where $C_{r:n}$ is defined as before.

Proof: From Equation (6), we have

$$\mu_{r:n}^{(k)} = C_{r:n} \int_0^d x^k [F(x)]^{r-1} [1-F(x)]^{n-r} f(x)dx.$$

$$= C_{r:n} \sum_{p=0}^{r-1} (-1)^p \binom{r-1}{p} \int_0^d x^k [1-F(x)]^{n-r+p} f(x)dx. \qquad (14)$$

Applying similar steps of Theorem 2.1 leads to (13).

Remark 2.1: (**a**) By setting $n = r = 1$ in (9) or (13), we obtain

$$\mu_{1:1}^{(k)} = E(X^k) = \alpha\, d^{\beta+k} B\Big(\frac{k}{\beta}+1, \frac{\alpha\, d^\beta}{2}+1\Big)\, {}_2F_1\Big(\frac{\alpha\, d^\beta}{2}+1, \frac{k}{\beta}+1; \frac{k}{\beta}+\frac{\alpha\, d^\beta}{2}+1; -1\Big), \qquad (15)$$

which is the kth moment of X reported by Okorie and Nadarajah (2019).

By setting $k = 1, 2, 3$ and $k = 4$ in Equation (15), one can obtain closed form expressions for the first four moments, variance, skewness, and kurtosis of X (Okorie and Nadarajah, 2019).

(**b**) Setting $r = 1$ in (13),

$$\mu_{1:n}^{(k)} = n\,\alpha\, d^{\beta+k} B\Big(\frac{k}{\beta}+1, \frac{n\,\alpha\, d^\beta}{2}+1\Big)\, {}_2F_1\Big(\frac{n\,\alpha\, d^\beta}{2}+1, \frac{k}{\beta}+1; \frac{k}{\beta}+\frac{n\,\alpha\, d^\beta}{2}+1; -1\Big), \qquad (16)$$

and, setting $r = n$ in (13),

$$\mu_{n:n}^{(k)} = n\,\alpha\, d^{\beta+k} \sum_{p=0}^{n-1} (-1)^p \binom{n-1}{p} B\Big(\frac{k}{\beta}+1, \frac{\alpha}{2}(p+1)d^\beta+1\Big)$$

$$\times {}_2F_1\Big(\frac{\alpha}{2}(p+1)d^\beta+1, \frac{k}{\beta}+1; \frac{k}{\beta}+\frac{\alpha}{2}(p+1)d^\beta+1; -1\Big), \qquad (17)$$

which are the kth moments of the minimum and maximum order statistics, respectively.

Recurrence relations for single moments of order statistics from the cdf (2) follow in the next theorem.

Theorem 2.3: For $1 \le r \le n$ and $k \in \mathbb{N}$, we have

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \frac{k}{n\,\alpha\,\beta}\left[\mu_{r:n}^{(k-\beta)} - \frac{\mu_{r:n}^{(k+\beta)}}{d^{2\beta}}\right] \tag{18}$$

and, consequently,

$$\mu_{r:n}^{(k)} - \mu_{r-1:n}^{(k)} = \frac{k}{(n-r+1)\,\alpha\,\beta}\left[\mu_{r:n}^{(k-\beta)} - \frac{\mu_{r:n}^{(k+\beta)}}{d^{2\beta}}\right]. \tag{19}$$

Proof: Khan *et al*. (1983a) proved that (for $1 \le r \le n$)

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} k \int_0^\infty x^{k-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx. \tag{20}$$

or, equivalently, from (3),

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \binom{n-1}{r-1}\frac{k}{\alpha\,\beta}\int_0^d x^{k-\beta}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$
$$-\binom{n-1}{r-1}\frac{k}{\alpha\,\beta\,d^{2\beta}}\int_0^d x^{k+\beta}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx,$$

which after simplification yields (18).

Using the well-known recurrence relation for moments (David and Nagaraja, 2003, p. 44)

$$n\mu_{r-1:n-1}^{(k)} = (n-r+1)\mu_{r-1:n}^{(k)} + (r-1)\mu_{r:n}^{(k)},$$

in Equation (18), we obtain (19). This completes the proof.

Remark 2.2: We obtain the negative moments in Theorem 2.3 when $k < \beta$. For applications, one may refer to Khan et al. (1984) and Ali and Khan (1987) and the papers referred therein.

Corollary 2.1: For $n \ge 1$ and $k \in \mathbb{N}$,

$$\mu_{1:n}^{(k)} = \frac{k}{n\,\alpha\,\beta}\left[\mu_{1:n}^{(k-\beta)} - \frac{\mu_{1:n}^{(k+\beta)}}{d^{2\beta}}\right] \tag{21}$$

and

$$\mu_{n:n}^{(k)} - \mu_{n-1:n}^{(k)} = \frac{k}{\alpha\,\beta}\left[\mu_{n:n}^{(k-\beta)} - \frac{\mu_{n:n}^{(k+\beta)}}{d^{2\beta}}\right]. \tag{22}$$

Proof: Equations (21) and (22) can be proved by setting $r = 1$ and $r = n$, respectively, in (19) and noting that $\mu_{0:m}^{(k)} = 0$ for $m \ge 1$ and $k \in \mathbb{N}$.

2.2. Product moments

Further, we derive the results for the product moment of order statistics from the omega distribution in the form of the next theorems.

Theorem 2.4: For $1 \le r < s \le n$ and $k, l \in \mathbb{N}$, we can write

$$\mu_{r,s:n}^{(k,l)} - \mu_{r,s-1:n}^{(k,l)} = \frac{l}{(n-s+1)\,\alpha\,\beta}\left[\mu_{r,s:n}^{(k,l-\beta)} - \frac{\mu_{r,s:n}^{(k,l+\beta)}}{d^{2\beta}}\right]. \tag{23}$$

Proof: Khan *et al*. (1983b) showed that (for $1 \le r < s \le n$)

$$\mu_{r,s:n}^{(k,l)} - \mu_{r,s-1:n}^{(k,l)} = C_{r,s:n}\frac{l}{(n-s+1)}\int_0^d x^k [F(x)]^{r-1} \mathrm{I}(x) f(x) dx, \tag{24}$$

where

$$\mathrm{I}(x) = \int_x^d y^{l-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy. \tag{25}$$

Using (3) in Equation (25), we obtain

$$\mathrm{I}(x) = \frac{1}{\alpha\,\beta}\int_x^d y^{l-\beta}\,[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y)\,dy$$
$$- \frac{1}{\alpha\,\beta\,d^{2\beta}}\int_x^d y^{l+\beta}\,[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(y)\,dy. \tag{26}$$

Substituting (26) in Equation (24), and simplifying, leads to (23).

Remark 2.3: The negative moments follow in Theorem (2.4) when $l < \beta$.

Corollary 2.2: For the omega distribution, we can write ($k, l \in \mathbb{N}$)

$$\mu_{r,r+1:n}^{(k,l)} - \mu_{r:n}^{(k+l)} = \frac{l}{(n-r)\,\alpha\,\beta}\left[\mu_{r,r+1:n}^{(k,l-\beta)} - \frac{\mu_{r,r+1:n}^{(k,l+\beta)}}{d^{2\beta}}\right],\ 1 \le r \le n-1, n \ge 3, \tag{27}$$

and

$$\mu_{n-1,n:n}^{(k,l)} - \mu_{n-1:n}^{(k+l)} = \frac{l}{\alpha\,\beta}\left[\mu_{n-1,n:n}^{(k,l-\beta)} - \frac{\mu_{n-1,n:n}^{(k,l+\beta)}}{d^{2\beta}}\right],\ n \ge 2. \tag{28}$$

Proof: Setting $s = r + 1$ in Equation (23) and noting that (Khan *et al*., 1983b)

$$\mu_{r,r:n}^{(k,l)} = E[X_{r:n}^k X_{r:n}^l] = E[X_{r:n}^{k+l}] = \mu_{r:n}^{(k+l)},$$

we obtain the recurrence relation (27).

Similarly, for $r = n-1$ and $s = n$ in Equation (23) and

$$\mu_{n-1,n-1:n}^{(k,l)} = E[X_{n-1:n}^k X_{n-1:n}^l] = E[X_{n-1:n}^{k+l}] = \mu_{n-1:n}^{(k+l)},$$

gives (28).

Corollary 2.3: For $k = 0$ in Theorem 2.4, Theorem 2.3 follows.

Remark 2.4: By setting $k = 1$ in (13), we calculate the means, second moments of the order statistics for the omega distribution (for $n = 1(1)5$) for selected parameter values. The computed values to six decimal places are reported in Table 1. It can be noted that the condition $\sum_{r=1}^{n} \mu_{r:n} = nE(X)$ holds (see David and Nagaraja, 2003).

The variance of $X_{r:n}$ $(1 \le r \le n)$ is $V(X_{r:n}) = \mu_{r:n}^{(2)} - \left[\mu_{r:n}^{(1)}\right]^2$, where $\mu_{r:n}^{(1)}$ and $\mu_{r:n}^{(2)}$ can be calculated by setting $k = 1$ and $k = 2$ in Equation (13), respectively. The R software (R Core Team, 2016) is used to compute the means, second moments and variances.

Table 1: Means, second moments and variances of order statistics.

		$d = 0.50$					
		$\alpha = 0.25,\ \beta = 0.75$			$\alpha = 0.75,\ \beta = 0.25$		
n	r	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$V[X_{r:n}]$	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$V[X_{r:n}]$
1	1	0.013726	0.004213	0.004024	0.013719	0.003157	0.002968
2	1	0.023849	0.007062	0.006493	0.013363	0.002548	0.002369
	2	0.003602	0.001363	0.001351	0.014075	0.003766	0.003568
3	1	0.031307	0.008945	0.007965	0.010607	0.001677	0.001565
	2	0.008935	0.003296	0.003216	0.018876	0.004288	0.003932
	3	0.000935	0.000397	0.000397	0.011675	0.003504	0.003368
4	1	0.036770	0.010140	0.008788	0.007984	0.001049	0.000985
	2	0.014916	0.005360	0.005137	0.018476	0.003562	0.003221
	3	0.002953	0.001231	0.001223	0.019277	0.005015	0.004643
	4	0.000263	0.000119	0.000119	0.009140	0.003001	0.002917
5	1	0.040730	0.010844	0.009185	0.005938	0.000649	0.000614
	2	0.020931	0.007325	0.006887	0.016170	0.002646	0.002385
	3	0.005894	0.002411	0.002377	0.021934	0.004936	0.004455
	4	0.000992	0.000445	0.000444	0.017505	0.005067	0.004761
	5	0.000081	0.000038	0.000038	0.007049	0.002484	0.002434
		$d = 0.90$					
		$\alpha = 0.25,\ \beta = 0.75$			$\alpha = 0.75,\ \beta = 0.25$		
n	r	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$V[X_{r:n}]$	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$V[X_{r:n}]$
1	1	0.035482	0.019216	0.017957	0.025358	0.010194	0.009551
2	1	0.057569	0.029494	0.026180	0.022597	0.007307	0.006797
	2	0.013394	0.008938	0.008758	0.028120	0.013081	0.012290
3	1	0.071164	0.034511	0.029446	0.016735	0.004361	0.004081
	2	0.030381	0.019462	0.018539	0.03432	0.013200	0.012023
	3	0.004901	0.003675	0.003651	0.02502	0.013021	0.012395
4	1	0.079262	0.036410	0.030127	0.011916	0.002509	0.002367
	2	0.046870	0.028814	0.026617	0.031192	0.009918	0.008945
	3	0.013891	0.010109	0.009916	0.037447	0.016483	0.015081
	4	0.001904	0.001531	0.001527	0.020878	0.011867	0.011431
5	1	0.083749	0.036471	0.029457	0.008468	0.001445	0.001373
	2	0.061312	0.036166	0.032406	0.025708	0.006763	0.006102
	3	0.025207	0.017787	0.017152	0.039418	0.014650	0.013096
	4	0.006347	0.004991	0.004951	0.036134	0.017706	0.016400
	5	0.000794	0.000666	0.000665	0.017064	0.010407	0.010116

3. Some statistical properties

3.1. L-Moments

The L-moments are expectations of certain linear combinations of order statistics (Hosking, 1990). The mth L-moment of a distribution can be defined as

$$\lambda_m = \frac{1}{m}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\mu_{m-j:m},\ m \geq 1, \tag{29}$$

where

$$\mu_{i:m} = \frac{m!}{(i-1)!(m-i)!}\int_0^d xF(x)^{i-1}\left[1-F(x)\right]^{m-i}f(x)dx.$$

L-moments are direct analogous to the conventional moments such as mean, variance, skewness, kurtosis, and so on. The properties and applications of L-moments were much explored by Hosking (1990). In comparison to conventional moments, L-moments have lower sample variances and are more robust against outliers. Apart from summarization of observed data, L-moments can also be used in model specification to characterize probability distributions, parameter estimation and hypothesis testing.

Setting $n = 1, 2, 3$ and 4 in Equation (29), the first four L-moments easily follow. The L-moments for the omega distribution can be written as $\lambda_1 = \mu_{1:1}$ $\lambda_2 = \mu_{2:2} - \mu_{1:1}$ $\lambda_3 = 2\mu_{3:3} - 3\mu_{2:2} + \mu_{1:1}$ and $\lambda_4 = 5\mu_{4:4} - 10\mu_{3:3} + 6\mu_{2:2} - \mu_{1:1}$, where

$$\mu_{i:i} = i\alpha\, d^{\beta+1}B\left(\frac{1}{\beta}+1, \frac{i\alpha d^{\beta}}{2}+1\right)\,{}_2F_1\left(\frac{i\alpha d^{\beta}}{2}+1,\ \frac{1}{\beta}+1;\ \frac{1}{\beta}+\frac{i\alpha d^{\beta}}{2}+1;\ -1\right).$$

Hosking (1990) also introduced some L-moment ratios. For example, L-coefficient of variation (L-CV) which is a dimensionless measure of variability given by λ_2/λ_1, and L-skewness and L-kurtosis which are dimensionless measures of asymmetry and kurtosis defined by $\tau_3 = \lambda_3/\lambda_2$ and $\tau_4 = \lambda_4/\lambda_2$, respectively. The L-moments of the omega distribution are computed to six decimal places for selected parameter values, and the results are reported in Table 2.

3.2. Incomplete moments

Okorie and Nadarajah (2019) derived closed form non-central moments of X, say $\mu'_r = E(X^r)$, which can be obtained from (16) with $k = 1$. The rth incomplete moment of X, say $\mu'_r(t)$, is

$$\mu'_r(t) = \int_0^t x^r f(x)dx, \tag{30}$$

and substituting from (1) gives

$$\mu'_r(t) = \alpha\beta d^{\beta}\int_0^t \frac{x^{r+\beta}}{d^{2\beta}-x^{2\beta}}\left(\frac{d^{\beta}+x^{\beta}}{d^{\beta}-x^{\beta}}\right)^{-\frac{\alpha d^{\beta}}{2}}dx. \tag{31}$$

It can be easily shown that

$$\int_t^{\infty} x^r f(x)dx = \mu'_r - \mu'_r(t). \tag{32}$$

The first incomplete moment of X follows from Equation (30) when $r = 1$, which also gives the mean deviations and the Bonferroni and Lorenz curves.

Table 2: L-moments of the omega distribution.

	$d = 0.20$		
	$\alpha = 0.25,\ \beta = 0.75$	$\alpha = 0.50,\ \beta = 0.50$	$\alpha = 0.75,\ \beta = 0.25$
λ_1	0.001859	0.004310	0.004390
λ_2	−0.001602	−0.002471	−0.000367
λ_3	0.001162	0.000246	−0.001655
λ_4	−0.000661	0.000943	0.000363
L-CV	−0.861414	−0.573207	−0.083695
L-skewness	−0.725598	−0.099589	4.504543
L-kurtosis	0.412664	−0.381677	−0.987604
	$d = 0.50$		
	$\alpha = 0.25,\ \beta = 0.75$	$\alpha = 0.50,\ \beta = 0.50$	$\alpha = 0.75,\ \beta = 0.25$
λ_1	0.008609	0.014787	0.011707
λ_2	−0.006418	−0.005931	0.000522
λ_3	0.00319	−0.002199	−0.004351
λ_4	−0.000395	0.003567	−0.000487
L-CV	−0.745481	−0.401099	0.044580
L-skewness	−0.497129	0.370738	−8.33625
L-kurtosis	0.061485	−0.601398	−0.932747
	$d = 0.90$		
	$\alpha = 0.25,\ \beta = 0.75$	$\alpha = 0.50,\ \beta = 0.50$	$\alpha = 0.75,\ \beta = 0.25$
λ_1	0.022307	0.031476	0.021566
λ_2	−0.01418	−0.008611	0.002748
λ_3	0.003914	−0.007655	−0.007433
λ_4	0.002497	0.005752	−0.002429
L-CV	−0.635665	−0.273568	0.127432
L-skewness	−0.276027	0.888975	−2.704682
L-kurtosis	−0.176061	−0.668018	−0.883949

4. Methods of Estimation

Dombi *et al.* (2019) proposed two approaches for practical statistical estimation of the omega parameters: the first one is the GLOBAL method to maximize the log-likelihood function, and the second depends on fitting its *cdf* to an empirical *cdf*. In this section, we discuss some other methods to estimate the parameters of the omega distribution.

4.1. Maximum likelihood estimation

Let $X_1, \cdots, X_n$ be a random sample from the omega distribution with corresponding observations $x_1, \cdots, x_n$. Also, let $X_{1:n} < \cdots < X_{r:n}$ $(r \leq n)$ be the first r order statistics of a random sample of size n, which represents the type-II right censored data. Note that if $r = n$, we deal with the complete data. The maximum likelihood estimate (MLE) of d follows by noting that $d > max\{x_i\}_{i=1,\dots,r}$. So, the MLE of d is $\hat{d} = max\{x_i\}_{i=1,\dots,r}$. We write the likelihood function in type-II right censored data to find the MLEs of α and β as

$$L(\theta) \equiv L(\alpha, \beta, d) = C\ [1 - F(x_r; \theta)]^{n-r} \prod_{i=1}^{r} f(x_i; \theta).$$

The statistical literature contains many papers for estimation under different censoring types and all the derivations in these papers are based on the MLE method. So, using (1) and (2), we have

$$L(\theta)=C\left(\frac{d^{\beta}+x_r^{\beta}}{d^{\beta}-x_r^{\beta}}\right)^{-\frac{1}{2}\alpha d^{\beta}(n-r)}\prod_{i=1}^{r}\frac{\alpha\beta d^{2\beta}x_i^{\beta-1}}{d^{2\beta}-x_i^{2\beta}}\left(\frac{d^{\beta}+x_i^{\beta}}{d^{\beta}-x_i^{\beta}}\right)^{-\frac{1}{2}\alpha d^{\beta}}.$$

The log-likelihood function, $\ell(\theta)$, is

$$\begin{aligned}\ell(\theta) &= \ln C+r\ln\alpha+r\ln\beta+(2r\ln d+\sum_{i=1}^{r}\ln x_i)\beta-\sum_{i=1}^{r}\ln x_i-\sum_{i=1}^{r}\ln(d^{2\beta}-x_i^{2\beta})\\ &- \frac{\alpha}{2}\left[d^{\beta}\left(\sum_{i=1}^{r}\ln\left(\frac{d^{\beta}+x_i^{\beta}}{d^{\beta}-x_i^{\beta}}\right)-(n-r)\ln\left(\frac{d^{\beta}+x_r^{\beta}}{d^{\beta}-x_r^{\beta}}\right)\right)\right].\end{aligned}$$

The first partial derivatives of ℓ with respect to α and β are

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial\alpha} &= \frac{r}{\alpha}-\frac{1}{2}\left[d^{\beta}\left(\sum_{i=1}^{r}\ln\left(\frac{d^{\beta}+x_i^{\beta}}{d^{\beta}-x_i^{\beta}}\right)-(n-r)\ln\left(\frac{d^{\beta}+x_r^{\beta}}{d^{\beta}-x_r^{\beta}}\right)\right)\right],\\ \frac{\partial\ell(\theta)}{\partial\beta} &= \frac{r}{\beta}+\frac{\alpha d^{\beta}(n-r)}{2}\left(\frac{2x_r^{\beta}(\ln x_r-\ln d)}{d^{2\beta}-x_r^{2\beta}}+\ln\left(\frac{d^{\beta}+x_r^{\beta}}{d^{\beta}-x_r^{\beta}}\ln d\right)\right)\\ &- \alpha d^{\beta}\sum_{i=1}^{r}\frac{x_i^{\beta}(\ln x_i-\ln d)}{d^{2\beta}-x_i^{2\beta}}+(2r\ln d+\sum_{i=1}^{r}\ln x_i)-2\sum_{i=1}^{r}\frac{d^{2\beta}\ln d-x_i^{2\beta}\ln x_i}{d^{2\beta}-x_i^{2\beta}}\\ &- \frac{\alpha d^{\beta}}{2}\ln d\sum_{i=1}^{r}\ln\left(\frac{d^{\beta}+x_i^{\beta}}{d^{\beta}-x_i^{\beta}}\right).\end{aligned}$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ can be derived by solving the previous nonlinear equations using the MLE of d therein.

4.2. Ordinary and weighted least-squares

Swain *et al.* (1988) proposed the least squares (LS) and weighted least squares (WLS) to obtain estimates of the parameters of the beta distribution.

Let $X_{1:n}<\cdots<X_{n:n}$ be the order statistics of a random sample of size n from the omega distribution with cdf (2). It is well known that $E\left[F(X_{i:n})\right]=\frac{i}{n+1}$ and $V\left[F(X_{i:n})\right]=\frac{i(n-i+1)}{(n+1)^2(n+2)}$.

The least square estimates (LSEs) $\hat{\alpha}$, $\hat{\beta}$ and $\hat{d}$ can be determined by minimizing

$$\sum_{j=1}^{n}\left[1-\frac{\alpha\beta d^{2\beta}x_{(j)}^{\beta-1}}{d^{2\beta}-x_{(j)}^{2\beta}}\left(\frac{d^{\beta}+x_{(j)}^{\beta}}{d^{\beta}-x_{(j)}^{\beta}}\right)^{-\frac{1}{2}\alpha d^{\beta}}-\frac{j}{n+1}\right]^2,$$

with respect to α, β, and d.

The weighted least squares estimates (WLSEs) of the unknown parameters can be determined by minimizing

$$\sum_{j=1}^{n}w_j\left[1-\frac{\alpha\beta d^{2\beta}x_{(j)}^{\beta-1}}{d^{2\beta}-x_{(j)}^{2\beta}}\left(\frac{d^{\beta}+x_{(j)}^{\beta}}{d^{\beta}-x_{(j)}^{\beta}}\right)^{-\frac{1}{2}\alpha d^{\beta}}-\frac{j}{n+1}\right]^2,$$

with respect to these parameters, where the weight function w_j at the jth point is $w_j = \frac{1}{V\left[F(X_{j:n})\right]} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$.

4.3. Maximum product of spacing

Alternative method to the ML method for estimating the parameters of a specific continuous distribution called maximum product of spacing (MPS) was introduced by Cheng and Amin (1983) and Ranneby (1984).

Let $x_{(1)} < \cdots < x_{(n)}$ denote the observed order statistics. Then, for the cdf of the omega distribution (2), we define the uniform spacing $D_i(\alpha, \beta)$ (for $i = 1, 2, \ldots, n$) by

$$D_i(\alpha, \beta) = F(x_{(i)}; \alpha, \beta) - F(x_{(i-1)}; \alpha, \beta),$$

where $F(x_{(0)}; \alpha, \beta) = 0$ and $F(x_{(n+1)}; \alpha, \beta) = 1$. Note that $\sum_{i=1}^{n+1} D_i(\alpha, \beta) = 1$. We can obtain the MPS estimates (MPSEs) of the parameters α and β (with fixed value for d) by maximizing the geometric mean of the uniform spacing

$$G(\alpha, \beta) = \left[\prod_{i=1}^{n} D_i(\alpha, \beta)\right]^{\frac{1}{n+1}}$$

with respect to α and β. This can be done equivalently by maximizing the function

$$M(\alpha, \beta) = \frac{1}{n+1} \sum_{i=1}^{n} \log\left[D_i(\alpha, \beta)\right].$$

The MPSE of the unknown vector parameter $\theta = (\alpha, \beta)$ can be found by solving the non-linear equations

$$\begin{aligned} \frac{\partial M(\theta)}{\partial \alpha} &= \frac{1}{n+1} \sum_{i=1}^{n} \frac{1}{D_i(\theta)} \left[g_1(x_{(i)}; \theta) - g_1(x_{(i-1)}; \theta)\right] = 0, \\ \frac{\partial M(\theta)}{\partial \beta} &= \frac{1}{n+1} \sum_{i=1}^{n} \frac{1}{D_i(\theta)} \left[g_2(x_{(i)}; \theta) - g_2(x_{(i-1)}; \theta)\right] = 0, \end{aligned}$$

where

$$g_1(x_{(\cdot)}; \theta) = \frac{\partial F(x_{(\cdot)}; \theta)}{\partial \alpha} \quad \text{and} \quad g_2(x_{(\cdot)}; \theta) = \frac{\partial F(x_{(\cdot)}; \theta)}{\partial \beta}.$$

4.4. Percentiles

Since the omega distribution has a closed form cdf, one can obtain estimates of α, β and d by equating the sample percentile points with the population percentiles which is known as the percentile method. If p_i denotes an estimate of $F(x_i : n_j; \alpha, \beta, d)$, then the percentile estimates $\hat{\alpha}_{PE}$, $\hat{\beta}_{PE}$ and $hatd_{PE}$ can be obtained by minimizing the function

$$P(\alpha, \beta, d) = \sum_{j=1}^{n} \left[x_i - Q(p_i)\right]^2,$$

where $Q(p_i)$ for the omega distribution is

$$Q(p_i) = d \left[\frac{(1-p_i)^{-2/\alpha d^\beta} - 1}{(1-p_i)^{-2/\alpha d^\beta} + 1} \right]^{\frac{1}{\beta}},$$

and $p_i = \frac{i}{n+1}$ is the unbiased estimator of $F(X_{i:n}; \alpha, \beta, d)$.

4.5. Anderson–Darling and right-tail Anderson–Darling

The Anderson-Darling estimates (ADEs) of the omega parameters can be found by minimizing

$$A(\alpha,\beta,d) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{(i)}|\alpha,\beta,d\right) + \log S\left(x_{(i)}|\alpha,\beta,d\right)\right],$$

with respect to α, β and d. These ADEs are also obtained by solving the non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s\left(x_{(i)}|\alpha,\beta,d\right)}{F\left(x_{(i)}|\alpha,\beta,d\right)} - \frac{\Delta_s\left(x_{(n+1-i)}|\alpha,\beta,d\right)}{S\left(x_{(n+1-i)}|\alpha,\beta,d\right)}\right] = 0, \ \ s = 1,2,3,$$

where

$$\Delta_1\left(x_{(i)}|\alpha,\beta,d\right) = \frac{\partial}{\partial\alpha}F\left(x_{(i)}|\alpha,\beta,d\right), \ \Delta_2\left(x_{(i)}|\alpha,\beta,d\right) = \frac{\partial}{\partial\beta}F\left(x_{(i)}|\alpha,\beta,d\right)$$

and

$$\Delta_3\left(x_{(i)}|\alpha,\beta,d\right) = \frac{\partial}{\partial d}F\left(x_{(i)}|\alpha,\beta,d\right).$$

The Right-tail Anderson–Darling estimates (RADEs) of the omega parameters are determined by minimizing

$$R(\alpha,\beta,d) = \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_{i:n}|\alpha,\beta,d\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i:n}|\alpha,\beta,d\right),$$

with respect to these parameters. The RADEs can also be calculated by solving the following non-linear equations

$$-2\sum_{i=1}^{n}\Delta_s\left(x_{i:n}|\alpha,\beta,d\right) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s\left(x_{n+1-i:n}|\alpha,\beta,d\right)}{S\left(x_{n+1-i:n}|\alpha,\beta,d\right)} = 0, \ \ s = 1,2,3.$$

5. Simulations

Samples of order statistics of sizes $n = 25, 50, 100$ are simulated from the Omg(α,β) model, where d has two values $d = 2.5, 5$. The other parameters are unknown and samples of sizes $n = 100, 200$ are simulated from the Omg(α,β,d) model. To compare the performance of the reported estimation methods in the previous section, we consider the following scenarios:

(i) Two unknown parameters: we use two actual values of the unknown parameters $\alpha = 0.87, 1.2$, and $\beta = 0.93, 1.13$. The results for the estimates of the parameters α and β by the seven described methods and their MSEs are reported in Tables 3-5.

Table 3: The MLEs, LSEs, WLSEs, MPSEs, PCEs, ADEs, RADEs and their MSEs.

$n=25, d=2.5$																			
Actual Value		MLE MSE						LSE MSE		WLSE MSE		MPSE MSE		PCE MSE		ADE MSE		RADE MSE	
		$r=19$		$r=22$		$r=25$													
α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.87	0.93	1.90361	0.97775	1.17247	0.98500	0.70901	0.93445	1.08478	1.04719	1.07467	1.03326	0.83278	0.84073	0.84178	0.90401	0.85494	0.92186	0.84801	0.94972
		1.06834	0.00228	0.09149	0.00303	0.02592	0.00002	0.04613	0.01373	0.04189	0.01066	0.01256	0.02182	0.01404	0.02933	0.01461	0.02129	0.01488	0.02885
	1.13	1.76199	1.19622	1.12939	1.20065	0.70856	1.13624	1.16218	1.08787	1.14776	1.06316	0.84693	1.02608	0.83936	1.08234	0.85357	1.11997	0.87280	1.15809
		0.79564	0.00439	0.06728	0.00499	0.02606	0.00004	0.08537	0.00177	0.07715	0.00447	0.01372	0.03125	0.01253	0.03498	0.01788	0.02374	0.01631	0.03774
1.2	0.93	2.69826	1.02435	1.64527	1.01351	0.97872	0.93556	0.98912	0.99923	1.03952	1.001991	1.12266	0.84731	1.14453	0.87031	1.18940	0.93200	1.17475	0.94700
		2.24480	0.00890	0.19827	0.00697	0.04897	0.00003	0.04426	0.00479	0.02575	0.00808	0.02816	0.02060	0.03028	0.02721	0.02773	0.01854	0.02763	0.02239
	1.13	2.49098	1.23895	1.58396	1.22766	0.97824	1.13743	1.07621	1.02107	1.11504	1.04399	1.12919	1.02693	1.15512	1.07141	1.18158	1.12426	1.19308	1.14571
		1.66662	0.01187	0.14743	0.00954	0.04918	0.00006	0.01532	0.01187	0.00722	0.00740	0.02911	0.02749	0.02515	0.03222	0.02687	0.02214	0.02713	0.02518
$n=25, d=5$																			
Actual Value		MLE MSE						LSE MSE		WLSE MSE		MPSE MSE		PCE MSE		ADE MSE		RADE MSE	
		$r=19$		$r=22$		$r=25$													
α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.87	0.93	1.54211	1.03280	1.05172	1.00841	0.70734	0.93657	1.08327	1.05595	1.05401	1.02839	0.83803	0.85779	0.84783	0.88259	0.85959	0.93044	0.85968	0.93157
		0.45173	0.01057	0.03302	0.00615	0.02646	0.00004	0.04548	0.01586	0.03386	0.00968	0.01207	0.01719	0.01629	0.02632	0.01506	0.01487	0.01665	0.01880
	1.13	1.45280	1.30177	1.01514	1.24375	0.70639	1.13942	1.16405	1.10418	1.13861	1.06261	0.85525	1.03388	0.86585	1.06827	0.88848	1.13508	0.85762	1.14933
		0.33965	0.02950	0.02107	0.01294	0.02677	0.00009	0.08647	0.00067	0.07215	0.00454	0.01174	0.02582	0.01738	0.02158	0.01738	0.02158	0.01746	0.02570
1.2	0.93	2.20326	1.04720	1.48388	1.01874	0.97695	0.93775	0.96939	0.99884	0.97100	1.00069	1.14440	0.85349	1.19189	0.92439	1.19189	0.92439	1.18214	0.95906
		1.00654	0.01373	0.08059	0.00788	0.04975	0.00006	0.05318	0.00474	0.05244	0.00500	0.02648	0.01456	0.02443	0.02621	0.02620	0.01366	0.02943	0.01693
	1.13	2.10115	1.30970	1.44094	1.25042	0.97603	1.14093	1.06191	1.02706	1.06082	1.02775	1.15861	1.03901	1.15246	1.04726	1.19892	1.13898	1.19322	1.13811
		0.81207	0.03229	0.05805	0.01450	0.05016	0.00012	0.01907	0.01060	0.01937	0.01045	0.02281	0.02187	0.02003	0.03058	0.02772	0.01767	0.02920	0.02455

Table 4: The MLEs, LSEs, WLSEs, MPSEs, PCEs, ADEs, RADEs and their MSEs.

$n=50, d=2.5$																			
Actual Value		MLE MSE						LSE MSE		WLSE MSE		MPSE MSE		PCE MSE		ADE MSE		RADE MSE	
		$r=43$		$r=47$		$r=50$													
α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.87	0.93	1.34853	1.00804	1.01406	1.02278	0.78926	0.97620	1.12365	1.05208	1.11842	1.03861	0.83136	0.87720	0.86727	0.91031	0.86670	0.93192	0.86835	0.92905
		0.22899	0.00609	0.02075	0.00861	0.00652	0.00213	0.06434	0.01490	0.06171	0.01180	0.00717	0.01052	0.00786	0.01416	0.00847	0.00912	0.00689	0.01213
	1.13	1.29182	1.23316	0.99197	1.25150	0.78534	1.19121	1.20831	1.09239	1.19600	1.06877	0.84425	1.05951	0.86358	1.09930	0.85660	1.13831	0.86605	1.13738
		0.17793	0.01064	0.01488	0.01476	0.00717	0.00375	0.11445	0.00141	0.10627	0.00375	0.00732	0.01473	0.00779	0.01810	0.00802	0.01366	0.00883	0.01779
1.2	0.93	1.91432	1.04520	1.43378	1.04552	1.09919	0.98271	1.00075	0.99885	1.08239	1.02611	1.15683	0.88122	1.16811	0.90605	1.19586	0.93076	1.18405	0.93916
		0.51025	0.01327	0.05465	0.01335	0.01016	0.00278	0.03970	0.00474	0.01383	0.00924	0.01169	0.00889	0.01300	0.01205	0.01282	0.00771	0.01480	0.00941
	1.13	1.83291	1.26779	1.40502	1.27320	1.09529	1.19679	1.09726	1.02205	1.16675	1.05265	1.15442	1.07899	1.16688	1.09085	1.19491	1.11544	1.19832	1.13403
		0.40057	0.01899	0.04203	0.02051	0.01097	0.00446	0.01056	0.01165	0.00111	0.00598	0.01425	0.01164	0.01252	0.01511	0.01377	0.01384	0.01243	0.01422
$n=50, d=5$																			
Actual Value		MLE MSE						LSE MSE		WLSE MSE		MPSE MSE		PCE MSE		ADE MSE		RADE MSE	
		$r=43$		$r=47$		$r=50$													
α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.87	0.93	1.19276	1.04442	0.94912	1.04919	0.77686	0.98620	1.10071	1.05384	1.06551	1.02568	0.85951	0.87735	0.86037	0.89362	0.86465	0.93171	0.86160	0.93042
		0.10418	0.01309	0.00626	0.01421	0.00868	0.00316	0.05323	0.01534	0.03822	0.00915	0.00773	0.00756	0.00906	0.01399	0.00770	0.00757	0.00718	0.00901
	1.13	1.14805	1.29397	0.92835	1.28804	0.77266	1.20042	1.19059	1.10049	1.15709	1.05861	0.85970	1.06870	0.87311	1.06892	0.87015	1.12682	0.86461	1.13474
		0.07731	0.02689	0.00341	0.02498	0.00496	0.00496	0.10278	0.00087	0.08242	0.00510	0.00674	0.01131	0.00821	0.01762	0.00704	0.01042	0.00886	0.01165
1.2	0.93	1.70553	1.05709	1.35483	1.05781	1.08732	0.98796	0.98537	1.00050	0.99420	1.00447	1.17078	0.88325	1.17816	0.89004	1.20293	0.92650	1.20355	0.93267
		0.25556	0.01615	0.02397	0.01634	0.01270	0.00336	0.04607	0.00497	0.04235	0.00555	0.01226	0.00662	0.01368	0.01235	0.01354	0.00688	0.01527	0.00945
	1.13	1.65395	1.30184	1.33259	1.29386	1.08356	1.20149	1.08266	1.02744	1.09014	1.03172	1.17186	1.06329	1.18024	1.07519	1.19245	1.12796	1.18118	1.14030
		0.20607	0.02953	0.01758	0.02685	0.01356	0.00511	0.01377	0.01052	0.01207	0.00966	0.01268	0.01144	0.01249	0.01649	0.01343	0.00903	0.01486	0.00992

(ii) Three unknown parameters: we use actual values of the parameters $\alpha = 0.87, 1.2$, $\beta = 0.93, 1.13$ and $d = 3.5, 5$. The results for the estimates of three parameters by the seven estimation methods and their MSEs are reported in Tables 6 and 7.

Based on the figures in Tables 6 and 7, we note that decreasing the α actual value improves the β estimates, while increasing the β actual value improves the α estimates. Further, we note that increasing d gives good estimates of α and β. The MLEs, MPSEs, ADEs, RADEs, WLSEs, LSEs, and PCEs are evaluated based on the following quantities including the average estimates and the MSEs for each sample size. The figures in Tables 3-7 indicate that the behaviors of the estimates of the omega parameters are good and show small MSEs in all cases, i.e., these estimates are quite reliable and very close to the actual parameter values. Moreover, the MSEs decrease when n increases, thus showing that these estimators are consistent for the parameters. On the other hand, the performance ordering of the proposed estimators, from best to worst, in terms of their MSEs is MLE, MPSE, ADE, RADE, WLSE, LSE, and PCE in most of these cases.

Table 5: The MLEs, LSEs, WLSEs, MPSEs, PCEs, ADEs, RADEs and their MSEs.

$n=100, d=2.5$																			
Actual Value		MLE MSE						LSE MSE		WLSE MSE		MPSE MSE		PCE MSE		ADE MSE		RADE MSE	
		$r=85$		$r=95$		$r=100$													
α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.87	0.93	1.45050	0.95173	0.97891	0.93983	0.74543	0.87881	1.04628	1.02686	1.05436	1.01859	0.85500	0.89777	0.86302	0.91781	0.86457	0.93082	0.86860	0.93064
		0.33698	0.00047	0.01186	0.00010	0.01552	0.00262	0.03107	0.00938	0.03399	0.00785	0.00330	0.00455	0.00315	0.00734	0.00346	0.00492	0.00373	0.00652
	1.13	1.38531	1.15687	0.96318	1.14421	0.74886	1.06856	1.12466	1.05904	1.12377	1.04149	0.86294	1.09025	0.86783	1.11965	0.86321	1.13137	0.86643	1.13577
		0.26554	0.00072	0.00868	0.00020	0.01468	0.00377	0.06485	0.00504	0.06440	0.00783	0.00311	0.00655	0.00367	0.00805	0.00315	0.00569	0.00337	0.00821
1.2	0.93	2.02646	0.98252	1.35536	0.95649	1.01827	0.87988	0.93840	0.97469	1.02700	1.01074	1.17643	0.89712	1.19057	0.91267	1.19480	0.93004	1.18940	0.93878
		0.68304	0.00276	0.02414	0.00070	0.03303	0.00251	0.06843	0.00200	0.02993	0.00652	0.00703	0.00412	0.00691	0.00725	0.00705	0.00386	0.00748	0.00581
	1.13	1.93060	1.18660	1.33287	1.16146	1.02209	1.06972	1.01866	0.98638	1.09410	1.02921	1.16278	1.09422	1.19074	1.11981	1.19849	1.12524	1.19452	1.13326
		0.53378	0.00320	0.01765	0.00099	0.03165	0.00363	0.03289	0.02063	0.01122	0.01016	0.00645	0.00585	0.00771	0.00673	0.00680	0.00490	0.00681	0.00637
$n=100, d=5$																			
Actual Value		MLE MSE						LSE MSE		WLSE MSE		MPSE MSE		PCE MSE		ADE MSE		RADE MSE	
		$r=85$		$r=95$		$r=100$													
α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.87	0.93	1.27440	0.97197	0.93217	0.95269	0.75563	0.88071	1.02716	1.02834	1.00295	1.00369	0.86789	0.89946	0.86565	0.90885	0.86897	0.93038	0.86926	0.93136
		0.16354	0.00176	0.00387	0.00051	0.01308	0.00243	0.02470	0.00967	0.01768	0.00543	0.00314	0.00386	0.00431	0.00614	0.00350	0.00352	0.00368	0.00514
	1.13	1.22416	1.20242	0.91604	1.16740	0.75906	1.07067	1.11346	1.07054	1.08868	1.03031	0.87108	1.08931	0.87014	1.10645	0.86386	1.13887	0.87051	1.12814
		0.12543	0.00525	0.00212	0.00140	0.01231	0.00352	0.05927	0.00354	0.04782	0.00994	0.00364	0.00541	0.00413	0.00803	0.00448	0.00433	0.00452	0.00566
1.2	0.93	1.78210	0.98400	1.29300	0.96102	1.02945	0.88117	0.91602	0.97229	0.93676	0.98497	1.17599	0.89597	1.19327	0.90448	1.19926	0.93377	1.19185	0.92922
		0.33884	0.00292	0.00865	0.00096	0.02909	0.00238	0.08064	0.00179	0.06929	0.00302	0.00572	0.00408	0.00668	0.00698	0.00648	0.00359	0.00688	0.00385
	1.13	1.72137	1.21054	1.27431	1.17384	1.03305	1.07085	1.00501	0.99249	1.02174	1.00571	1.17254	1.09691	1.18779	1.09645	1.19416	1.13131	1.19843	1.13062
		0.27183	0.00649	0.00552	0.00192	0.02787	0.00350	0.03802	0.01891	0.03178	0.01545	0.00581	0.00423	0.00599	0.00672	0.00641	0.00485	0.00680	0.00630

Table 6: The MLEs, LSEs, WLSEs, MPSEs, PCEs, ADEs, RADEs and their MSEs.

Actual Value			MLE MSE			LSE MSE			WLSE MSE			MPSE MSE			PCE MSE			ADE MSE			RADE MSE		
												$n=25$											
α	β	d	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$
0.87	0.93	2.5	1.16973	0.97058	2.42667	0.85797	0.89910	3.61047	0.86182	0.91653	2.57385	0.82138	0.85666	2.49664	0.84120	0.87920	2.56599	0.86459	0.94039	2.50937	0.85276	0.94735	2.50230
			0.08984	0.00165	0.00538	0.01904	0.02049	0.72080	0.01924	0.02529	0.05422	0.01618	0.02311	0.02626	0.01783	0.03123	0.05753	0.01818	0.02216	0.04243	0.0190	0.03011	0.03212
		3.5	1.10448	0.96495	3.18576	0.84108	0.90352	3.71845	0.86217	0.91164	3.85118	0.84486	0.85089	3.50032	0.83404	0.87348	3.56767	0.86606	0.92755	3.53227	0.84532	0.95100	3.55824
			0.05498	0.00122	0.09875	0.01790	0.02386	0.69853	0.01836	0.02006	0.67096	0.01564	0.02038	0.36521	0.01730	0.02721	0.50768	0.01761	0.01801	0.52519	0.01867	0.02500	0.44699
	1.13	2.5	1.12800	1.17398	2.37765	0.85638	1.10046	2.62488	0.85913	1.11324	2.62249	0.82903	1.05127	2.50626	0.83602	1.10019	2.54137	0.85524	1.14335	2.50127	0.85331	1.14198	2.49006
			0.06656	0.00193	0.01497	0.01898	0.03815	0.15516	0.01913	0.03248	0.12087	0.01773	0.03359	0.05968	0.01710	0.03425	0.09475	0.01630	0.02971	0.08296	0.01722	0.04375	0.07120
		3.5	1.07008	1.17165	2.99676	0.85381	1.07378	3.70903	0.84669	1.09664	3.92017	0.83840	1.02877	3.50698	0.84160	1.05252	3.63691	0.85842	1.12629	3.50504	0.85856	1.14217	3.58477
			0.04003	0.00174	0.25325	0.01652	0.03082	1.10097	0.01760	0.02850	1.22602	0.01700	0.03263	0.76098	0.01944	0.03263	0.92533	0.01968	0.02351	1.02916	0.01917	0.03000	0.81197
1.2	0.93	2.5	1.64792	0.99520	2.26968	1.17214	0.89429	2.67821	1.18009	0.91221	2.70901	1.10971	0.83781	2.49436	1.12330	0.87145	2.53378	1.19145	0.92422	2.49067	1.18732	0.94025	2.53305
			0.20063	0.00425	0.05305	0.03792	0.02375	0.43638	0.03870	0.02129	0.34603	0.04219	0.02233	0.22219	0.03861	0.03252	0.27448	0.03922	0.01984	0.22524	0.04339	0.02599	0.22528
		3.5	1.56192	0.97964	2.80003	1.17319	0.89346	3.78217	1.16352	0.90406	4.15396	1.12557	0.85339	3.51248	1.13180	0.84244	3.93617	1.18766	0.93411	3.54796	1.17033	0.9263	3.57184
			0.13099	0.00246	0.48996	0.03507	0.02001	2.45763	0.03566	0.01755	2.35785	0.03856	0.01838	1.48395	0.03495	0.02396	1.67699	0.04030	0.01893	1.69027	0.03634	0.01800	1.54125
	1.13	2.5	1.59148	1.19621	2.18668	1.16595	1.08608	2.54017	1.16945	1.09705	2.75583	1.14230	1.03035	2.51986	1.15041	1.06205	2.66932	1.16812	1.11271	2.45261	1.18475	1.14355	2.48243
			0.15326	0.00438	0.09817	0.04165	0.03334	0.50741	0.03596	0.02842	0.53997	0.03661	0.03082	0.31262	0.02908	0.03019	0.42781	0.03455	0.02656	0.39271	0.03716	0.03067	0.34325
		3.5	1.52164	1.18331	2.57844	1.16117	1.06498	3.50364	1.19403	1.12632	3.73279	1.11321	1.01263	3.50056	1.12579	1.01083	3.85679	1.15473	1.11011	3.63090	1.16324	1.10965	3.73028
			0.10345	0.00284	0.84928	0.03721	0.02858	2.66621	0.00876	0.00569	0.76719	0.03436	0.02751	1.84959	0.03110	0.02952	2.60923	0.02909	0.02539	2.42241	0.03863	0.02604	2.41540

Table 7: The MLEs, LSEs, WLSEs, MPSEs, PCEs, ADEs, RADEs and their MSEs.

Actual Value			MLE MSE			LSE MSE			WLSE MSE			MPSE MSE			PCE MSE			ADE MSE			RADE MSE		
												$n=100$											
α	β	d	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{d}$
0.87	0.93	2.5	0.97880	0.93964	2.49926	0.87104	0.92841	3.54490	0.86923	0.93406	2.51990	0.85230	0.90074	2.50732	0.86208	0.90887	2.50505	0.86507	0.93647	2.50449	0.87170	0.94490	2.51283
			0.01184	0.00009	0.00000	0.00457	0.00576	0.10668	0.00466	0.00542	0.00330	0.00446	0.00519	0.00135	0.00382	0.00929	0.00719	0.00377	0.00486	0.00237	0.0045	0.00578	0.00214
		3.5	0.95295	0.94328	3.48873	0.86833	0.93153	3.52863	0.87338	0.92823	3.56461	0.86499	0.90094	3.53805	0.86773	0.92735	3.56388	0.86270	0.93799	3.52931	0.87093	0.95082	3.54010
			0.00688	0.00018	0.00013	0.00484	0.00624	0.08283	0.00456	0.00521	0.05955	0.00382	0.00470	0.03365	0.00411	0.00584	0.06303	0.00435	0.00491	0.04358	0.00379	0.00656	0.03929
	1.13	2.5	0.96291	1.14355	2.49730	0.87024	1.14195	2.53072	0.87266	1.13342	2.52735	0.86193	1.09156	2.52224	0.86641	1.10883	2.51962	0.87231	1.13790	2.50952	0.86975	1.14612	2.51740
			0.00863	0.00018	0.00001	0.00522	0.00827	0.01681	0.00442	0.00758	0.00896	0.00384	0.00708	0.00462	0.00437	0.01013	0.01299	0.00451	0.00725	0.00723	0.00483	0.00964	0.00548
		3.5	0.93611	1.15178	3.46097	0.86646	1.13147	3.61355	0.87020	1.13168	3.61366	0.86378	1.10062	3.59934	0.86611	1.11925	3.64973	0.86698	1.13708	3.53451	0.86855	1.14506	3.53665
			0.00437	0.00047	0.00152	0.00500	0.00872	0.31372	0.00442	0.00710	0.18080	0.00407	0.00677	0.10902	0.00393	0.00876	0.16244	0.00464	0.00743	0.13304	0.00495	0.00773	0.11993
1.2	0.93	2.5	1.35447	0.95561	2.49149	1.20989	0.92921	2.54467	1.20209	0.93366	2.55229	1.17249	0.89724	2.53444	1.19713	0.92500	2.55575	1.19314	0.93896	2.52120	1.20592	0.94595	2.51118
			0.02386	0.00066	0.00007	0.01142	0.00582	0.06289	0.00962	0.00499	0.03502	0.00898	0.00498	0.01908	0.00918	0.00679	0.03923	0.00948	0.00448	0.02371	0.00862	0.00643	0.02200
		3.5	1.31762	0.95387	3.43316	1.19365	0.92637	3.56898	1.19924	0.93133	3.65960	1.18870	0.90496	3.60242	1.19213	0.91665	3.66724	1.21087	0.93322	3.55809	1.19687	0.9449	3.58595
			0.01384	0.00057	0.00447	0.01008	0.00435	0.54059	0.00909	0.00472	0.35629	0.00831	0.00465	0.22486	0.00943	0.00741	0.27246	0.00902	0.00412	0.31724	0.00911	0.00557	0.23988
	1.13	2.5	1.33131	1.15920	2.47943	1.19828	1.12827	2.53089	1.20480	1.13201	2.57526	1.18642	1.09261	2.54373	1.18622	1.11948	2.54547	1.20212	1.13189	2.52018	1.21200	1.14271	2.53404
			0.01724	0.00085	0.00042	0.01001	0.00719	0.10014	0.00958	0.00705	0.06865	0.00840	0.00711	0.04119	0.00950	0.00872	0.05489	0.01020	0.00803	0.04895	0.00884	0.00893	0.05142
		3.5	1.29618	1.15965	3.34558	1.19697	1.11334	3.63626	1.19403	1.12632	3.73279	1.19482	1.10559	3.51937	1.19388	1.11944	3.77965	1.20138	1.12515	3.53402	1.19888	1.13129	3.65714
			0.00925	0.00088	0.02384	0.00941	0.00749	1.00722	0.00876	0.00569	0.76719	0.00760	0.00528	0.57646	0.00958	0.00709	0.57924	0.00843	0.00650	0.58128	0.00903	0.00723	0.68413

6. Real data illustration

This section discusses the flexibility of the omega distribution in fitting a real data set and compared it with other competing distributions based on the Kolmogorov–Smirnov (K-S) statistic with its associated p-value. The data set consists of 72 exceedances of flood peaks (in m^3/s) of the Wheaton river near Carcross in Yukon Territory, Canada for the years 1958-1984. These data analyzed by Choulakian and Stephens (2001) are: 0.4, 0.7, 1.7, 1.1, 1.9, 1.1, 2.2, 2.2, 14.4, 20.6, 5.3, 12.0, 13.0, 9.3, 1.4, 18.7, 8.5, 22.9, 1.7, 0.1, 25.5, 2.5, 14.4, 1.7, 37.6, 0.6, 11.6, 14.1, 22.1, 39.0, 0.3, 15.0, 36.4, 2.7, 64.0, 1.5, 11.0, 7.3, 1.1, 0.6, 9.0, 1.7, 7.0, 14.1, 3.6, 5.6, 30.8, 13.3, 9.9, 10.4, 10.7, 20.1, 0.4, 2.8, 30.0, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 2.5, 27.4, 1.0, 27.1, 5.3, 9.7, 20.2, 16.8, 27.5, 2.5, 27.0. For computational stability with fitting of the distributions, each observation is divided by 65, and hence the estimate of the parameter d is $\hat{d} = 0.985$.

The analyzed data are used to show the flexibility of the omega model as compared with some well-known distributions such as the modified Weibull (MW), transmuted complementary Weibull-geometric (TCWG), Lindley Weibull (LiW), power generalized Weibull (PGW), alpha power Weibull (APW), alpha power exponentiated-Weibull (APEW), exponentiated-Weibull (EW), extended odd Weibull exponential (EOWE), logarithmic transformed Weibull (LTW), and Weibull (W) distributions.

Table 8 reports parameter estimates using the ML method with their corresponding standard errors (SEs), K-S statistic (K-S (stat)) with its associated p-value (K-S (p-value)) for some models fitted to the current data. The figures in this table reveal that the omega model provides the closest fit to the current data as compared to other competing distributions.

The fitted pdf, cdf, survival function, and probability-probability (PP) plots of the omega distribution are displayed in Figure 1. The PP plots of the omega model and other fitted models are depicted in Figure 2. The parameters of the omega distribution are estimated using several estimation methods as listed in Table 9. The PP plots of the omega model using different estimation methods are depicted in Figure 3.

Table 8. Results from the fitted distributions to Wheaton river data.

Distribution	Estimates	SEs	K-S (stat)	K-S (p-value)
OMEGA	$\hat{\alpha} = 3.0305765$	0.4824807	0.0889468	0.6191953
	$\hat{\beta} = 0.7377699$	0.0810828		
MW	$\hat{\alpha} = 2.0878218$	6.2615708	0.1046855	0.4091079
	$\hat{\beta} = 0.8120513$	0.4037047		
	$\hat{\lambda} = 2.6780859$	6.6030829		
TCWG	$\hat{\alpha} = 0.8679701$	0.8151071	0.1071177	0.3805323
	$\hat{\beta} = 0.8762272$	0.1689987		
	$\hat{\lambda} = 0.0000006$	0.4452576		
	$\hat{\sigma} = 6.1091600$	3.3411493		
LiW	$\hat{\alpha} = 2.5936284$	5.7555398	0.1066129	0.3863592
	$\hat{\beta} = 0.8705367$	0.1103413		
	$\hat{\theta} = 2.5365106$	4.2907423		
PGW	$\hat{\lambda} = 0.8141584$	1.7813975	0.1062762	0.3902766
	$\hat{\theta} = 0.7440689$	0.1473203		
	$\hat{\alpha} = 3.2245731$	5.5840937		
APW	$\hat{\alpha} = 1.1397937$	1.269151	0.1061151	0.3921589
	$\hat{\beta} = 0.8894862$	0.132235		
	$\hat{\lambda} = 4.7910660$	0.983672		
APEW	$\hat{\alpha} = 0.0426735$	0.0339949	0.09805297	0.4930507
	$\hat{\beta} = 4.2287394$	0.3614408		
	$\hat{\lambda} = 2.5130669$	0.3860032		
	$\hat{\theta} = 0.1978522$	0.0289383		
EW	$\hat{\beta} = 1.3867142$	0.5896521	0.1073935	0.377372
	$\hat{\lambda} = 5.1557944$	1.2449955		
	$\hat{\theta} = 0.5185501$	0.3116688		
EOWE	$\hat{\alpha} = 0.7618609$	0.1200032	0.09914981	0.4786092
	$\hat{\beta} = 0.4522144$	0.5052505		
	$\hat{\lambda} = 4.4213827$	1.4535656		
LTW	$\hat{\alpha} = 1.1506548$	0.9457565	0.1070967	0.380773
	$\hat{\beta} = 0.8764569$	0.1681637		
	$\hat{\lambda} = 4.8816417$	1.2212993		
W	$\hat{\beta} = 0.9011665$	0.08555716	0.1052065	0.402882
	$\hat{\lambda} = 4.7140919$	0.75473964		

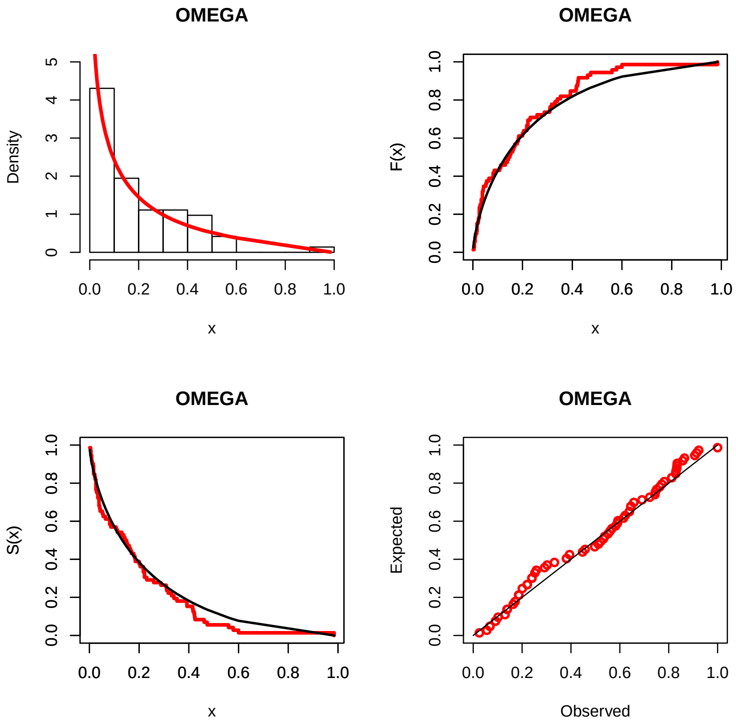

Figure 1. The fitted pdf, cdf, survival function, and PP plots of the omega distribution for Wheaton river data.

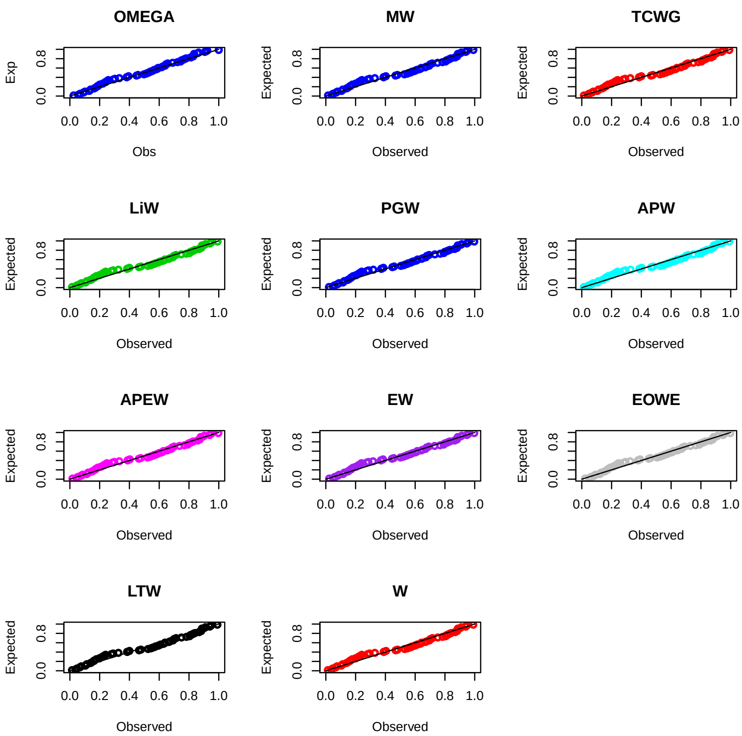

Figure 2. The PP plots of the fitted omega and other distributions to Wheaton river data.

Table 9. Estimates of the omega parameters, K-S (stat) with its associated p-value for Wheaton river data using all methods.

Method	Estimates	K-S (stat)	K-S (p-value)
MLE	$\hat{\alpha} = 3.0305765$ $\hat{\beta} = 0.7377699$	0.0889468	0.6191953
LSE	$\hat{\alpha} = 3.319753$ $\hat{\beta} = 0.8492076$	0.07090078	0.9125179
WLSE	$\hat{\alpha} = 3.565044$ $\hat{\beta} = 0.7814565$	0.14252334	0.1719133
MPSE	$\hat{\alpha} = 2.554748$ $\hat{\beta} = 0.7323336$	0.11491242	0.3978472
PCE	$\hat{\alpha} = 3.880584$ $\hat{\beta} = 0.9426041$	0.07701275	0.8553612
ADE	$\hat{\alpha} = 3.473845$ $\hat{\beta} = 0.8618174$	0.07626228	0.8630764
RADE	$\hat{\alpha} = 3.817778$ $\hat{\beta} = 0.9260090$	0.07102754	0.9114741

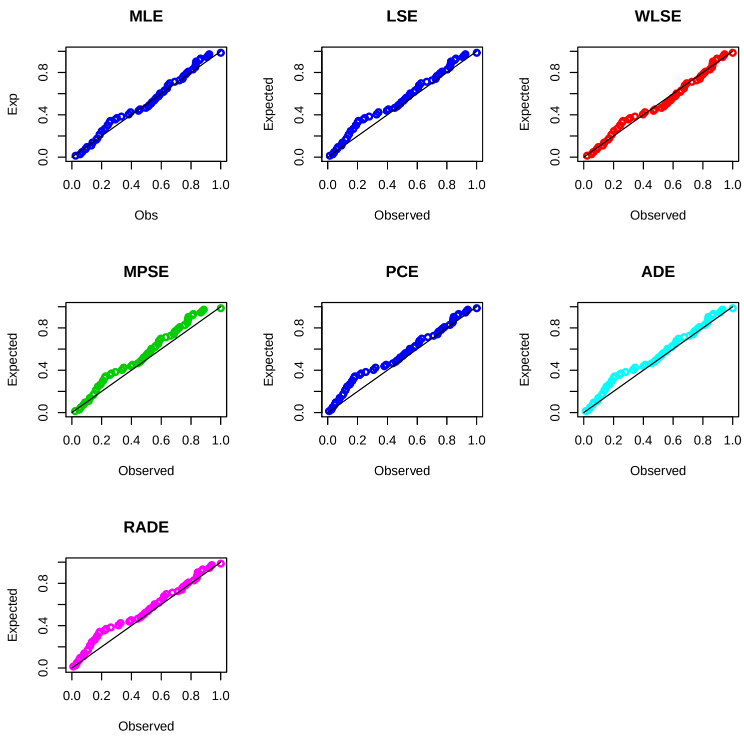

Figure 3. The PP plots of the omega distribution for Wheaton river data based on seven methods of estimation.

259 ## 7. Conclusion

260 The omega distribution was pioneered by Dombi *et al*. (2019) to model reliability data and its
261 basic properties were studied by Okorie and Nadarajah (2019). In this paper, we obtain the moment
262 properties from the order statistics viewpoint including explicit expressions for single moments,
263 recurrence relations for single and product moments of order statistics of this distribution along

with L-moments, which may be useful to the practitioners. This will encourage researchers to do further works about the omega distribution, and also the order statistics. We present seven estimation methods, namely: maximum likelihood, maximum product of spacings, ordinary least-squares and weighted least-squares, percentiles, Anderson–Darling and right–tail Anderson–Darling, to determine estimates of the parameters of the omega distribution and provide a simulation study to illustrate the performance of the different estimators. Hence, based on our simulation study, we show that the maximum likelihood method gives consistent estimates of the omega parameters. An application to real data proves the flexibility of the omega distribution, which gives superior fit then ten other distributions.

It is worth mentioning that the research in this article can be extended in many ways. For example, exponentiated version of the omega distribution can be established, among other extensions, several properties of order statistics from the distribution can be explored and their relations to well-known stochastic orders, and a bivariate or multivariate omega distribution can also be proposed. Furthermore, the parameters of the omega distribution can be estimated using the Bayesian approach under different losses functions.

References

1. Afify, A.Z. and Mohamed, O.A. (2020): A new three-parameter exponential distribution with variable shapes for the hazard rate: estimation and applications. Mathematics, 8, p.135.
2. Afify, A.Z., Nofal, Z.M. and Butt, N.S. (2014): Transmuted complementary Weibull geometric distribution. Pakistan Journal of Statistics and Operation Research, 10, 435–454.
3. Akhter, Z., MirMostafaee and Athar, H. (2019): On the moments of order statistics from the standard two-sided power distribution. *J. Math. Model.*, **7**(4), 381–398.
4. Akhter, Z., Saran, J., Verma, K. and Pushkarna, N. (2020): Moments of order statistics from length-biased exponential distribution and associated inference. *Ann. Data. Sci.*, https://doi.org/10.1007/s40745-020-00245-5
5. Ali M.M. and Khan, A.H. (1987): On order statistics from the log-logistic distribution. *Journal of Statistical Planning and Inference*, **17**, 103-108.
6. Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (2008): *A First Course in Order Statistics*, SIAM Publishers, Philadelphia, USA.
7. Bagdonovacius, V. and Nikulin, M. (2002): *Accelerated Life Models: Modeling and Statistical Analysis.* Chapman and Hall/CRC, Boca Raton, FL, USA.
8. Balakrishnan, N., and Cohen, A. C. (1991): *Order Statistics and Inference: Estimation Methods.* San Diego, CA: Academic Press.
9. Balakrishnan, N., Malik, H.J. and Ahmed, S.E. (1988): Recurrence relations and identities for moments of order statistics II: Specific continuous distributions. *Comm. Statist. Theory Methods*, **17**(8), 2657-2694.
10. Balakrishnan, N. and Sultan, K.S. (1998): Recurrence relations and identities for moments of order statistics, In: N. Balakrishnan and C. R. Rao, eds., Handbook of Statistics, vol. 16, 149-228, Order Statistics: Theory and Methods, North-Holland, Amsterdam, The Netherlands.
11. Cheng, R.C.H. and N.A.K. Amin. (1983): Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society* **B 45**: 394–403.
12. Choulakian, V. and Stephens, M.A. (2001): Goodness-of-fit tests for the generalized Pareto distribution. *Technometrics*, **43**(4), 478-484.
13. Cordeiro, G.M., Afify, A.Z., Yousof, H.M., Cakmakyapan, S. and Ozel, G. (2018): The Lindley Weibull distribution: properties and applications. Anais da Academia Brasileira de Ciências, 90, 2579–2598.
14. David, H.A. and Nagaraja, H.N. (2003): *Order Statistics*. 3rd Edn. John Wiley, New York.
15. Dombi, J., Jónás, T., Tóth, E.Z. and Árva, G. (2019): The omega probability distribution and its applications in reliability theory. *Qual. Reliab. Engng. Int.*, **35**(2), 600-626.
16. Çetinkaya, Ç. and Genç, A.İ. (2018): Moments of order statistics of the standard two-sided power distribution. *Comm. Statist. Theory Methods*,**47**(17), 4311-4328.

17. Gradshteyn, I.S. and Ryzhik, I.M. (2007): Tables of Integrals, Series and Products. Edited by Jeffrey, A. and Zwillinger, D. 7th Ed., Academic Press, San Diego.
18. Hosking, J.R.M. (1990): L-moments: analysis and estimation of distributions using linear combinations of order statistics. *J. Roy. Statist. Soc. B*, **52**(1), 105–124.
19. Kamps, U. (1998): *Characterizations of Distributions by Recurrence Relations and Identities for Moments of Order Statistics.* In: N. Balakrishnan and C.R. Rao, ed. Handbook of statistics, Vol 16. Order statistics: theory and methods, North-Holland, Amsterdam, 291-311.
20. Khan, A.H., Yaqub, M. and Parvez, S. (1983a): Recurrence relations between moments of order statistics. *Naval Research Logistics Quarterly*, **30**, 419-441. [Corrections, **32**, (1985) 693]
21. Khan, A.H., Parvez, S. and Yaqub, M. (1983b): Recurrence relations between product moments of order statistics. *Journal of Statistical Planning and Inference*, **95**, 175-183.
22. Khan, A.H., Khan, R.U. and Parvez, S. (1984): Inverse moments of order statistics from Weibull distribution. *Scand. Actuarial J.*, 91-94.
23. Lin, G.D. (1989): *Characterizations of distributions via moments of order statistics: A survey and comparison of methods.* In: Y. Dodge, ed., Statistical Data Analysis and Inference. North-Holland, Amsterdam, 297-307.
24. Malik, H.J., Balakrishnan, N. and Ahmed, S.E. (1988): Recurrence relations and identities for moments of order statistics I: Arbitrary continuous distribution. *Comm. Statist. Theory Methods*, **17**(8), 2632-2655.
25. Mead, M.E., Cordeiro, G.M., Afify, A.Z. and Al Mofleh, H. (2019): The alpha power transformation family: properties and applications. Pakistan Journal of Statistics and Operation Research, **15**, 525–545.
26. Mudholkar, G. and Srivastava, D. (1993): Exponentiated Weibull family for analyzing bathtub failure-real data. *IEEE Transactions on Reliability*, **42**(2), 299–302.
27. Nadarajah, S. (2008): Explicit Expressions for Moments of Order Statistics. *Statist. Probab. Lett.*, **78**, 196–205.
28. Nagaraja, H.N. (2013): Moments of order statistics and L-moments for the symmetric triangular distribution. *Statist. Probab. Lett.*, **83**, 2357–2363.
29. Nassar, M., Afify, A.Z., Shakhatreh, M.K. and Dey, S. (2020): On a new extension of Weibull distribution: properties, estimation, and applications to one and two causes of failures. *Qual. Reliab. Engng. Int.*, **36**, 2019–2043.
30. Nassar, M., Alzaatreh, A., Mead, M. and Abo-Kasem, O. (2017): Alpha power Weibull distribution: properties and applications. *Comm. Statist. Theory Methods*, **46**, 10236–10252.
31. Okorie, I.E. and Nadarajah, S. (2019): On the omega probability distribution. *Qual. Reliab. Engng. Int.*, **35**(6), 2045-2050.
32. Ranneby, B. (1984): The maximum spacing method: An estimation method related to the maximum likelihood method. *Scand. J. Stat.*, **11**(2), 93–112.
33. R Core Team (2016): R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. url: http://www.R-project.org/.
34. Sarhan, A. and Zaindin, M. (2009): Modified Weibull distribution. *Applied Sciences*, **11**, 123–136.
35. Swain, J.J., Venkatraman, S. and Wilson, J.R. (1988): Least squares estimation of distribution function in Johnson's translation system. *J. Stat Comput Simul.*, **29**, 271–297.