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Some Fixed Point Theorems for $F(\psi, \varphi)$-Contractions and Their Application to Fractional Differential Equations

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Abstract. The main object of this paper is to establish some fixed point results for $F(\psi, \varphi)$-contractions in partially-ordered metric spaces. As an application of one of these fixed point theorems, we discuss the existence of a unique solution for a coupled system of higher-order fractional differential equations with multi-point boundary conditions. The results presented in this paper are shown to extend many recent results appearing in the literature.

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1. INTRODUCTION AND MOTIVATION

Existence of fixed points for contractive mappings in partially-ordered metric spaces has been considered in many recent works (see, for example, [3, 8, 13, 15, 28]), in which applications to matrix, ordinary differential and integral equations are also presented.

Zhou *et al.* [30] considered an interesting class $\mathcal{A}$ of functions, which includes all bounded functions $\beta : [0, \infty) \to [0, K)$ with upper bound $K > 0$, in order to refine the Banach contraction principle.

Theorem 1. (see [30]) *Let $(X, \preceq, d)$ be a partially-ordered complete metric space. Suppose that $f : X \to X$ is a nondecreasing mapping and that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose also that there exist a constant $\theta \in (0, 1/K)$ and a function $\mathfrak{h} \in \mathcal{A}$ such that*

$$d\big(f(x), f(y)\big) \leqq \theta \mathfrak{h}\big(\theta d(x, y)\big) d(x, y) \qquad (\textit{for each } x, y \in X \ \textit{ with } \ x \succeq y).$$

Assume that either f is continuous or that, if an increasing sequence $\{x_n\}$ tends to $x \in X$ then

$$x_n \preceq x \qquad (\forall\, n \in \mathbb{N}).$$

If, for any $x, y \in X$, there exists $z \in X$, which is comparable to x and y, then f has a unique fixed point.

Fractional calculus is a generalization of the ordinary calculus of differentiation and integration to arbitrary noninteger order. It significantly aids in describing various natural phenomena and modelling them more accurately. In particular, fractional differential equations (FDEs) play an important rôle in many fields of science and engineering (see, for details, [9, 10, 14]).

Recently, the existence and uniqueness of a solution of the initial and boundary value problems for nonlinear fractional differential equations were studied systematically by using different types of fixed point theorems such as the Banach contraction principle, Schauder and Leray-Schauder fixed point theorems, and so on (see [2, 5, 6, 7, 26, 27]). There are a few papers which considered multi-point boundary value problems for fractional differential equations when both the nonlinearity and the boundary conditions involve fractional-order derivatives of the unknown functions (see, for example, [16, 19, 29]). Zhou *et al.* [30] used Theorem

1 to prove the existence of a unique positive solution of the following multi-term fractional differential equation with multi-point boundary conditions (see also the related recent works [18, 22, 23, 24, 25] involving fractional-order derivatives of the unknown functions in various applied problems):

$$\begin{gathered}
\mathcal{D}^{\alpha}x(t) = f\big(t, x(t), \mathcal{D}^{\mu_1}x(t), \mathcal{D}^{\mu_2}x(t), \cdots, \mathcal{D}^{\mu_{n-1}}x(t)\big), \\
\mathcal{D}^{\mu_i}x(0) = 0 \quad (i = 1, 2, \cdots, n-1), \\
\mathcal{D}^{\mu_{n-1}+1}x(0) = 0 \qquad \text{and} \qquad \mathcal{D}^{\mu_{n-1}}x(1) = \sum_{j=1}^{m-2} a_j \mathcal{D}^{\mu_{n-1}}x(\xi_j),
\end{gathered} \tag{1.1}$$

where $\alpha \in (n-1, n]$, $3 \leqq n \in \mathbb{N}$, $0 < \mu_1 < \cdots < \mu_{n-1}$, $n-3 < \mu_{n-1} < \alpha-2$, $a_j \in \mathbb{R}$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$,

$$\sum_{j=1}^{m-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} < 1 \qquad \text{and} \qquad f \in C\big([0,1] \times \mathbb{R}^n[0,\infty)\big).$$

By the help of the Schauder fixed point theorem, Rehman and Khan [17] established sufficient conditions for the existence of solutions to multi-point boundary value problems for a coupled system of higher-order fractional differential equations.

Motivated by the above-mentioned recent works, we generalize Theorem 1 by using the class F of functions, which was introduced by Ansari [4]. We then study the existence of a unique solution for the following coupled system of multi-term nonlinear fractional differential equations with multi-point boundary conditions by using a markedly different technique based on finding one fixed point for the $F(\psi, \varphi)$-contractions in partially-ordered metric spaces:

$$\begin{gathered}
\mathcal{D}^{\alpha}x(t) = f\big(t, y(t), \mathcal{D}^{\nu_1}y(t), \cdots, \mathcal{D}^{\nu_{m-1}}y(t)\big), \ \mathcal{D}^{\beta}y(t) = g\big(t, x(t), \mathcal{D}^{\mu_1}x(t), \cdots, \mathcal{D}^{\mu_{n-1}}x(t)\big), \\
D^{\alpha-i}x(0) = 0 \quad (\forall\, i = 1, 2, \cdots, n), \ D^{\beta-j}y(0) = 0 \quad (\forall\, j = 1, 2, \cdots, m), \\
\mathcal{D}^{\mu_{n-1}}x(0) = 0 = \mathcal{D}^{\nu_{m-1}}y(0), \ \mathcal{D}^{\mu_{n-1}+1}x(0) = 0 = \mathcal{D}^{\nu_{m-1}+1}y(0), \\
\mathcal{D}^{\mu_{n-1}}x(1) = \sum_{j=1}^{p-2} a_j \mathcal{D}^{\mu_{n-1}}x(\xi_j) \qquad \text{and} \qquad \mathcal{D}^{\nu_{m-1}}y(1) = \sum_{j=1}^{q-2} b_j \mathcal{D}^{\nu_{m-1}}y(\eta_j),
\end{gathered} \tag{1.2}$$

where $n = [\alpha] + 1$, $m = [\beta] + 1$, $[\alpha]$ and $[\beta]$ denote the integer parts of the real numbers α and β,

$$0 < \mu_1 < \cdots < \mu_{n-1}, \quad 0 < \nu_1 < \cdots < \nu_{m-1}, \quad n-3 \leqq \mu_{n-1} < \alpha - 2, \quad m-3 \leqq \nu_{m-1} < \beta - 2,$$

$$0 < \xi_1 < \cdots < \xi_{p-2} < 1 \qquad \text{and} \qquad 0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} < 1,$$

$$0 < \eta_1 < \cdots < \eta_{q-2} < 1 \qquad \text{and} \qquad 0 < \sum_{j=1}^{q-2} b_j \eta_j^{\beta-\nu_{m-1}-1} < 1$$

and $f, g \in C[0,1] \cap L[0,1]$.

2. DEFINITIONS AND PRELIMINARIES

Definition 1. (see [4]) An *ultra altering distance function* is a continuous and nondecreasing mapping $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for $t > 0$. Let Φ denote the class of ultra altering distance functions.

Definition 2. (see [4]) A mapping $F : [0,\infty)^2 \to R$ is called a C-*class function* if it is continuous and satisfies the following axioms:

1. $F(s,t) \leqq s$;
2. $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $t \in [0,\infty)$.

Let $\mathcal{C}$ denote the set of C-class functions.

Definition 3. (see [10] and [14]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $x : [0,\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, x(s)ds,$$

provided that the right-hand side is pointwise defined on $[0,\infty)$.

Definition 4. (see [10] and [14]) The *Riemann–Liouville fractional derivative of order* $\alpha > 0$ of a continuous function $x : [0,\infty) \to \mathbb{R}$ is given by

$$\mathcal{D}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n}\left\{\int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}\, ds\right\},$$

where $n = [\alpha]+1$, provided that the right-hand side is pointwise defined on $[0,\infty)$.

Lemma 1. (see [26]) *The Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative satisfy each of the following properties*:

(1) $I^{\alpha}I^{\beta}x(t) = I^{\alpha+\beta}x(t)$ *and* $\mathcal{D}^{\alpha}I^{\beta}x(t) = I^{\beta-\alpha}x(t)$ *for all* $\beta \geqq \alpha > 0$, $x \in L[0,1]$;

(2) $I^{\alpha}\mathcal{D}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + \cdots + c_nt^{\alpha-n}$, *where* $n = [\alpha]+1$ *and* $x, \mathcal{D}^{\alpha}x \in C[0,1] \cap L[0,1]$;

(3) $I^{\alpha} : C[0,1] \to C[0,1]$, *where* $\alpha > 0$.

Lemma 2. *Assume that the function* $x : [0,1] \to [0,\infty)$ *and its fractional derivative of order* $\alpha > 0$ *belongs to* $C[0,1] \cap L[0,1]$. *If* x *satisfies the following fractional differential equation*:

$$\mathcal{D}^{\alpha}x(t) = h(t) \ \text{ and } \ D^{\alpha-i}x(0) = 0 \qquad (\forall\, i = 1,2,\cdots,n;\ n = [\alpha]+1), \tag{2.1}$$

then x *can be written as* $x(t) = I^{\mu}u(t)$ *for some continuous function* u.

Proof. In fact, by Eq. (2.1), we have

$$I^{\alpha}\mathcal{D}^{\alpha}x(t) = I^{\alpha}h(t).$$

Thus, by applying Lemma 1 and the following relations (see [5, Remark 2.1]):

$$\mathcal{D}^{\alpha}t^{\lambda} = \begin{cases} \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}\, t^{\lambda-\alpha} & (\lambda > -1;\ \lambda \geqq \alpha > 0) \\ 0 & (\lambda < \alpha), \end{cases}$$

we obtain

$$\begin{aligned} & x(t) + c_1t^{\alpha-1} + \cdots + c_nt^{\alpha-n} = I^{\alpha}h(t), \\ & \mathcal{D}^{\alpha-1}x(t) + \Gamma(\alpha)c_1 = I^1h(t), \ \text{ at } t=0 \implies c_1 = 0, \\ & \mathcal{D}^{\alpha-2}x(t) + \Gamma(\alpha-1)c_2 = I^2h(t), \ \text{ at } t=0 \implies c_2 = 0, \\ & \vdots \\ & \mathcal{D}^{\alpha-n}x(t) + \Gamma(\alpha-n+1)c_n = I^nh(t), \ \text{ at } t=0 \implies c_n = 0, \end{aligned}$$

so that

$$x(t) = I^{\alpha}h(t) = I^{\mu}I^{\alpha-\mu}h(t) = I^{\mu}u(t).$$

Our demonstration of Lemma 2 is thus completed. □

3. A SET OF FIXED POINT RESULTS

We begin this section by stating and proving Theorem 2 below.

Theorem 2. *Let* $(X,\preceq,d)$ *be a partially-ordered complete metric space and let* $T : X \to X$ *be a nondecreasing mapping such that*

$$\psi\big(d(Tx,Ty)\big) \leqq F\Big(\psi\big(d(x,y)\big), \varphi\big(d(x,y)\big)\Big) \qquad (\forall\, x \succeq y), \tag{3.1}$$

where $\psi,\varphi \in \Phi$ *and* $F \in \mathcal{C}$. *Suppose that either*

(a) *T is continuous or*

(b) *X satisfies following property:*

$$\textit{If a nondecreasing sequence } \{x_n\} \to x \in X, \textit{ then } x_n \preceq x \quad (\forall\, n \in \mathbb{N}). \tag{3.2}$$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$. Define a sequence $\{x_n\} \in X$ by

$$x_n = T^n x_0 = Tx_{n-1} \qquad (\forall\, n \in \mathbb{N}).$$

Since T is monotone nondecreasing and $x_0 \preceq x_1$, so we have

$$x_1 = Tx_0 \preceq Tx_1 = x_2.$$

Continuing the above process, we obtain

$$x_n \preceq x_{n+1} \qquad (\forall\, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{3.3}$$

Now, from (3.1) and (3.3), we have

$$\begin{aligned}\psi\big(d(x_{n+1}, x_n)\big) = \psi\big(d(Tx_n, Tx_{n-1})\big) &\leqq F\Big(\psi\big(d(x_n, x_{n-1})\big), \varphi\big(d(x_n, x_{n-1})\big)\Big)\\ &\leqq \psi\big(d(x_{n-1}, x_n)\big).\end{aligned} \tag{3.4}$$

We want to prove that

$$d(x_{n+1}, x_n) \to 0 \qquad (n \to \infty).$$

If $d(x_{n+1}, x_n) = 0$ for some $n \in \mathbb{N}_0$, then

$$x_n = x_{n+1} = Tx_n,$$

that is, x_n is a fixed point of T. So, we consider

$$d(x_{n+1}, x_n) > 0 \qquad (\forall\, n \in \mathbb{N}_0).$$

The inequality (3.4) and the properties of ψ imply that

$$d(x_{n+1}, x_n) \leqq d(x_n, x_{n-1}) \qquad (\forall\, n \in \mathbb{N}_0).$$

It follows that the sequence $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence of positive numbers. Thus, clearly, there exists $r \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{3.5}$$

We next proceed to prove that $r = 0$. Indeed, by letting $n \to \infty$ in (3.4), we find that

$$\limsup_{n\to\infty} \psi\big(d(x_n, x_{n+1})\big) \leqq \limsup_{n\to\infty} F\Big(\psi\big(d(x_{n-1}, x_n)\big), \varphi\big(d(x_{n-1}, x_n)\big)\Big) \leqq \limsup_{n\to\infty} \psi\big(d(x_{n-1}, x_n)\big).$$

Using Eq. (3.5) and the properties of the functions F, ψ and φ, we obtain

$$\psi(r) \leqq F\big(\psi(r), \varphi(r)\big) \leqq \psi(r) \Longrightarrow F\big(\psi(r), \varphi(r)\big) = \psi(r) \Longrightarrow \psi(r) = 0 \text{ or } \varphi(r) = 0 \Longrightarrow r = 0.$$

Therefore, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.6}$$

We now prove that $\{x_n\}$ is a Cauchy sequence in (X, d). Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$n_k > m_k > k, \;\; d(x_{n_k}, x_{m_k}) \geqq \epsilon. \tag{3.7}$$

Suppose that k is the smallest integer which satisfies (3.7). Then

$$d(x_{n_k-1},x_{m_k})<\epsilon. \tag{3.8}$$

By the triangle inequality of the metric d, we obtain

$$\epsilon \leqq d(x_{n_k},x_{m_k}) \leqq d(x_{n_k},x_{n_k-1})+d(x_{n_k-1},x_{m_k}) < d(x_{n_k},x_{n_k-1})+\epsilon.$$

Letting k tend to infinity and using (3.6), we have

$$\lim_{k\to\infty} d(x_{n_k},x_{m_k})=\epsilon. \tag{3.9}$$

On the other hand, the triangle inequality implies that

$$\begin{aligned}
&d(x_{n_k},x_{m_k}) \leqq d(x_{n_k},x_{n_k+1})+d(x_{n_k+1},x_{m_k+1})+d(x_{m_k+1},x_{m_k}),\\
&d(x_{n_k+1},x_{m_k+1}) \leqq d(x_{n_k+1},x_{n_k})+d(x_{n_k},x_{m_k})+d(x_{m_k},x_{m_k+1})\\
&\Longrightarrow\ 0 \leqq \left|d(x_{n_k+1},x_{m_k+1})-d(x_{n_k},x_{m_k})\right| \leqq d(x_{n_k+1},x_{n_k})+d(x_{m_k},x_{m_k+1})\\
&\Longrightarrow\ \lim_{k\to\infty}\left|d(x_{n_k+1},x_{m_k+1})-d(x_{n_k},x_{m_k})\right|=0.
\end{aligned}$$

So, we have

$$\lim_{k\to\infty} d(x_{n_k+1},x_{m_k+1})=\epsilon. \tag{3.10}$$

As $x_{n_k} \geqq x_{m_k}$, we can apply (3.1) to obtain

$$\psi\big(d(x_{n_k+1},x_{m_k+1})\big)=\psi\big(d(Tx_{n_k},Tx_{m_k})\big) \leqq F\Big(\psi\big(d(x_{n_k},x_{m_k})\big),\varphi\big(d(x_{n_k},x_{m_k})\big)\Big) \leqq \psi\big(d(x_{n_k},x_{m_k})\big).$$

Letting $k\to\infty$ and taking into account (3.9) and (3.10), we obtain

$$\psi(\epsilon) \leqq F\big(\psi(\epsilon),\varphi(\epsilon)\big) \leqq \psi(\epsilon) \Longrightarrow\ F\big(\psi(\epsilon),\varphi(\epsilon)\big)=\psi(\epsilon) \Longrightarrow\ \psi(\epsilon)=0 \ \text{ or } \ \varphi(\epsilon) \Longrightarrow\ \epsilon=0,$$

which is a contradiction, so our assumption is not true, that is, $\{x_n\}$ is a Cauchy sequence. Therefore, by the completeness of the space X, we have

$$\exists\, x\in X: x_n \to x \text{ as } n\to\infty.$$

Now, if T is continuous, then

$$x=\lim_{n\to\infty} x_{n+1}=\lim_{n\to\infty} Tx_n = Tx.$$

In the case when (3.2) holds true, we claim that $x=Tx$ still holds true. Applying (3.1), we obtain

$$\begin{aligned}
\psi\big(d(Tx,x_{n+1})\big) =&\psi\big(d(Tx,Tx_n)\big) \leqq F\Big(\psi\big(d(x,x_n)\big),\varphi\big(d(x,x_n)\big)\Big) \leqq \psi\big(d(x,x_n)\big)\\
&\Longrightarrow\ \psi\big(d(Tx,x)\big) \leqq F\big(\psi(0),\varphi(0)\big) \leqq \psi(0).
\end{aligned}$$

Since ψ is nondecreasing, we have $d(Tx,x)=0$ and this proves that x is a fixed point. □

For the uniqueness of the fixed point in Theorem 2, we consider the following condition:

$$\text{For } \ x,y\in X, \ \text{ there exists } z\in X \ \text{ which is comparable to } x \text{ and } y. \tag{3.11}$$

Theorem 3. *Under the added condition* (3.11) *to the hypotheses of Theorem* 2, *the fixed point of* T *is unique.*

Proof. Suppose that x and y are two fixed points. We distinguish the following two cases:

Case 1. If x and y are comparable (say $x\succeq y$), then

$$\begin{aligned}
\psi\big(d(x,y)\big)=\psi\big(d(Tx,Ty)\big) &\leqq F\Big(\psi\big(d(x,y)\big),\varphi\big(d(x,y)\big)\Big) \leqq \psi\big(d(x,y)\big)\\
&\Longrightarrow\ F\Big(\psi\big(d(x,y)\big),\varphi\big(d(x,y)\big)\Big)=\psi\big(d(x,y)\big)\\
&\Longrightarrow\ \psi\big(d(x,y)\big)=0 \ \text{ or } \ \varphi\big(d(x,y)\big)=0 \Longrightarrow\ d(x,y)=0 \ \Longrightarrow\ x=y.
\end{aligned}$$

Case 2. If x and y are not comparable, then there exists $z\in X$, which is comparable to x and y. Therefore, one of the following conditions holds:

- $z \succeq x$ and $z \succeq y$;
- $z \succeq x$ and $z \preceq y$;
- $z \preceq x$ and $z \succeq y$;
- $z \preceq x$ and $z \preceq y$.

We consider the case when the first relation to hold true and the other relations remain the same. Then the monotonicity of T implies that

$$T^n z \succeq T^n x = x \quad \text{and} \quad T^n z \succeq T^n y = y \qquad (\forall\, n \in \mathbb{N} \cup \{0\})$$

and

$$\begin{aligned} \psi\big(d(T^n z, x)\big) &= \psi\big(d(T^n z, T^n x)\big) \leqq F\Big(\psi\big(d(T^{n-1}z, T^{n-1}x)\big),\ \varphi\big(d(T^{n-1}z, T^{n-1}x)\big)\Big) \\ &\leqq \psi\big(d(T^{n-1}z, x)\big). \end{aligned} \tag{3.12}$$

Thus, clearly, $\{d(T^n z, x)\}$ is a monotone nonincreasing sequence of positive numbers. So

$$\exists\, \gamma \geqq 0 : \{d(T^n z, x)\} \to \gamma.$$

Now, from (3.12), we obtain

$$\lim_{n\to\infty} \psi\big(d(T^n z, x)\big) \leqq \lim_{n\to\infty} F\Big(\psi\big(d(T^{n-1}z, x)\big), \varphi\big(d(T^{n-1}z, x)\big)\Big) \leqq \lim_{n\to\infty} \psi\big(d(T^{n-1}z, x)\big)$$

and

$$\psi(\gamma) \leqq F\big(\psi(\gamma), \varphi(\gamma)\big) \leqq \psi(\gamma),$$

which imply that

$$F\big(\psi(\gamma), \varphi(\gamma)\big) = \psi(\gamma) \Longrightarrow \gamma = 0.$$

Consequently, we have

$$\lim_{n\to\infty} d(T^n z, x) = 0. \tag{3.13}$$

Analogously, one can prove that

$$\lim_{n\to\infty} d(T^n z, y) = 0. \tag{3.14}$$

The uniqueness of the limit gives us $x = y$, thereby completing our proof of Theorem 3. □

4. FRACTIONAL DIFFERENTIAL EQUATIONS

Consider the following modified form of the system given by (1.2),

$$\begin{aligned} &\mathcal{D}^{\alpha-\mu_{n-1}} u(t) = f\big(t, I^{\nu_{m-1}} v(t), I^{\nu_{m-1}-\nu_1} v(t), \cdots, v(t)\big), \\ &\mathcal{D}^{\beta-\nu_{m-1}} v(t) = g\big(t, I^{\mu_{n-1}} u(t), I^{\mu_{n-1}-\mu_1} u(t), \cdots, u(t)\big), \\ &u(0) = 0 = v(0), \quad u'(0) = 0 = v'(0), \\ &u(1) = \sum_{j=1}^{p-2} a_j u(\xi_j), \quad v(1) = \sum_{j=1}^{q-2} b_j v(\eta_j), \\ &0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1} < 1 \quad \text{and} \quad 0 < \sum_{j=1}^{q-2} b_j \eta_j^{\beta-\nu_{m-1}-1} < 1. \end{aligned} \tag{4.1}$$

Lemma 3. *The boundary value problem* (1.2) *is equivalent to the one mentioned in* (4.1). *Moreover, if* $(u, v) \in C^2[0,1]$ *is a solution of problem* (4.1), *then the pair of functions given by*

$$\big(x(t), y(t)\big) = \big(I^{\mu_{n-1}} u(t), I^{\nu_{m-1}} v(t)\big)$$

is a solution of the problem (1.2).

Proof. Consider the problem (1.2) and put

$$x(t) = I^{\mu_{n-1}}u(t) \qquad \text{and} \qquad y(t) = I^{\nu_{m-1}}v(t).$$

This can be done according to Lemma 2. Then, by the definition of the Riemann-Liouville fractional derivative and Lemma 1, we obtain

$$\begin{aligned}
&\mathcal{D}^{\alpha}x(t) = \frac{d^n}{dt^n}I^{n-\alpha}I^{\mu_{n-1}}u(t) = \mathcal{D}^{\alpha-\mu_{n-1}}u(t), \ \ \mathcal{D}^{\beta}y(t) = \mathcal{D}^{\beta-\nu_{m-1}}v(t),\\
&\mathcal{D}^{\mu_1}x(t) = \mathcal{D}^{\mu_1}I^{\mu_{n-1}}u(t) = I^{\mu_{n-1}-\mu_1}u(t), \ \ \mathcal{D}^{\nu_1}y(t) = I^{\nu_{m-1}-\nu_1}v(t),\\
&\vdots\\
&\mathcal{D}^{\mu_{n-1}}x(t) = \mathcal{D}^{\mu_{n-1}}I^{\mu_{n-1}}u(t) = u(t), \ \ \mathcal{D}^{\nu_{m-1}}y(t) = v(t),\\
&\mathcal{D}^{\mu_{n-1}+1}x(t) = \frac{d}{dt}\mathcal{D}^{\mu_{n-1}}x(t) = u'(t), \ \ \mathcal{D}^{\nu_{m-1}+1}y(t) = v'(t).
\end{aligned}$$

We also have

$$\begin{aligned}
&u(0) = 0 = v(0), \ \ u'(0) = 0 = v'(0),\\
&u(1) = \sum_{j=1}^{p-2} a_j u(\xi_j) \qquad \text{and} \qquad v(1) = \sum_{j=1}^{q-2} b_j u(\eta_j).
\end{aligned}$$

Hence, by the following choice:

$$x(t) = I^{\mu_{n-1}}u(t) \qquad \text{and} \qquad y(t) = I^{\nu_{m-1}}v(t),$$

the equations (1.2) can be transformed into (4.1).

Now, let $u, v \in C[0,1]$ be a solution of the system (4.1). Then, by Lemma 1, we obtain

$$\begin{aligned}
\mathcal{D}^{\alpha}I^{\mu_{n-1}}u(t) &= f\big(t, I^{\nu_{m-1}}v(t), \mathcal{D}^{\nu_1}I^{\nu_{m-1}}v(t), \cdots, \mathcal{D}^{\nu_{m-1}}I^{\nu_{m-1}}v(t)\big)\\
\mathcal{D}^{\beta}I^{\nu_{m-1}}v(t) &= g\big(t, I^{\mu_{n-1}}u(t), \mathcal{D}^{\mu_1}I^{\mu_{n-1}}u(t), \cdots, \mathcal{D}^{\mu_{n-1}}I^{\mu_{n-1}}u(t)\big).
\end{aligned}$$

Upon setting

$$I^{\mu_{n-1}}u(t) = x(t) \qquad \text{and} \qquad I^{\nu_{m-1}}v(t) = y(t),$$

we obtain (1.2). Also, the boundary conditions in (4.1) implies those in (1.2). We note here that

$$\begin{aligned}
&\mathcal{D}^{\alpha-i}x(t) = I^{\mu_{n-1}-\alpha+i}u(t) = \frac{1}{\Gamma(\mu_{n-1}-\alpha+i)}\int_0^t (t-s)^{\mu_{n-1}-\alpha+i-1}\, u(s)ds\\
&\Longrightarrow \ \mathcal{D}^{\alpha-i}x(0) = 0 \qquad (i = 1, 2, \cdots, n).
\end{aligned}$$

In a similar way, we have

$$\mathcal{D}^{\beta-j}y(0) = 0 \qquad (j = 1, 2, \cdots, m).$$

Consequently, $x, y : [0,1] \to [0,\infty)$ are solutions of (1.2). □

Lemma 4. *If $n-1 < \alpha \leqq n$, $n-3 \leqq \mu_{n-1} < \alpha - 2$ and $h \in L[0,1]$, then the following boundary value problem:*

$$\mathcal{D}^{\alpha-\mu_{n-1}}u(t) = h(t), \ \ u(0) = u'(0) = 0, \ \ u(1) = \sum_{j=1}^{r-2} d_j u(\xi_j) \tag{4.2}$$

has the unique solution given by

$$u(t) = \int_0^1 K(t,s)h(s)ds,$$

where

$$K(t,s) = k(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}\sum\limits_{j=1}^{r-2} d_j k(\xi_j, s)}{1 - \sum\limits_{j=1}^{r-2} d_j \xi_j^{\alpha-\mu_{n-1}-1}}$$

and

$$k(t,s) = \begin{cases} \dfrac{\big(t(1-s)\big)^{\alpha-\mu_{n-1}-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} & (0 \leqq s \leqq t \leqq 1) \\[2ex] \dfrac{\big(t(1-s)\big)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} & (1 \geqq s \geqq t > 0), \end{cases}$$

$K(t,s)$ being the Green function of the boundary value problem (4.2).

Proof. Applying Lemma 1, we can reduce (4.2) to an equivalent integral equation given by

$$I^{\alpha-\mu_{n-1}}\mathcal{D}^{\alpha-\mu_{n-1}}u(t) = I^{\alpha-\mu_{n-1}}h(t),$$

that is,

$$u(t) + c_1 t^{\alpha-\mu_{n-1}-1} + c_2 t^{\alpha-\mu_{n-1}-2} + c_3 t^{\alpha-\mu_{n-1}-3} = \int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds,$$

which, for $t=0$, implies eventually that $c_3 = 0$.

We also have

$$\begin{aligned} &u'(t) + c_1(\alpha-\mu_{n-1}-1)t^{\alpha-\mu_{n-1}-2} + c_2(\alpha-\mu_{n-1}-2)t^{\alpha-\mu_{n-1}-3} \\ &\quad = \frac{d}{dt}\left\{\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1}-1)}\, h(s)ds\right\} = \int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-2}}{\Gamma(\alpha-\mu_{n-1}-1)}\, h(s)ds, \end{aligned}$$

which, for $t=0$, implies eventually that $c_2 = 0$. Hence we find that

$$u(t) + c_1 t^{\alpha-\mu_{n-1}-1} = \int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds.$$

For $t=1$ and $t=\xi_j$ $(j=1,2,\cdots,p-2)$, we have

$$u(1) + c_1 = \int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds$$

and

$$u(\xi_j) + c_1 \xi_j^{\alpha-\mu_{n-1}-1} = \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds,$$

respectively. So, we obtain

$$\begin{aligned} c_1 &= \int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds - \sum_{j=1}^{r-2} d_j u(\xi_j) \\ &= \int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds - \sum_{j=1}^{r-2} d_j \left(\int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds - c_1\xi_j^{\alpha-\mu_{n-1}-1}\right) \\ &= \frac{1}{1-\sum\limits_{j=1}^{r-2} d_j\xi_j^{\alpha-\mu_{n-1}-1}} \left(\int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds - \sum_{j=1}^{r-2} d_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\, h(s)ds\right). \end{aligned}$$

Consequently, the general solution of (4.2) is given

$$u(t)=\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds-\frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum\limits_{j=1}^{r-2} d_j\xi_j^{\alpha-\mu_{n-1}-1}}$$

$$\times\left(\int_0^1 \frac{(1-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds-\sum_{j=1}^{r-2} d_j\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds\right)$$

$$=\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds-\int_0^1\frac{(t(1-s))^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds-\frac{t^{\alpha-\mu_{n-1}-1}\sum\limits_{j=1}^{r-2} d_j}{1-\sum\limits_{j=1}^{r-2} d_j\xi_j^{\alpha-\mu_{n-1}-1}}$$

$$\times\left(\int_0^1 \frac{(\xi_j(1-s))^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds-\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}\,h(s)ds\right),$$

that is,

$$u(t)=-\int_0^1 k(t,s)h(s)ds-\frac{t^{\alpha-\mu_{n-1}-1}\sum\limits_{j=1}^{r-2} d_j}{1-\sum\limits_{j=1}^{r-2} d_j\xi_j^{\alpha-\mu_{n-1}-1}}\int_0^1 k(\xi_j,s)h(s)ds$$

$$=-\int_0^1\left(k(t,s)+\frac{t^{\alpha-\mu_{n-1}-1}\sum\limits_{j=1}^{r-2} d_j k(\xi_j,s)}{1-\sum\limits_{j=1}^{r-2} d_j\xi_j^{\alpha-\mu_{n-1}-1}}\right)h(s)ds=-\int_0^1 K(t,s)h(s)ds.$$

□

We now consider the space $C[0,1]$ defined by

$$C[0,1]:=\big\{x:[0,1]\to\mathbb{R}\ \ (x \text{ is continuous})\big\}$$

with the classical metric and order given by

$$d(x,y)=\sup_{t\in[0,1]}\big|x(t)-y(t)\big|;\quad x\preceq y\Leftrightarrow x(t)\leqq y(t)\qquad(\forall\, t\in[0,1]).$$

The completeness of the space $(C[0,1],d)$ is discussed in details in the earlier work [11].

We now prove that the space $(C[0,1],\preceq,d)$ satisfies the condition (3.2). Let $\{x_n\}$ be a convergent nondecreasing sequence in $C[0,1]$, that is,

$$x_n\preceq x_{n+1}\to x\in C[0,1].$$

Then, for any $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\big|x_n(t)-x(t)\big|\leqq\sup_{t\in[0,1]}\big|x_n(t)-x(t)\big|=d(x_n,x)<\varepsilon\qquad(\forall\, n>n_0)$$

and

$$x_n(t)\leqq x_{n+1}(t)\qquad(\forall\, t\in[0,1]).$$

Therefore, for a fixed t, the sequence $\big(x_1(t),x_2(t),\cdots\big)$ is a convergent nondecreasing sequence of real numbers. Then the least upper bound of this sequence is the limit $x(t)\in\mathbb{R}$ or $x_n(t)\leqq x(t)$ $(\forall\, n)$. Thus, for each $t\in[0,1]$, we have

$$x_n(t)\leqq x(t)\implies x_n\preceq x,$$

which verifies our claim. Moreover, for $x,y\in C[0,1]$, the function $\max\{x,y\}\in C[0,1]$ is that function which is comparable at x and y. That is, $(C[0,1],\preceq)$ satisfies the condition (3.11) of Theorem 3.

For our further investigation, we will work in the space $C^2[0,1]$ with the metric $\bar{d}$ and the partial order $\preceq_p$ defined for (x,y) and $(u,v) \in C^2[0,1]$ by

$$\bar{d}\big((x,y),(u,v)\big) = d(x,u) + d(y,v)$$

and

$$(x,y) \preceq_p (u,v) \iff x \preceq u \qquad \text{and} \qquad y \preceq v.$$

For simplicity, the symbol X stands for the space $\big(C^2[0,1], \preceq_p, \bar{d}\big)$.

Lemma 5. *The partially-ordered metric space X is complete and satisfies the conditions* (3.2) *and* (3.11).

Proof. Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $\big(C^2[0,1], \bar{d}\big)$. Then, for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m)\big(\text{or, } d(y_n, y_m)\big) \leqq \bar{d}\big((x_n, y_n), (x_m, y_m)\big) < \varepsilon \qquad (\forall\, n, m > n_0).$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $\big(C[0,1], d\big)$ which is complete. Thus there are $x, y \in C[0,1]$ such that

$$x_n \to x \text{ and } y_n \to y \ \text{ as } \ n \to \infty.$$

Therefore, we obtain

$$(x_n, y_n) \to (x,y) \in C^2[0,1] \ \text{ as } \ n \to \infty.$$

So, the metric space $\big(C^2[0,1], \bar{d}\big)$ is complete.

Furthermore, by assuming that $\{(x_n, y_n)\}$ is a convergent nondecreasing sequence in X, we have

$$\begin{gathered}(x_n, y_n) \preceq_p (x_{n+1}, y_{n+1}) \to (x,y) \in C^2[0,1] \\ \Longrightarrow x_n \preceq x_{n+1} \to x \in C[0,1] \ \text{ and } \ y_n \preceq y_{n+1} \to y \in C[0,1].\end{gathered}$$

This shows that x_n and y_n are convergent nondecreasing sequences in $\big(C[0,1], \preceq, d\big)$, which satisfies the condition (3.2). So, we obtain

$$x_n \preceq x \ \text{ and } \ y_n \preceq y \implies (x_n, y_n) \preceq_p (x,y) \qquad (\forall\, n \in \mathbb{N}).$$

Thus the space X satisfies the condition (3.2). Moreover, for (x,y) and $(u,v) \in C^2[0,1]$, the function $\max\{(x,y),(u,v)\} \in C^2[0,1]$ is that function which is comparable at (x,y) and (u,v). That is, $(C^2[0,1], \preceq_p)$ satisfies the condition (3.11), too. □

As a consequence of Lemma 4, the solution of the system (4.1) coincides with that of the integral equations given by

$$\begin{aligned} u(t) &= \int_0^1 K_1(t,s) f\big(s, I^{\nu_{m-1}} v(s),\ I^{\nu_{m-1}-\nu_1} v(s), \cdots, v(s)\big) ds \\ v(t) &= \int_0^1 K_2(t,s) g\big(s, I^{\mu_{n-1}} u(s),\ I^{\mu_{n-1}-\mu_1} u(s), \cdots, u(s)\big) ds, \end{aligned} \tag{4.3}$$

where

$$K_1(t,s) = k_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1} \sum\limits_{j=1}^{p-2} a_j k_1(\xi_j, s)}{1 - \sum\limits_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}}$$

and

$$k_1(t,s) = \begin{cases} \dfrac{\big(t(1-s)\big)^{\alpha-\mu_{n-1}-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} & (0 \leqq s \leqq t \leqq 1) \\[2ex] \dfrac{\big(t(1-s)\big)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} & (1 \geqq s > t \geqq 0); \end{cases}$$

$$K_2(t,s) = k_2(t,s) + \frac{t^{\beta-\nu_{m-1}-1}\sum\limits_{j=1}^{q-2} b_j k_2(\eta_j, s)}{1-\sum\limits_{j=1}^{q-2} b_j \eta_j^{\beta-\nu_{m-1}-1}}$$

and

$$k_2(t,s) = \begin{cases} \dfrac{\big(t(1-s)\big)^{\beta-\nu_{m-1}-1} - (t-s)^{\beta-\nu_{m-1}-1}}{\Gamma(\beta-\nu_{m-1})}, & (0 \leqq s \leqq t \leqq 1) \\[2ex] \dfrac{\big(t(1-s)\big)^{\beta-\nu_{m-1}-1}}{\Gamma(\beta-\nu_{m-1})} & (1 \geqq s > t \geqq 0). \end{cases}$$

We note for all $0 \leqq s, t \leqq 1$ that

$$k_1(t,s) \leqq \frac{\big(t(1-s)\big)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} \leqq \frac{1}{\Gamma(\alpha-\mu_{n-1})}$$

and

$$k_2(t,s) \leqq \frac{\big(t(1-s)\big)^{\beta-\nu_{m-1}-1}}{\Gamma(\beta-\nu_{m-1})} \leqq \frac{1}{\Gamma(\beta-\nu_{m-1})}.$$

We thus find that

$$K_1(t,s) \leqq \frac{1}{\Gamma(\alpha-\mu_{n-1})}\left(1 + \frac{\sum\limits_{j=1}^{p-2} a_j}{1-\sum\limits_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu_{n-1}-1}}\right) =: \rho_1$$

and

$$K_2(t,s) \leqq \frac{1}{\Gamma(\beta-\nu_{m-1})}\left(1 + \frac{\sum\limits_{j=1}^{q-2} b_j}{1-\sum\limits_{j=1}^{q-2} b_j \eta_j^{\beta-\nu_{m-1}-1}}\right) =: \rho_2.$$

We now define the operator $T : C^2[0,1] \to C^2[0,1]$ by

$$\begin{aligned} T(u,v)(t) &= \left(\int_0^1 K_1(t,s) f\big(s, I^{\nu_{m-1}}v(s), \cdots, v(s)\big)ds,\ \int_0^1 K_2(t,s) g\big(s, I^{\mu_{n-1}}u(s), \cdots, u(s)\big)ds\right) \\ &= \big(T_1 v(t), T_2 u(t)\big). \end{aligned}$$

Then, by Lemma 3 and Lemma 4, the fixed point of the operator T coincides with the solution of the system (1.2).

For the following discussion, let us define

$$\begin{aligned} \theta_1 &= \max\left\{\frac{1}{\Gamma(\nu_{m-1}+1)}, \frac{1}{\Gamma(\nu_{m-1}-\nu_1+1)}, \cdots, \frac{1}{\Gamma(1)}\right\}, \\ \theta_2 &= \max\left\{\frac{1}{\Gamma(\mu_{n-1}+1)}, \frac{1}{\Gamma(\mu_{n-1}-\mu_1+1)}, \cdots, \frac{1}{\Gamma(1)}\right\}, \\ \theta &= \max\{m\theta_1, n\theta_2\}, \\ \rho &= \max\{\rho_1, \rho_2\}. \end{aligned}$$

We also consider

$$\psi(t) = \theta t \qquad \text{and} \qquad F(s,t) = \sqrt[\mathfrak{n}]{\ln\left(1+s^{\mathfrak{n}}\right)} \qquad (\mathfrak{N} \in \mathbb{N})$$

and let $\varphi(t)$ be any ultra altering distance function.

Theorem 4. *Suppose that $f(t, x_1, \cdots, x_m)$ and $g(t, y_1, \cdots, y_n)$ are nondecreasing functions in x_i $(i = 1, 2, \cdots, n)$ and y_j $(j = 1, 2, \cdots, m)$. Also let f and g satisfy the following conditions*:

$$\left|f\left(t, y_1, \cdots, y_m\right) - f\left(t, v_1, \cdots, v_m\right)\right| = \frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\left(\sum_{j=1}^{m}|y_j - v_j|\right)^{\mathfrak{n}}\right)}$$

and

$$\left|g\left(t, x_1, \cdots, x_n\right) - g\left(t, u_1, \cdots, u_n\right)\right| = \frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\left(\sum_{i=1}^{n}|x_i - u_i|\right)^{\mathfrak{n}}\right)}$$

for

$$\mathfrak{n} \in \mathbb{N}, \quad x_i \succeq u_i \qquad \text{and} \qquad y_j \succeq v_j \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m).$$

Then the problem (1.2) *has a unique solution.*

Proof. We verify that the hypotheses in Theorem 2 are satisfied. Firstly, since the operator T is nondecreasing, we find for $(x, y) \preceq_p (u, v)$ that

$$\begin{aligned}
T(x,y)(t) &= \left(\int_0^1 K_1(t,s) f\left(s, I^{\nu_{m-1}}y(s), \cdots, y(s)\right)ds,\ \int_0^1 K_2(t,s) g\left(s, I^{\mu_{n-1}}x(s), \cdots, x(s)\right)ds\right) \\
&\leqq \left(\int_0^1 K_1(t,s) f\left(s, I^{\nu_{m-1}}v(s), \cdots, v(s)\right)ds,\ \int_0^1 K_2(t,s) g\left(s, I^{\mu_{n-1}}u(s), \cdots, u(s)\right)ds\right) \\
&\leqq (T_1 v, T_2 u) = T(u,v)(t).
\end{aligned}$$

Also, for $x \succeq u$ and $y \succeq v$, we have

$$\begin{aligned}
d(T_2x(t), T_2u(t)) &= \sup_{t\in[0,1]} \left|T_2x(t) - T_2u(t)\right| \\
&= \sup_{t\in[0,1]} \left|\int_0^1 K_2(t,s)\left[g\left(s, I^{\mu_{n-1}}x(s), \cdots, x(s)\right) - g\left(s, I^{\mu_{n-1}}u(s), \cdots, u(s)\right)\right]ds\right| \\
&\leqq \rho_2 \int_0^1 \left|g\left(s, I^{\mu_{n-1}}x(s), \cdots, x(s)\right) - g\left(s, I^{\mu_{n-1}}u(s), \cdots, u(s)\right)\right|ds \\
&\leqq \rho_2 \frac{1}{2\rho\theta}\int_0^1 \sqrt[\mathfrak{n}]{\ln\left(1+\left(\left|I^{\mu_{n-1}}x(s) - I^{\mu_{n-1}}u(s)\right| + \cdots + |x(s) - u(s)|\right)^{\mathfrak{n}}\right)}ds \\
&\leqq \rho_2 \frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\left(\frac{d(x,u)}{\Gamma(\mu_{n-1}+1)} + \cdots + d(x,u)\right)^{\mathfrak{n}}\right)} \\
&\leqq \rho_2 \frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\left(n\theta_2 d(x,u)\right)^{\mathfrak{n}}\right)}.
\end{aligned} \tag{4.4}$$

We note here that

$$I^{\alpha}x(t) - I^{\alpha}u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s) - u(s)|ds \leqq \frac{d(x,u)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds \leqq \frac{d(x,u)}{\Gamma(\alpha+1)}.$$

In a similar way, we obtain

$$d(T_1y(t), T_1v(t)) \leqq \rho_1 \frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\left(m\theta_1 d(y,v)\right)^{\mathfrak{n}}\right)}\,ds. \tag{4.5}$$

The inequalities (4.4) and (4.5) imply, for $(x,y)\succeq_p(u,v)$, that

$$\begin{aligned}\bar{d}\big(T(x,y),T(u,v)\big)&=d(T_1y(t),T_1v(t))+d(T_2x(t),T_2u(t))\\&\leqq\rho_1\frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\big(m\theta_1d(y,v)\big)^{\mathfrak{n}}\right)}+\rho_2\frac{1}{2\rho\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\big(n\theta_2d(x,u)\big)^{\mathfrak{n}}\right)}\\&\leqq\frac{1}{2\theta}\left[\sqrt[\mathfrak{n}]{\ln\left(1+\big(m\theta_1d(y,v)\big)^{\mathfrak{n}}\right)}+\sqrt[\mathfrak{n}]{\ln\left(1+\big(n\theta_2d(x,u)\big)^{\mathfrak{n}}\right)}\right]\\&\leqq\frac{1}{\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\big(m\theta_1d(y,v)\big)^{\mathfrak{n}}+\big(n\theta_2d(x,u)\big)^{\mathfrak{n}}\right)}\\&\leqq\frac{1}{\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\big(m\theta_1d(y,v)+n\theta_2d(x,u)\big)^{\mathfrak{n}}\right)},\end{aligned}$$

that is,

$$\begin{aligned}\bar{d}\big(T(x,y),T(u,v)\big)=d(T_1y(t),T_1v(t))+d(T_2x(t),T_2u(t))&\leqq\frac{1}{\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\big(\theta\bar{d}((x,y),(u,v))\big)^{\mathfrak{n}}\right)}\\&\leqq\frac{1}{\theta}\sqrt[\mathfrak{n}]{\ln\left(1+\psi\big(\bar{d}((x,y),(u,v))\big)^{\mathfrak{n}}\right)}\leqq\frac{1}{\theta}F\left(\psi\left(\bar{d}\big((x,y),(u,v)\big)\right),\varphi\left(\bar{d}\big((x,y),(u,v)\big)\right)\right).\end{aligned}$$

Thus we obtain

$$\psi\left(\bar{d}\big(T(x,y),T(u,v)\big)\right)\leqq F\left(\psi\left(\bar{d}\big((x,y),(u,v)\big)\right),\varphi\left(\bar{d}\big((x,y),(u,v)\big)\right)\right),$$

that is, the operator T satisfies the hypothesis (3.1) of Theorem 2. Moreover, the zero function $(0,0)\in C^2[0,1]$ satisfies $(0,0)\preceq T(0,0)$. Then, clearly, all of the hypotheses of Theorems 2 and 3 are satisfied. Consequently, the operator T has a unique fixed point or, equivalently, the system (1.2) has a unique positive solution in $C^2[0,1]$. We have thus completed our proof of Theorem 4. □

5. CONCLUDING REMARKS AND OBSERVATIONS

In our present investigation, we have established several fixed point results for the $F(\psi,\varphi)$-contractions in partially-ordered metric spaces. As an application of one of these fixed point theorems, we have discussed the existence of a unique solution for a coupled system of higher-order fractional differential equations which are equipped with multi-point boundary conditions. We have also shown that the results presented in this paper extend many recent results appearing in the literature on the subject-matter of this paper. It is believed that several recent works (see, for example, [1], [20] and [21]) will possibly motivate further research on mathematical modelling and analysis of applied problems along the lines which we have developed in this article.

REFERENCES

[1] B. Ahmad, M. Alghanmi, A. Alsaedi, H. M. Srivastava, and S. K. Ntouyas, "The Langevin Equation in Terms of Generalized Liouville-Caputo Derivatives with Nonlocal Boundary Conditions Involving a Generalized Fractional Integral," Mathematics **7** (2019), Article ID 533, 1–10.

[2] B. Ahmad and A. Alsaedi, "Existence and Uniqueness of Solutions for Coupled Systems of Higher-Order Nonlinear Fractional Differential Equations," Fixed Point Theory Appl, **2010**, Article ID 364560, 1–17 (2010).

[3] A. Amini-Harandi and H. Emami, "A Fixed Point Theorem for Contraction Type Maps in Partially-Ordered Metric Spaces and Application to Ordinary Differential Equations," Nonlinear Anal.: Theory Methods Appl. **72**, 2238–2242 (2010).

[4] A. H. Ansari, "Note on φ-Ψ-Contractive Type Mappings and Related Fixed Point," Second Regional Conference on Mathematical Sciences and Applications (Payame Noor University, Tehran, Iran; October 02, 2014), 377–380 (2014).

[5] Z.-B. Bai and H.-S. Lü, "Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation," J. Math. Anal. Appl. **311**, 495–505 (2005).

[6] I. J. Cabrera, J. Harjani, and K. B. Sadarangani, “Existence and Uniqueness of Positive Solutions for a Singular Fractional Three-Point Boundary Value Problem,” Abstr. Appl. Anal. **2012**, Article ID 803417, 1–18 (2012).

[7] M.-R. Hao and C.-B. Zhai, “Application of Schauder Fixed Point Theorem to a Coupled System of Differential Equations of Fractional Order,” J. Nonlinear Sci. Appl. **7**, 131–137 (2014).

[8] J. Harjani and K. B. Sadarangani, “Generalized Contractions in Partially-Ordered Metric Spaces and Applications to Ordinary Differential Equations,” Nonlinear Anal.: Theory Methods Appl. **72**, 1188–1197 (2010).

[9] R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific Publishing Company, Singapore, New Jersey, Londin and Hong Kong, 2000).

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006).

[11] E. Kreyszig, *Introductory Functional Analysis with Applications* (John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989).

[12] H.-S. Liang and J.-H. Zhang, “Positive Solutions for Boundary Value Problems of Nonlinear Fractional Differential Equation,” Nonlinear Anal.: Theory Methods Appl. **71**, 5545–5550 (2009).

[13] J. J. Nieto and R. Rodríguez-López, “Contractive Mapping Theorems in Partially Ordered Sets and Applications to Ordinary Differential Equations,” Order **22**, 223–239 (2005).

[14] I. Podlubny, *Fractional Differential Equations*: *An Introduction to Fractional Derivatives*, *Fractional Differential Equations*, *to Methods of Their Solution and Some of Their Applications* (Mathematics in Science and Engineering, Vol. **198**, Academic Press, New York, London, Sydney, Tokyo and Toronto, 1999).

[15] A. C. M. Ran and M. C. B. Reurings, “A Fixed Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations,” Proc. Amer. Math. Soc. **132**, 1435–1443 (2004).

[16] M. Rehman and R. A. Khan, “Existence and Uniqueness of Solutions for Multi-Point Boundary Value Problems for Fractional Differential Equations,” Appl. Math. Lett. **23**, 1038–1044 (2010).

[17] M. Rehman and R. A. Khan, “A Note on Boundary Value Problems for a Coupled System of Fractional Differential Equations,” Comput. Math. Appl. **61**, 2630–2637 (2011).

[18] K. M. Saad, H. M. Srivastava, and J. F. Gómez-Aguilar, “A Fractional Quadratic Autocatalysis Associated with Chemical Clock Reactions Involving Linear Inhibition,” Chaos Solitons Fractals **132** (2020), Article ID 109557, 1–9.

[19] M. Shoaib, K. Shah, and R. A. Khan, “On Applications of Coupled Fixed-Point Theorem in Hybrid Differential Equations of Arbitrary Order,” Matrix Sci. Math. **1** (2), 17–21 (2017).

[20] H. M. Srivastava, H. I. Abdel-Gawad, and K. M. Saad, “Stability of Traveling Waves Based upon the Evans Function and Legendre Polynomials,” Appl. Sci. **10**, Article ID 846, 1–16 (2020).

[21] H. M. Srivastava, R. S. Dubey, and M. Jain, “A Study of the Fractional-Order Mathematical Model of Diabetes and Its Resulting Complications,” Math. Methods Appl. Sci. **42**, 4570–4583 (2019).

[22] H. M. Srivastava, A. M. A. El-Sayed, and F. M. Gaafar, “A Class of Nonlinear Boundary Value Problems for an Arbitrary Fractional-Order Differential Equation with the Riemann–Stieltjes Functional Integral and Infinite-Point Boundary Conditions,” Symmetry **10**, Article ID 508, 1–13 (2018).

[23] H. M. Srivastava and K. M. Saad, “Some New Models of the Time-Fractional Gas Dynamics Equation,” Adv. Math. Models Appl. **3** (1), 5–17 (2018).

[24] H. M. Srivastava and K. M. Saad, “New Approximate Solution of the Time-Fractional Nagumo Equation Involving Fractional Integrals Without Singular Kernel,” Appl. Math. Inform. Sci., **14**, 1–8 (2020).

[25] H. M. Srivastava, K. M. Saad, and E. H. F. Al-Sharif, “New Analysis of the Time-Fractional and Space-Time Fractional-Order Nagumo Equation,” J. Inform. Math. Sci. **10**, 545–561 (2018).

[26] X.-W. Su, “Boundary Value Problem for a Coupled System of Nonlinear Fractional Differential Equations,” Appl. Math. Lett. **22**, 64–69 (2009).

[27] S.-R. Sun, Q.-P. Li, and Y.-N. Li, “Existence and Uniqueness of Solutions for a Coupled System of Multi-Term Nonlinear Fractional Differential Equations,” Comput. Math. Appl. **64**, 3310–3320 (2012).

[28] F.-F. Yan, Y.-F. Su, and Q.-S. Feng, “A New Contraction Mapping Principle in Partially Ordered Metric Spaces and Applications to Ordinary Differential Equations,” Fixed Point Theory Appl. **2012**, Article ID 152, 1–13 (2012).

[29] X.-G. Zhang, L.-S. Liu, and Y.-H. Wu, “Existence Results for Multiple Positive Solutions of Nonlinear Higher Order Perturbed Fractional Differential Equations with Derivatives,” Appl. Math. Comput. **219**, 1420–1433 (2012).

[30] X.-B. Zhou, W.-Q. Wu, and H.-J. Ma, “A Contraction Fixed Point Theorem in Partially Ordered Metric Spaces and Application to Fractional Differential Equations,” Abstr. Appl. Anal. **2012**, Article ID 856302, 1–11 (2012).