Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

Bull. Pure Appl. Sci. Sect. E Math. Stat.
38E(2), 625–635 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.5958/2320-3226.2019.00063.8

Some relations on ultraspherical matrix polynomials *

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Abstract The main aim of this work is to give a different approach to the proof of some properties for Ultraspherical matrix polynomials (UMPs). We obtain the connections between Laguerre, Hermite and Ultraspherical matrix polynomials. Some definitions of new families of Ultraspherical matrix polynomials are given. Finally, various families of linear, bilinear and bilateral generating matrix functions (GMFs) for UMPs are given.

Key words Functional matrix calculus, Hermite, Laguerre, Ultraspherical, matrix polynomials.

2010 Mathematics Subject Classification 15A60, 33C50, 42C05, 33D45.

1 Introduction

In mathematics the matrix analogues of the classical orthogonal matrix polynomials, called the orthogonal matrix polynomials, that are most widely used include the Hermite, the Laguerre, the Jacobi, the Gegenbauer, the Chebyshev, the Konhauser, the Humbert and the Legendre matrix polynomials for matrices in $\mathbb{C}^{N\times N}$ (see, [1, 4–6, 8, 9, 13, 15, 16, 18, 21–24]). They have many important applications in such areas as mathematical physics, statistics, engineering, group representation theory, approximation theory, numerical analysis, number theory and many others. Our main aim in this work is to establish new properties for the Ultraspherical matrix polynomials. The outline of this work is as follows. The relations between the Laguerre, the Hermite and the Ultraspherical matrix polynomials in this study are proven and a new type of UMPs is given in section 2. Finally, various families of linear, bilinear and bilateral GMFs of the UMPs are presented in section 3.

1.1 Preliminaries

First of all, we start with some basic concepts, theorems, definitions and terminology for this paper. The complex space $\mathbb{C}^{N\times N}$ of all square complex matrices of common order N is considered throughout this paper. The null and identity matrix of $\mathbb{C}^{N\times N}$ will be denoted by $\mathbf{0}$ and I, respectively.

* Communicated by *Prof. Dr. A.K. Sharma* (Managing Editor).
Received March 16, 2019 / Revised October 29, 2019 / Accepted November 27, 2019. Online First Published on December 24, 2019 at https://www.bpasjournals.com/.
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For the purpose of this work, we recall:

Definition 1.1. (Defez and Jódar [7]) For $P \in \mathbb{C}^{N\times N}$. We say that P is a positive stable matrix if

$$Re(\mu) > 0 \text{ for all } \ \mu \in \sigma(P), \ \ \sigma(P) := \text{spectrum of} P, \tag{1.1}$$

where $\sigma(P)$ is the set of all the eigenvalues of P.

Definition 1.2. (Jódar and Cortés [10]) The Gamma matrix function $\Gamma(P)$ is defined by

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I} dt; \quad t^{P-I} = \exp\Big((P-I)\ln t\Big), \tag{1.2}$$

where P is a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.1).

Definition 1.3. (Jódar and Company [11]) The Hermite matrix polynomials (HMPs) are defined by

$$H_n(x, A) = n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{k!(n-2k)!} (x\sqrt{2A})^{n-2k}, n \geq 0 \tag{1.3}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.1).

Definition 1.4. (Jódar et. al. [12]) Let A be a matrix in $\mathbb{C}^{N\times N}$ such that

$$-k \notin \sigma(A) \text{ for all integers } k > 0. \tag{1.4}$$

Then the Laguerre matrix polynomials (LMPs) are defined by

$$L_n^{(A,\lambda)}(x) = \sum_{r=0}^{n} \frac{(-nI)_r (A+I)_n [(A+I)_r]^{-1} (\lambda x)^r}{r!n!} \tag{1.5}$$

where λ is a complex number with $Re(\lambda) > 0$.

Definition 1.5. [19, 20] Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$Re(\lambda) > -\frac{1}{2}, \ \forall \ \lambda \in \sigma(A). \tag{1.6}$$

For $n \geq 0$, the UMPs $P_n^A(x)$ are defined by the hypergeometric matrix function

$$P_n^A(x) = \frac{(A+I)_n}{n!} \ {}_2F_1\Big(-nI, 2A+(n+1)I; A+I; \frac{1-x}{2}\Big) \tag{1.7}$$

such that $A + (n+1)I$ is an invertible matrix for all integers $n \geq -1$ and for $\left|\frac{1-x}{2}\right| < 1$.

Theorem 1.6. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.6). Then a generating matrix function (GMF) representation for the UMPs is the following:*

$$F(x, t, A) = \sum_{n=0}^{\infty} (2A+I)_n [(A+I)_n]^{-1} P_n^A(x) t^n = \left[1 - 2tx + t^2\right]^{-(A+\frac{1}{2}I)}; \ |t| < r, \ |x| < 1, |2tx - t^2| < 1. \tag{1.8}$$

If r_1 and r_2 are the roots of the quadratic equation $1 - 2xt + t^2 = 0$ and if r is the minimum of the set $\{r_1, r_2\}$, then the matrix function $F(x, t, A)$ regarded as a function of t, is analytic in the disk $|t| < r$, for every real number for which $|x| < 1$.

Theorem 1.7. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.6). Then the GMFs are derived in the form*

$${}_0F_1\Big(-; A+I; \frac{t(x-1)}{2}\Big) \ {}_0F_1\Big(-; A+I; \frac{t(x+1)}{2}\Big) = \sum_{n=0}^{\infty} [(2A+I)_n]^{-1} [(A+I)_n]^{-1} P_n^A(x) t^n \tag{1.9}$$

where $A + (k+1)I$ are invertible matrices for all integers $k \geq -1$.

2 Connections between the Laguerre, the Hermite and the Ultraspherical matrix polynomials

Here, we deduce the relations between the Ultraspherical and the Laguerre matrix polynomials.

Theorem 2.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.6). Then the connection between the Laguerre and the Ultraspherical matrix polynomials is*

$$P_n^A(x) = \Gamma^{-1}(2A+(n+1)I)\int_0^\infty t^{2A+nI}e^{-t}L_n^A\left(\frac{1-x}{2\lambda}t\right)dt. \tag{2.1}$$

Proof. Using (1.5), the right hand side of (2.1) can be written as

$$\begin{aligned}&\Gamma^{-1}(2A+(n+1)I)\int_0^\infty t^{2A+nI}e^{-t}L_n^A\left(t\frac{1-x}{2\lambda}\right)dt\\ &=\Gamma^{-1}(2A+(n+1)I)\sum_{r=0}^{n}\frac{(-nI)_r(A+I)_n[(A+I)_r]^{-1}}{r!n!}\left(\frac{1-x}{2}\right)^r\Gamma^{-1}(2A+(n+r+1)I).\end{aligned} \tag{2.2}$$

Further by using the GMFs to evaluate the integral and by making the necessary arrangements we obtain the required relation. □

Theorem 2.2. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.4). The Laguerre and the Ultraspherical matrix polynomials satisfy the following interesting formula:*

$$L_n^{A,\lambda}(x) = \lim_{r\to\infty} P_n^{rA}\left(1-\frac{2}{r}x\lambda\right). \tag{2.3}$$

Proof. Using (1.7) in the left-hand side of (2.2) the theorem can be proved. □

Here, the HMPs will be utilized to define another version of the new types of generalization of UMPs with two and three matrices.

Theorem 2.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.6) and B be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.1). Then the integral representation for UMPs with two matrices is given by*

$$P_n^A(x,B) = \frac{1}{n!}(A+I)_n[(2A+I)_n]^{-1}\Gamma^{-1}(A+\frac{1}{2}I)\int_0^\infty e^{-u}\ u^{A+\frac{n-1}{2}}\ {}_HH_n(x\sqrt{u},B)du. \tag{2.4}$$

Proof. Using (1.3) in the right hand side of (2.1), we have

$$\frac{1}{n!}\int_0^\infty e^{-u}u^{A+\frac{n-1}{2}I}\ H_n(x\sqrt{u},B)du = \sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2B})^{n-2k}\int_0^\infty e^{-u}u^{A+(n-k-\frac{1}{2})I}du.$$

Using the Gamma matrix function, we can write

$$\Gamma\left(A+(n-k+\frac{1}{2})I\right) = \int_0^\infty e^{-u}u^{A+(n-k-\frac{1}{2})I}du.$$

□

Hence, we find that the UMPs with two matrices are defined by the following series

$$P_n^A(x;B) = (A+I)_n[(2A+I)_n]^{-1}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k y^k\left(A+\frac{1}{2}I\right)_{n-k}}{k!(n-2k)!}(x\sqrt{2B})^{n-2k}. \tag{2.5}$$

Thus, we have the following main theorem.

Theorem 2.4. *Let B be a matrix in $\mathbb{C}^{N\times N}$, where $Re(\mu) > 0$ for all eigenvalues $\mu \in \sigma(B)$, and let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $(-\frac{z}{2}) \in \sigma(A)$ for all $z \in \mathbb{Z}^+ \cup \{0\}$, $AB = BA$ and $\|B\| < \frac{1}{\sqrt{2}}$. Then the GMF for UMPs with two matrices is*

$$\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x;B)t^n = (I - xt\sqrt{2B} + t^2 I)^{-A-\frac{1}{2}I} \tag{2.6}$$

where $\|xt\sqrt{2B} - t^2 I\| < 1$.

Proof. In [7], the HMPs $H_n(x,B)$ are defined as

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}H_n(x,B) = \exp\left(xu\sqrt{2B} - u^2 I\right). \tag{2.7}$$

Multiplying t^n on both sides of (2.3) and summing up over n, using (2.6) and them integrating over u, we obtain (2.5). The theorem is thus proved. □

In [22], the generalized Hermite matrix polynomials $H_{n,m,\nu}(x,y,B)$ is defined as

$$H_{n,m,\nu}(x,y,B) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k}{k!\Gamma\left(\frac{n-mk}{\nu}+1\right)}\left(x\sqrt{mB}\right)^{\frac{n-mk}{\nu}} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n,m,\nu}(x,y,B) = \exp\left(xt^{\nu}\sqrt{mB} - yt^m I\right). \tag{2.9}$$

In general, we can introduce the new generalized Ultraspherical-type matrix polynomials with three matrices by using the integral representation:

$$P_{n,m}^A(x,y;B,P) = \frac{1}{n!}(A+I)_n[(2A+I)_n]^{-1}\Gamma^{-1}(A+\frac{1}{2}I)\int_0^{\infty} e^{-Pu}\, u^{A-\frac{1}{2}I}\, H_{n,m}(xu,yu,B)du, \tag{2.10}$$

Or

$$P_{n,m}^A(x,y;B,P) = (A+I)_n[(2A+I)_n]^{-1}\sum_{k=0}^{[\frac{1}{m}n]}\frac{(-1)^k P^{(m-1)kI-nI-A}y^k(A+\frac{1}{2}I)_{n-(m-1)k}}{k!(n-mk)!}(x\sqrt{mB})^{n-mk}. \tag{2.11}$$

Theorem 2.5. *Let B be a matrix in $\mathbb{C}^{N\times N}$, where $Re(\mu) > 0$ for all eigenvalues $\mu \in \sigma(B)$, and A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $(-\frac{z}{2}) \in \sigma(A)$ for all $z \in \mathbb{Z}^+ \cup \{0\}$, $AB = BA$ and $\|B\| < \frac{1}{\sqrt{2}}$. Then the GMF for generalized Ultraspherical-type matrix polynomials with three matrices is*

$$\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_{n,m}^A(x,y;B,P)t^n = (P - xt\sqrt{mB} + yt^m I)^{-A-\frac{1}{2}I} \tag{2.12}$$

where $\|xt\sqrt{mB} - yt^m I\| < 1$.

Proof. Multiplying t^n on both sides of (2.9) and summing up over n, using (2.8) and them integrating over u, we find (2.11) and thus the relation is established. □

3 New kind of Ultraspherical matrix polynomials

In the forthcoming concluding section, we will present further interesting consequences of the point of view developed in our study. Now, we define a new kind of UMPs with the help of the GMFs (1.9) and establish some of their properties.
In (1.9) taking $\Phi_n^A(x) = (1+x)^n[(2A+I)_n]^{-1}[(A+I)_n]^{-1}P_n^A(\frac{1-x}{1+x})$, we obtain the result which is summarized in the following theorem:

Theorem 3.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.6). Then we have the derived generating matrix function*

$$ {}_0F_1(-;A+I;t)\ {}_0F_1(-;A+I;-xt) = \sum_{n=0}^{\infty} \Phi_n^A(x)t^n. \tag{3.1}$$

Now, we prove the following interesting relation for $\Phi_n^A(x)$.

Theorem 3.2. *The new matrix polynomials satisfy the differential equation*

$$\theta(\theta\ I+A)\Phi_n^A(x)+x\Phi_{n-1}^A(x)=\boldsymbol{0}; n\geq 1, \tag{3.2}$$

where $\theta = z\frac{d}{dz}$ is differential operator.

Proof. Starting from (3.1), we write ${}_0F_1(-;A+I;z)$, then

$$\theta(\theta\ I+A)\ {}_0F_1(-;A+I;z) = \sum_{k=1}^{\infty}\frac{(k\ I+A)}{(k-1)!}z^k[(A+I)_k]^{-1}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}z^{k+1}[(A+I)_k]^{-1} = z\ {}_0F_1(-;A+I;z).$$

The matrix function $Y = {}_0F_1(-;A+I;-z)$ is a solution of matrix differential equation

$$\Big[\theta(\theta\ I+A)-z\ I\Big]\ {}_0F_1(-;A+I;z)=\mathbf{0}.$$

Put $z=-xt$. Then the differential operator $\theta = x\frac{d}{dx}$ becomes

$$\Big[\theta(\theta\ I+A)+xt\ I\Big]\ {}_0F_1(-;A+I;-xt)=\mathbf{0}; \theta = x\frac{d}{dx}.$$

On the other hand, we can write

$$\theta(\theta\ I+A)\ {}_0F_1(-;A+I;-xt) = -xt\ {}_0F_1(-;A+I;-xt).$$

Now, we consider the effect of differential operator on both sides of the above relation on ${}_0F_1(-;A+I;t)$ to get

$$\theta(\theta\ I+A)\ {}_0F_1(-;A+I;-xt)\ {}_0F_1(-;A+I;t) = -xt\ {}_0F_1(-;A+I;-xt)\ {}_0F_1(-;A+I;t).$$

or,

$$\theta(\theta\ I+A)\sum_{n=0}^{\infty}\Phi_n^A(x)t^n = -x\sum_{n=0}^{\infty}\Phi_n^A(x)t^{n+1} = -x\sum_{n=1}^{\infty}\Phi_{n-1}^A(x)t^n.$$

Therefore,

$$\theta(\theta\ I+A)\Phi_n^A(x)+x\Phi_{n-1}^A(x)=\mathbf{0}; n\geq 1; \theta(\theta\ I+A)\Phi_0^A(x)=\mathbf{0}.$$

This ends the proof. □

4 Bilinear and bilateral GMPs for the UMPs

These major properties are completed to develop several families of bilinear and bilateral GMFs for the UMPs derived from the GMPs (1.9), then using Theorem 1.7 and given explicitly by (1.7) without using Lie algebraic techniques but, with the help of the similar method as considered in [1–5, 21]. We state our results as the follows:

Theorem 4.1. *Corresponding to a non-vanishing matrix function $\Omega_\mu(y_1, y_2, \dots, y_s)$ of s complex variables $y_1, y_2, \dots, y_s$, $s \in \mathbb{N}$ and involving a complex parameter μ, called the order, let us consider the following*

$$\Lambda_{\mu,\nu}(y_1, y_2, \dots, y_s; z) = \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, y_2, \dots, y_s) z^k; a_k \neq 0, \mu, \nu \in \mathbb{C} \tag{4.1}$$

where the coefficients a_k are assumed to be non-vanishing in order for the matrix function on the L.H.S to be non-null. Suppose that

$$\begin{aligned}&\Psi_{n,m,\mu,\nu}(x; y_1, y_2, \dots, y_s; \eta)\\ &= \sum_{k=0}^{[\frac{1}{m}n]} a_k [(2A+I)_{n-mk}]^{-1} [(A+I)_{n-mk}]^{-1} P^A_{n-mk}(x) \Omega_{\mu+\nu k}(y_1, y_2, \dots, y_s) \eta^k; n, m \in \mathbb{N}\end{aligned} \tag{4.2}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition $Re(\lambda) > -\frac{1}{2}$, for all eigenvalues $\lambda \in \sigma(A)$ and (as usual) $[\alpha]$ represents the greatest integer in $\alpha \in \mathbb{R}$. Then we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \dots, y_s; \frac{\eta}{t^m}\right) t^n\\ &= {}_0F_1\left(-; A+I; \frac{1}{2}t(x-1)\right) {}_0F_1\left(-; A+I; \frac{1}{2}t(x+1)\right) \Lambda_{\mu,\nu}(y_1, y_2, ..., y_s; \eta).\end{aligned} \tag{4.3}$$

Proof. For convenience, let S denote the first member of the assertion (4.3) of the Theorem 4.1. Then, plugging the matrix polynomials $\Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \dots, y_s; \frac{\eta}{t^m}\right)$ from (4.2) into the L.H.S. of (4.3), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \dots, y_s; \frac{\eta}{t^m}\right) t^n\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{m}n]} a_k [(2A+I)_{n-mk}]^{-1} [(A+I)_{n-mk}]^{-1} P^A_{n-mk}(x) \Omega_{\mu+\nu k}(y_1, y_2, \dots, y_s) \eta^k t^{n-mk}.\end{aligned} \tag{4.4}$$

Upon changing the order of summation in (4.4), if we replace n by $n = n + mk$, we can write

$$\begin{aligned}&\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \dots, y_s; \frac{\eta}{t^m}\right) t^n\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \left[(2A+I)_n\right]^{-1} \left[(A+I)_n\right]^{-1} P_n^{(A)}(x)\, \Omega_{\mu+\nu k}(y_1, y_2, \dots, y_s) \eta^k t^n\\ &= \left[\sum_{n=0}^{\infty} \left[(2A+I)_n\right]^{-1} \left[(A+I)_n\right]^{-1} P_n^{(A)}(x)\, t^n\right] \left[\sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, y_2, \dots, y_s) \eta^k\right]\\ &= {}_0F_1\left(-; A+I; \frac{1}{2}t\,(x-1)\right) {}_0F_1\left(-; A+I; \frac{1}{2}t\,(x+1)\right) \Lambda_{\mu\nu}\,(y_1, y_2, \dots, y_s; \eta).\end{aligned}$$

□

By expressing the multivariable matrix function $\Omega_{\mu+\nu k}(y_1, y_2, \dots, y_s)$, $k \in \mathbb{N}_0$ and $s \in \mathbb{N}$ in terms of simpler matrix function of one and more variables, we can give further applications of the Theorem 4.1. In the following, we obtain the results which provide a class of bilateral GMFs for the UMPs.

Corollary 4.2. *Let*

$$\Lambda_{\mu,\nu}(y; z) = \sum_{k=0}^{\infty} a_k [(2B+I)_k]^{-1} [(B+I)_k]^{-1} P^B_{\mu+\nu k}(y) z^k; a_k \neq 0, \mu, \nu \in \mathbb{N}_0$$

and

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k[(2A+I)_{n-mk}]^{-1}[(2B+I)_k]^{-1}[(A+I)_{n-mk}]^{-1}[(B+I)_k]^{-1}\times P^A_{n-mk}(x)P^B_{\mu+\nu k}(y)\eta^k;n,m\in\mathbb{N}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (1.6). Then we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta) \quad (4.5)$$

provided that each member of (4.5) exists.

Proof. Equation (4.5) can be proved using the same method in the Theorem 4.1. □

Remark 4.3. For the UMPs given by the GMPs (1.9) on taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\left[(2A+I)_{n-mk}\right]^{-1}\left[(2B+I)_k\right]^{-1}\left[(A+I)_{n-mk}\right]^{-1}\left[(B+I)_k\right]^{-1}P^A_{n-mk}(x)\,P^B_k(y)\,\eta^k t^{n-mk}$$
$$={}_0F_1\left(-;A+I;\frac{1}{2}t\,(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t\,(x+1)\right){}_0F_1\left(-;B+I;\frac{1}{2}\eta\,(y-1)\right)\times {}_0F_1\left(-;B+I;\frac{1}{2}\eta\,(y+1)\right).$$

Corollary 4.4. *Let $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_k\Phi^B_{\mu+\nu k}(y)z^k; a_k\neq 0,\mu,\nu\in\mathbb{N}_0$ and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k[(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1}P^A_{n-mk}(x)\Phi^B_{\mu+\nu k}(y)\eta^k;n,m\in\mathbb{N}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying (1.6). Then we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta) \quad (4.6)$$

provided that each member of (4.6) exists.

Remark 4.5. Using the GMPs (3.1) for the $\Phi^B_k(y)$ and taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\left[(2A+I)_{n-mk}\right]^{-1}\left[(A+I)_{n-mk}\right]^{-1}P^A_{n-mk}(x)\,\Phi^B_k(y)\eta^k t^{n-mk}$$
$$={}_0F_1\left(-;A+I;\frac{1}{2}t\,(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t\,(x+1)\right){}_0F_1\left(-;B+I;-y\eta\right){}_0F_1\left(-;B+I;\eta\right)$$

The Rice's matrix polynomials (RMPs) $H_n(A,B,z)$ are defined by

$$\begin{aligned}H_n(A,B,z)&=\sum_{k=0}^{\infty}\frac{z^k}{k!}(-nI)_k((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1};0\leqslant k\leqslant n\\&=\sum_{k=0}^{\infty}\frac{(-1)^kz^k}{k!(n-k)!}(I)_n((1+n)I)_k(A)_k[(I)_k]^{-1}[(B)_k]^{-1}\end{aligned}\quad (4.7)$$

where A and B in $\mathbb{C}^{N\times N}$ are commutative matrices in $\mathbb{C}^{N\times N}$ satisfying $B+kI$ is invertible for all integer $k\geq 0$ (see [14]). Here the RMPs are generated by

$$\sum_{n=0}^{\infty}H_n(A,B,z)t^n=(1-t)^{-I}\,{}_2F_1\left(\frac{1}{2}I,A;B;-\frac{4zt}{(1-t)^2}\right),\left|\frac{4zt}{(1-t)^2}\right|<1,|t|<1. \quad (4.8)$$

If we take $\Omega_{\mu+\nu k}(z)=H_{\mu+\nu k}(A,B,z)$ for the case $s=1$ in Theorem 4.1 then we obtain the following result which provides a class of bilateral GMPs for the Ultraspherical and the Rice's matrix polynomials.

Corollary 4.6. *Let* $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_kH_{\mu+\nu k}(B,C,y)z^k; a_k\neq 0,\mu,\nu\in\mathbb{N}_0$ *and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k[(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1}P^A_{n-mk}(x)H_{\mu+\nu k}(B,C,y)\eta^k;n,m\in\mathbb{N}$$

where A *is a matrix in* $\mathbb{C}^{N\times N}$ *satisfying the spectral condition* $Re(\lambda)>-\frac{1}{2}$ *for all eigenvalues* $\lambda\in\sigma(A)$. *Then we have*

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta)\quad(4.9)$$

provided that each member of (4.9) *exists.*

Remark 4.7. Using the GMPs (4.8) for the RMPs $H_k(B,C,y)$ and taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\left[(2A+I)_{n-mk}\right]^{-1}\left[(A+I)_{n-mk}\right]^{-1}P^A_{n-mk}(x)\,H_k(B,C,y)\,\eta^kt^{n-mk}$$
$$={}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)(1-\eta)^{-I}{}_2F_1\left(\frac{1}{2}I,B;C;-\frac{4y\eta}{(1-\eta)^2}\right),$$

for $\left|\frac{4y\eta}{(1-\eta)^2}\right|<1$,$|\eta|<1$.

According to [12], LMPs are generated by

$$\sum_{n=0}^{\infty}L_n^{(A,\lambda)}(x)t^n=(1-t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1-t}\right),\quad(4.10)$$

where $t,x\in\mathbb{C}$ and $|t|<1$.

Corollary 4.8. *Let* $\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_kL^{(B,\lambda)}_{\mu+\nu k}(y)z^k; a_k\neq 0,\mu,\nu\in\mathbb{N}_0$ *and*

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k[(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1}P^A_{n-mk}(x)L^{(B,\lambda)}_{\mu+\nu k}(y)\eta^k;n,m\in\mathbb{N}$$

where A *is a matrix satisfying the condition* $Re(\lambda)>-\frac{1}{2}$ *for all eigenvalues* $\lambda\in\sigma(A)$, *and* B *is a matrix satisfying the condition* (1.4). *Then we have*

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n={}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y;\eta)\quad(4.11)$$

provided that each member of (4.11) *exists.*

Remark 4.9. For the LMPs $L_k^{(B,\lambda)}(y)$, by the GMFs (4.10) on taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}[(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1}P^A_{n-mk}(x)L_k^{(B,\lambda)}(y)\eta^kt^{n-mk}$$
$$={}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)(1-\eta)^{-(A+I)}\exp\left(\frac{-\lambda\,y\eta}{1-\eta}\right).$$

Choose $s=1$ and $\Omega_{\mu+\nu k}(y)=\mathbf{P}_{\mu+\nu k}(y,C)$ in the Theorem 4.1, where the Legendre matrix polynomials $\mathbf{P}_n(x,C)$ are defined as [17]:

$$\mathbf{P}_n(x,C)=\sum_{k=0}^{n}\frac{(-1)^k(n+k)!}{k!(n-k)!}\left(\frac{1-x}{2}\right)^k\Gamma^{-1}(C+kI)\Gamma(C),n\geq 0\quad(4.12)$$

where C is a matrix in $\mathbb{C}^{N\times N}$ satisfying

$$0 < \Re(\lambda) < 1, \quad \text{for all } \lambda \in \sigma(C). \tag{4.13}$$

and $C + kI$ are invertible matrices for all integers $k \geq 0$ and $\left|\frac{1-x}{2}\right| < 1$. Here, the Legendre matrix polynomials $\mathbf{P}_n(x, C)$ are generated as follows:

$$\sum_{n=0}^{\infty} \mathbf{P}_n(x, C)t^n = (1-t)^{-1}\,{}_1F_1\left(\frac{1}{2}I; C; \frac{2t(x-1)}{(1-t)^2}\right); |t| < 1, \left|\frac{2t(x-1)}{(1-t)^2}\right| < 1. \tag{4.14}$$

Then we obtain a class of bilateral GMFs for the Ultraspherical and Legendre matrix polynomials.

Corollary 4.10. *Let $\Lambda_{\mu,\nu}(y; z) = \sum_{k=0}^{\infty} a_k \boldsymbol{P}_{\mu+\nu k}(y, C)z^k; a_k \neq 0, \mu, \nu \in \mathbb{N}_0$ and*

$$\Psi_{n,m,\mu,\nu}(x; y; \eta) = \sum_{k=0}^{[\frac{1}{m}n]} a_k [(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1} P_{n-mk}^{A}(x) \boldsymbol{P}_{\mu+\nu k}(y, C)\eta^k; n, m \in \mathbb{N}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying $Re(\lambda) > -\frac{1}{2}$ for all eigenvalues $\lambda \in \sigma(A)$, and C is a matrix in $\mathbb{C}^{N\times N}$ satisfying (4.13). Then we have

$$\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y; \frac{\eta}{t^m}\right)t^n = {}_0F_1\left(-; A+I; \frac{1}{2}t(x-1)\right)\,{}_0F_1\left(-; A+I; \frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y; \eta) \tag{4.15}$$

provided that each member of (4.15) exists.

Remark 4.11. For the $\mathbf{P}_k(y, C)$ generated by the GMFs (4.14) on taking $a_k = 1$, $\mu = 0$ and $\nu = 1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]} [(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1} P_{n-mk}^{A}(x)\mathbf{P}_k(y, C)\eta^k t^{n-mk} \\
&= {}_0F_1\left(-; A+I; \frac{1}{2}t(x-1)\right)\,{}_0F_1\left(-; A+I; \frac{1}{2}t(x+1)\right)\Big) \\
&\times (1-\eta)^{-1}\,{}_1F_1\left(\frac{1}{2}I; C; \frac{2\eta(y-1)}{(1-\eta)^2}\right); |\eta| < 1, \left|\frac{2\eta(y-1)}{(1-\eta)^2}\right| < 1.
\end{aligned}$$

For example, if we set $s = 2$ and $\Lambda_{\mu,\nu}(y, z) = \sum_{k=0}^{\infty} a_k H_{\mu+\nu k}(y, z, B)z^k; a_k \neq 0, \mu, \nu \in \mathbb{N}_0$ in Theorem 4.1, where the HMPs are defined by (see [22])

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(y, z, B) = \exp\left(yt\sqrt{2A} - zt^2 I\right); |t| < \infty. \tag{4.16}$$

where B is a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying (1.1), then we obtain the result which provides a family of bilateral GMFs for a matrix version of the multivariable Hermite matrix polynomials of two variables and the UMPs.

Corollary 4.12. *Let $\Lambda_{\mu,\nu}(y; z) = \sum_{k=0}^{\infty} a_k H_{\mu+\nu k}(y, z, B)z^k; a_k \neq 0, \mu, \nu \in \mathbb{N}_0$ and*

$$\Psi_{n,m,\mu,\nu}(x; y, z; \eta) = \sum_{k=0}^{[\frac{1}{m}n]} a_k [(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1} P_{n-mk}^{A}(x) H_{\mu+\nu k}(y, z, B)\eta^k; n, m \in \mathbb{N}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.6) and B is a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying $Re(\lambda) > 0$ for all eigenvalues $\lambda \in \sigma(B)$. Then we have

$$\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y; \frac{\eta}{t^m}\right)t^n = {}_0F_1\left(-; A+I; \frac{1}{2}t(x-1)\right)\,{}_0F_1\left(-; A+I; \frac{1}{2}t(x+1)\right)\Lambda_{\mu,\nu}(y, z; \eta) \tag{4.17}$$

provided that each member of (4.17) exists.

Remark 4.13. Using the GMFs (4.16) for the HMPs $H_k(y,z,B)$ and taking $a_k = \frac{1}{k!}$, $\mu = 0$ and $\nu = 1$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}[(2A+I)_{n-mk}]^{-1}[(A+I)_{n-mk}]^{-1}P^{A}_{n-mk}(x)\frac{1}{k!}H_k(y,z,B)\eta^k t^{n-mk}\\&= {}_0F_1\left(-;A+I;\frac{1}{2}t(x-1)\right){}_0F_1\left(-;A+I;\frac{1}{2}t(x+1)\right)\exp\left(y\eta\sqrt{2A}-z\eta^2 I\right).\end{aligned}$$

Acknowledgments The authors are grateful to the anonymous referees for their valuable comments on the original draft of this manuscript which they have found much beneficial for improving the presentation of this paper and they also thank them for suggesting certain modifications in some of the results deduced in this paper.

Dedicatory The second author (LMU) hereby most gratefully acknowledges the most gracious and benevolent support, patronage and the incessant encouragement received by him from The Best Master (Employer) of his entire teaching and research career - The Most Hon'ble Professor Dr. S.P. Joshi - the Principal and Professor of this author's institution, who, himself being a well known international authority on Himalayan Plant Botany and the Himalayan Environmental Issues, having fully understood the importance of the extreme hard work and perseverance of this most humble subordinate servant of him, has for the first time ever in the entire service career of this author, left no stone unturned to provide brand new computing facilities to this author in the Department of Mathematics of this author's institution, where this author has not only typeset this manuscript in Latex but also almost all of this issue of this Journal in his capacity as the Editor-in-Chief of this Journal. The second author (LMU) has no words at all to express his lifelong indebtedness to his Greatest Master - The Hon'ble Professor Dr. S.P. Joshi - for all the countless pains that he has taken despite facing some of the greatest challenges and grave threats and the most intense hostility and animosity from many quarters for all his endeavors towards the career progression (promotion) of this author and for the overall academic, research and holistic development of this author's institution. *The second author (LMU), therefore, takes this most humble opportunity to hereby dedicate this first ever issue of this Journal, which is typeset in Latex by him, to the following two leading revered academicians and researchers of his life: firstly, to his life's Best Master - The Most Hon'ble Professor Dr. S.P. Joshi and secondly, to The Most Hon'ble Professor Dr. A.K. Sharma, the Founder Editor-in-Chief and the Current Managing Editor of this Journal, who has always shown full trust on this humble author's capabilities in bringing out this Journal in Latex. In the conclusion the second author (LMU) also expresses his extreme indebtedness and thanks with his utmost devotion to the Lord God for all His extreme grace, benevolence and blessings showering upon this author forever!*

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